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BEST APPROXIMATION IN SPACES WITH ASYMMETRIC NORM

S. COBZAŞ* and C. MUSTĂŢA†

Abstract. In this paper we shall present some results on spaces with asymmetric seminorms, with emphasis on best approximation problems in such spaces.

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1. INTRODUCTION

Let X be a real vector space. An *asymmetric seminorm* on X is a positive sublinear functional $p : X \to [0, \infty)$, i.e. p satisfies the conditions:

(AN1) $p(x) \geq 0$;
(AN2) $p(tx) = tp(x)$;
(AN3) $p(x + y) \leq p(x) + p(y)$,

for all $x, y \in X$ and $t \geq 0$.

The function $\bar{p} : X \to [0, \infty)$ defined by $\bar{p}(x) = p(-x)$, $x \in X$, is another positive sublinear functional on X, called the *conjugate* of p, and

$$p^s(x) = \max\{p(x), p(-x)\},\ x \in X, \tag{1.1}$$

is a seminorm on $X$and the inequalities

$$|p(x) - p(y)| \leq p^s(x - y) \quad \text{and} \quad |\bar{p}(x) - \bar{p}(y)| \leq p^s(x - y) \tag{1.2}$$

hold for all $x, y \in X$. If the seminorm p^s is a norm on X then we say that p is an *asymmetric norm* on X. This means that, beside (AN1)–(AN3), it satisfies also the condition

(AN4) $p(x) = 0$ and $p(-x) = 0$ imply $x = 0$.

The pair (X, p), where X is a linear space and p is an asymmetric seminorm on X is called a *space with asymmetric seminorm*, respectively a *space with asymmetric norm*, if p is an asymmetric norm.

*"Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, 400084 Cluj-Napoca, Romania, e-mail: scobzas@math.ubbcluj.ro.
†"T. Popoviciu" Institute of Numerical Analysis, P.O. Box 68-1, Cluj-Napoca, Romania, e-mail: cmustata@ictp.acad.ro.

The function $\rho : X \times X \to [0;\infty)$ defined by $\rho(x,y) = p(y-x),\ x,y \in X$, is an asymmetric semimetric on X. Denote by

$$B'_p(x,r) = \{x' \in X : p(x'-x) < r\} \text{ and } B_p(x,r) = \{x' \in X : p(x'-x) \le r\},$$

the open, respectively closed, ball in X of center x and radius $r > 0$. Denoting by

$$B'_p = B'_p(0,1) \text{ and } B_p = B_p(0,1),$$

the corresponding unit balls then

$$B'_p(x,r) = x + rB'_p \text{ and } B_p(x,r) = x + rB_p.$$

The unit balls B'_p and B_p are convex absorbing subsets of the space X and p agrees with the Minkowski functional associated to any of them. Recall that for an absorbing subset C of X the Minkowski functional $p_C : X \to [0;\infty)$ is defined by

$$p_C(x) = \inf\{t > 0 : x \in tC\}.$$

If C is absorbing and convex, then p_C is a positive sublinear functional, and

$$\{x \in X : p_C(x) < 1\} \subset C \subset \{x \in X : p_C(x) \le 1\}.$$

An asymmetric seminorm p generates a topology τ_p on X, having as basis of neighborhoods of a point $x \in X$ the family $\{B'_p(x,r) : r > 0\}$ of open p-balls. The family $\{B_p(x,r) : r > 0\}$ of closed p-balls is also a neighborhood basis at x for τ_p.

The topology τ_p is translation invariant, i.e. the addition $+ : X \times X \to X$ is continuous, but the multiplication by scalars $\cdot : \mathbb{R} \times X \to X$ need not be continuous, as it is shown by some examples (see [7]).

The ball $B'_p(x,r)$ is τ_p-open but the ball $B_p(x,r)$ need not to be τ_p-closed, as can be seen from the following typical example.

Example 1.1. Consider on $\mathbb{R}$ the asymmetric seminorm $u(\alpha) = \max\{\alpha, 0\}$, $\alpha \in \mathbb{R}$, and denote by $\mathbb{R}_u$ the space $\mathbb{R}$ equipped with the topology τ_u generated by u. The conjugate seminorm is $\bar{u}(\alpha) = -\min\{\alpha, 0\}$, and $u^s(\alpha) = \max\{u(\alpha), \bar{u}(\alpha)\} = |\alpha|$. The topology τ_u, called the *upper topology* of $\mathbb{R}$, is generated by the intervals of the form $(-\infty; a)$, $a \in \mathbb{R}$, and the family $\{(-\infty; \alpha + \epsilon) : \epsilon > 0\}$ is a neighborhood basis of every point $\alpha \in \mathbb{R}$. The set $(-\infty; 1) = B'_u(0,1)$ is τ_u-open, and the ball $B_u(0,1) = (-\infty; 1]$ is not τ_u-closed because $\mathbb{R} \setminus B_u(0,1) = (1;\infty)$ is not τ_u-open. □

The topology τ_p could not be Hausdorff even if p is an asymmetric norm on X. A necessary and sufficient condition in order that τ_p be Hausdorff was given in [22]. Putting

$$\tilde{p}(x) = \inf\{p(x') + p(x'-x) : x' \in X\},\ x \in X, \tag{1.3}$$

it follows that $\tilde{p}$ is the greatest (symmetric) seminorm majorized by p and the topology τ_p is Hausdorff if and only if $\tilde{p}(x) > 0$ for every $x \neq 0$. Changing x

to $-x$ and taking $x' = 0$ it follows that, in this case, $p(x) > 0$ for every $x \neq 0$, but this condition is not sufficient for τ_p to be Hausdorff, see [22].

Spaces with asymmetric seminorms were investigated in a series of papers, emphasizing similarities with seminormed spaces as well as differences, see [1, 2, 3, 7, 17, 18, 19, 21, 22], and the references quoted therein. Among the differences we mention the fact that the dual of a space with asymmetric seminorm is not a linear space but merely a convex cone in the algebraic dual $X^{\#}$ of X. This is due to the fact that the continuity of a linear functional φ on (X, p) does not imply the continuity of $-\varphi$. For instance, $\varphi(u) = u$ is continuous on $(\mathbb{R}, u)$ but $\psi(u) = -u$ is not continuous. For an other example see [7]. The study of spaces with asymmetric norm was motivated and stimulated also by their applications in the complexity of algorithms, see [20, 40].

Some continuity properties of linear functionals in the symmetric case have their analogs in the asymmetric one.

PROPOSITION 1.2. [21] *Let (X, p) be a space with asymmetric seminorm and $\varphi : X \to \mathbb{R}$ a linear functional. Then the following are equivalent.*

(1) *φ is τ_p-τ_u-continuous at $0 \in X$.*
(2) *φ is τ_p-τ_u-continuous on X.*
(3) *There exists $L \geq 0$ such that*

$$\forall x \in X, \quad \varphi(x) \leq L\, p(x). \tag{1.4}$$

(4) *φ is upper semi-continuous from (X, τ_p) to $(\mathbb{R}, |\,|)$.*

A linear functional satisfying (1.4) is called *semi-Lipschitz* (or *p-bounded*) and L a *semi-Lipschitz constant.* Denote by $X_p^\flat$ the set of all bounded linear functionals on the space with asymmetric seminorm (X, p). As we did mention, $X_p^\flat$ is a convex cone in $X^{\#}$.

One can define a norm $\|\,|_p$ on $X_p^\flat$ by

$$\|\varphi|_p = \sup\{\varphi(x) : x \in B_p\},\ \varphi \in X_p^\flat . \tag{1.5}$$

Some useful properties of this norm, whose proofs can be found in [9, 12], are collected in the following proposition. We agree to call a linear functional φ on (X, p), $(p, \bar{p})$-bounded if it is both p- and $\bar{p}$-bounded, where $\bar{p}$ is the seminorm conjugate to p.

PROPOSITION 1.3. *If φ is a bounded linear functional on a space with asymmetric seminorm (X, p), $p \neq 0$, then the following assertions hold.*

(1) *$\|\varphi|_p$ is the smallest of the numbers $L \geq 0$ for which the inequality* (1.4) *holds.*
(2) *We have*

$$\|\varphi|_p = \sup\{\varphi(x)/p(x) : x \in X,\ p(x) > 0\} \tag{1.6}$$
$$= \sup\{\varphi(x) : x \in X,\ p(x) < 1\} \tag{1.7}$$
$$= \sup\{\varphi(x) : x \in X,\ p(x) = 1\}. \tag{1.8}$$

(3) *If* $\varphi \neq 0$, *then* $\|\varphi|_p > 0$. *Also, if* $\varphi \neq 0$ *and* $\varphi(x_0) = \|\varphi|_p$ *for some* $x_0 \in B_p$, *then* $p(x_0) = 1$.

(4) *If* φ *is* $(p, \bar{p})$*-bounded, then*

$$\varphi(rB'_p) = (-r\|\varphi|_{\bar{p}}, r\|\varphi|_p) \quad \textit{and} \quad \varphi(rB'_{\bar{p}}) = (-r\|\varphi|_p, r\|\varphi|_{\bar{p}})$$

where $B'_p = \{x \in X : p(x) < 1\}$, $B'_{\bar{p}} = \{x \in X : \bar{p}(x) < 1\}$ *and* $r > 0$.

(5) *If* φ *is* p*-bounded but not* $\bar{p}$*-bounded, then*

$$\varphi(rB'_p) = (-\infty, r\|\varphi|_p).$$

Remark 1.4. A linear functional $\varphi : X \to \mathbb{R}$ is $(p, \bar{p})$-bounded if and only if

$$\forall x \in X, \quad |\varphi(x)| \leq Lp(x), \tag{1.9}$$

for some $L \geq 0$. □

Indeed, if $L_1, L_2 \geq 0$ are such that

$$\varphi(x) \leq L_1 p(x) \quad \text{and} \quad \varphi(x) \leq L_2 p(-x),$$

for all $x \in X$, then $-\varphi(x) = \varphi(-x) \leq L_2 p(x), x \in X$, so (1.9) holds with $L = \max\{L_1, L_2\}$.

Denote by $X^\flat_{\bar{p}}$ the dual cone to $(X, \bar{p})$ and let X^* be the conjugate of the seminormed space (X, p^s), where p^s is the symmetric seminorm associated to p and $\bar{p}$ (see (1.1)).

Since

$$\varphi(x) \leq Lp(x) \leq Lp^s(x), \, x \in X,$$

implies $|\varphi(x)| \leq Lp(x)$, $x \in X$, it follows that $X^\flat_p$ is contained in the dual X^* of (X, p^s). Similarly, $X^\flat_{\bar{p}}$ is contained in X^* too.

For $x^* \in X^*$ put

$$\|x^*\| = \sup\{x^*(x) : x \in X, \, p^s(x) \leq 1\}.$$

Then $\| \, \|$ is a norm on X^* and X^* is complete with respect to this norm, i.e. is a Banach space (even if p^s is not a norm, see [11]).

Proposition 1.5. *Let* (X, p) *be a space with asymmetric seminorm.*

(1) *The cones* $X^\flat_p$ *and* $X^\flat_{\bar{p}}$ *are contained in* X^* *and*

$$\|\varphi|_p = \|\varphi\|, \, \varphi \in X^\flat_p \quad \textit{and} \quad \|\psi|_{\bar{p}} = \|\psi\|, \, \psi \in X^\flat_{\bar{p}}.$$

(2) *We have* $\|\varphi|_p = \|-\varphi|_{\bar{p}}$, *so that*

$$\varphi \in X^\flat_p \textit{ and } \|\varphi|_p \leq r \iff -\varphi \in X^\flat_{\bar{p}} \textit{ and } \|-\varphi|_{\bar{p}} \leq r.$$

The properties of the dual space $X^\flat_p$ were investigated in [21] where, among other things, the analog of the weak* topology of X was defined. This is denoted by $w^\flat$ and has a neighborhood basis at a point $\varphi \in X^\flat_p$, the family

$$V_{x_1,...,x_n;\epsilon}(\varphi) = \{\psi \in X^\flat_p : \psi(x_k) - \varphi(x_k) < \epsilon, \; k = 1, ..., n\},$$

for $n \in \mathbb{N}$, $x_1, ..., x_n \in X$ and $\epsilon > 0$. The $w^\flat$-convergence of a net $(\varphi_i : i \in I)$ in $X_p^\flat$ to $\varphi \in X_p^\flat$ can be characterized in the following way

$$\varphi_i \xrightarrow{w^\flat} \varphi \iff \forall x \in X,\ \varphi_i(x) \to \varphi(x) \text{ in } (\mathbb{R}, u).$$

It was shown that $w^\flat$ is the restriction of the topology $w^* = \sigma(X^*, X)$ on X^* to $X_p^\flat$ (see [21]). This study was continued in [9] where separation theorems for convex sets and a Krein-Milman type theorem were proved. In [10] asymmetric locally convex spaces were introduced and their basic properties were studied.

Another direction of investigation is that of best approximation in spaces with asymmetric seminorm. Due to the asymmetry of the seminorm we have two distances. For a nonempty subset Y of a space with asymmetric seminorm (X, p) and $x \in X$ put

$$d_p(x, Y) = \inf\{p(y - x) : y \in Y\}, \tag{1.10}$$

and

$$d_p(Y, x) = \inf\{p(x - y) : y \in Y\}. \tag{1.11}$$

Note that $d_p(Y, x) = d_{\bar{p}}(x, Y)$.

Duality formulae and characterization results for best approximation in spaces with asymmetric norm were obtained in [5, 6, 9, 12, 34, 35]. The papers [32, 33, 39] are concerned with best approximation in spaces of semi-Lipschitz functions defined on asymmetric metric spaces (called quasi-metric spaces) in connection with the extension properties of these functions. In the papers [13, 24, 25, 36], supposing that p is the Minkowski functional p_C of a bounded convex body C in a normed space $(X, \|\ \|)$, some generic existence results for best approximation with respect to the asymmetric norm p_C were proved, extending similar results from the normed case. As in the symmetric case, the geometric properties of the body C (or, equivalently, of the functional p_C) are essential. A study of the moduli of convexity and smoothness corresponding to p_C is done in [43].

Best approximation with respect to some asymmetric norms in concrete function spaces of continuous or of integrable functions, called sign-sensitive approximation, was also studied in a series of papers, see [14, 15, 16, 41], the references quoted therein, and the monograph by Krein and Nudelman [23, Ch. 9, §5]).

The present paper, which can be viewed as a sequel to [12] and [9], is concerned mainly with characterizations of the elements of best approximation in a subspace Y of a space with asymmetric norm (X, p) and duality results for best approximation. As in the case of (symmetric) normed spaces the characterizations will be done in terms of some linear bounded functionals vanishing on Y. The duality results will involve the annihilator in $X_p^\flat$ of the subspace Y. For this reason we start by recalling some extension results for bounded linear functionals on spaces with asymmetric seminorm. For proofs, all resorting to the classical Hahn-Banach extension theorem, see [9, 12].

Theorem 1.6. *Let (X,p) be a space with asymmetric seminorm and Y a linear subspace of X. If $\varphi_0 : Y \to \mathbb{R}$ is a linear p-bounded functional on Y then there exists a p-bounded linear functional φ defined on the whole X such that*

$$\varphi|_Y = \varphi_0 \quad \textit{and} \quad \|\varphi|_p = \|\varphi_0|_p.$$

We agree to call a functional φ satisfying the conclusions of the above theorem a *norm preserving extension* of φ_0.

Based on this extension result one can prove the following existence result.

Proposition 1.7. *Let (X,p) be a space with asymmetric seminorm and $x_0 \in X$ such that $p(x_0) > 0$. Then there exists a p-bounded linear functional $\varphi : X \to \mathbb{R}$ such that*

$$\|\varphi|_p = 1 \quad \textit{and} \quad \varphi(x_0) = p(x_0).$$

In its turn, this proposition has the following corollary.

Corollary 1.8. *If $p(x_0) > 0$ then*

$$p(x_0) = \sup\{\varphi(x_0) : \varphi \in X_p^\flat,\ \|\varphi|_p \le 1\}.$$

Moreover, there exists $\varphi_0 \in X_p^\flat,\ \|\varphi_0|_p = 1,$ such that $\varphi_0(x_0) = p(x_0)$.

The following proposition is the asymmetric analog of a well known result of Hahn.

Proposition 1.9. ([12]) *Let Y be a subspace of a space with asymmetric seminorm (X,p) and $x_0 \in X$.*

(1) *If $d := d_p(x_0, Y) > 0$, then there exists $\varphi \in X_p^\flat$ such that*

$$\text{(i) } \varphi|_Y = 0, \quad \text{(ii) } \|\varphi|_p = 1, \quad \textit{and} \quad \text{(iii) } \varphi(-x_0) = d.$$

(2) *If $\bar{d} := d_p(Y, x_0) > 0$, then there exists $\psi \in X_p^\flat$ such that*

$$\text{(j) } \psi|_Y = 0, \quad \text{(jj) } \|\psi|_p = 1, \quad \textit{and} \quad \text{(jjj) } \psi(x_0) = \bar{d}.$$

2. BEST APPROXIMATION IN SPACES WITH ASYMMETRIC SEMINORM

Let (X,p) be a space with asymmetric seminorm, $\bar{p}$ the seminorm conjugate to p and Y a nonempty subset of X. The distances $d_p(x,Y)$ and $d_p(Y,x)$ from an element $x \in X$ to Y are defined by the formulae (1.10) and (1.11). An element $y_0 \in Y$ such that $p(y_0 - x) = d_p(x,Y)$ will be called a *p-nearest point* to x in Y, and an element $y_1 \in Y$ such that $p(x - y_1) = \bar{p}(y_1 - x) = d_{\bar{p}}(x,Y)$ is called a *$\bar{p}$-nearest point* to x in Y.

Denote by

$$\begin{aligned} P_Y(x) &= \{y \in Y : p(y - x) = d_p(x,Y)\}, \text{ and} \\ \bar{P}_Y(x) &= \{y \in Y : p(x - y) = d_p(Y,x)\}, \end{aligned} \tag{2.1}$$

the possibly empty sets of p-nearest points, respectively $\bar{p}$-nearest points, to x in Y. The set Y is called *p-proximinal, p-semi-Chebyshev, p-Chebyshev* if for every $x \in X$ the set $P_Y(x)$ is nonempty, contains at most one element, contains exactly one element, respectively. Similar definitions are given in the case of $\bar{p}$-nearest points. A semi-Chebyshev set is called also a *uniqueness set.*

For a nonempty subset Y of a space with asymmetric seminorm (X,p), denote by $Y_p^\perp$ *the annihilator* of Y in $X_p^\flat$, i.e.

$$Y_p^\perp = \{\varphi \in X_p^\flat : \varphi|_Y = 0\}.$$

We start by a characterization of nearest points given in [12] we shall need in the sequel.

PROPOSITION 2.1. ([12]) *Let (X,p) be a space with asymmetric seminorm, Y a subspace of X and x_0 a point in X.*

(1) *Suppose that $d := d_p(x_0, Y) > 0$. An element $y_0 \in Y$ is a p-nearest point to x_0 in Y if and only if there exists a bounded linear functional $\varphi : X \to \mathbb{R}$ such that*

$$\text{(i) } \varphi|_Y = 0, \quad \text{(ii) } \|\varphi|_p = 1, \quad \text{(iii) } \varphi(-x_0) = p(y_0 - x_0).$$

(2) *Suppose that $\bar{d} := d_p(Y, x_0) > 0$. An element $y_0 \in Y$ is a $\bar{p}$-nearest point to x_0 in Y if and only if there exists a bounded linear functional $\psi : X \to \mathbb{R}$ such that*

$$\text{(j) } \psi|_Y = 0, \quad \text{(jj) } \|\psi|_p = 1, \quad \text{(jjj) } \psi(x_0) = p(x_0 - y_0).$$

From this theorem one can obtain characterizations of sets of nearest points.

COROLLARY 2.2. *Let (X,p) be a space with asymmetric seminorm, Y a subspace of X, $x \in X$, and Z a nonempty subset of Y.*

(1) *If $d = d_p(x_0, Y) > 0$ then $Z \subset P_Y(x)$ if and only if there exists a functional $\varphi \in X_p^\flat$ such that*

$$\text{(i) } \varphi|_Y = 0, \quad \text{(ii) } \|\varphi|_p = 1, \quad \text{(iii) } \forall y \in Z,\ \varphi(-x_0) = p(y - x_0).$$

(2) *If $\bar{d} = d_p(Y, x_0) > 0$ then $Z \subset \bar{P}_Y(x)$ if and only if there exists a functional $\psi \in X_p^\flat$ such that*

$$\text{(j) } \psi|_Y = 0, \quad \text{(jj) } \|\psi|_p = 1, \quad \text{(jjj) } \forall y \in Z,\ \psi(x_0) = p(x_0 - y).$$

In the next proposition we extend to the asymmetric case some characterization results for semi-Chebyshev subspaces (see [42, Chapter I, Theorem 3.2]).

THEOREM 2.3. *Let Y be a subspace of a space with asymmetric norm (X,p) such that $p(x) > 0$ for every $x \neq 0$. Then the following assertions are equivalent.*

(1) *Y is a p-semi-Chebyshev subspace of X.*
(2) *There are no $\varphi \in Y_p^\perp$ and $x_1, x_2 \in X$ with $x_1 - x_2 \in Y \setminus \{0\}$, such that*

$$\text{(i) } \|\varphi|_p = 1 \quad \text{and} \quad \text{(ii) } \varphi(-x_i) = p(-x_i),\ i = 1, 2.$$

(3) *There are no* $\psi \in Y_p^\perp,\ x \in X,$ *and* $y_0 \in Y \setminus \{0\}$ *such that*

$$\text{(j)}\quad \|\psi|_p = 1 \quad \text{and} \quad \text{(jj)}\ \ \psi(-x) = p(-x) = p(y_0 - x).$$

Proof. (1) ⇒ (2) Suppose that (2) does not hold. Let $\varphi \in Y_p^\perp$ and $x_1, x_2 \in X$ with $x_1 - x_2 \in Y \setminus \{0\}$, such that the conditions (i) and (ii) of the assertion (2) are satisfied, and put $y_0 = x_1 - x_2$. Then

$$\varphi(-x_2) = p(-x_2) \iff \varphi(y_0 - x_1) = p(y_0 - x_1),$$

and

$$\varphi(-x_1) = p(-x_1) \iff \varphi(0 - x_1) = p(0 - x_1).$$

By Proposition 2.1, it follows that 0 and y_0 are p-nearest points to x_1 in Y.

(2) ⇒ (3) Suppose that (3) does not hold. Then there exist $\psi \in Y_p^\perp,\ x \in X,$ and $y_0 \in Y \setminus \{0\}$ such that the conditions (j) and (jj) of the assertion (3) are fulfilled. It follows that the conditions (i) and (ii) of the assertion (2) are satisfied by $\varphi = \psi,\ x_1 = x$ and $x_2 = y_0 - x,$ i.e. (2) does not hold.

(3) ⇒ (1) Supposing that (1) does not hold, there exist $z \in X \setminus Y$ and $y_1, y_2 \in Y,\ y_1 \neq y_2,$ such that

$$p(y_1 - z) = p(y_2 - z) = d_p(z, Y).$$

If $d_p(z, Y) = 0$, then $y_1 = y_2 = z$, a contradiction which shows that $d_p(z, Y) > 0$.

If $x := z - y_1,$ then

$$\begin{aligned} d_p(z - y_1, Y) &= \inf\{p(y + y_1 - z) : y \in Y\} \\ &= \inf\{p(y' - z) : y' \in Y\} \\ &= d_p(z, Y) \\ &= p(y_1 - z) \\ &= p(y_2 - z). \end{aligned}$$

By Proposition 2.1, there exists $\psi \in Y_p^\perp,\ \|\psi|_p = 1,$ such that

$$\psi(y_1 - z) = p(y_1 - z) = p(y_2 - z),$$

or, denoting $y_0 := y_2 - y_1$, this is equivalent to

$$\psi(-x) = p(-x) = p(y_0 - x),$$

showing that (3) does not hold. □

Remark 2.4. Obviously that a similar characterization result holds for $\bar{p}$-semi-Chebyshev subspaces. □

Using Corollary 2.2, one can extend Theorem 2.3 to obtain characterizations of pseudo-Chebyshev subspaces, a notion introduced by Mohebi [28] in the case of normed spaces. Concerning other weaker notions of Chebyshev spaces – quasi-Chebyshev subspaces, weak-Chebyshev subspaces, as well as for their behaviour in concrete function spaces, see the papers [26, 27, 29, 31]. For a subset Z of a vector space X denote by aff(Z) the affine hull of the set Z, i.e.

$\operatorname{aff}(Z) = \{x \in X : \exists n \in \mathbb{N}, \exists\, z_1, ..., z_n \in Z, \exists\, a_1, ..., a_n \in \mathbb{R},\ a_1 + ... + a_n = 1$ such that $x = a_1 z_1 + ... + a_n z_n\}$. There exists a unique subspace Y of X such that $\operatorname{aff}(Z) = z + Y,$ for an arbitrary $z \in Z$. By definition, the *affine dimension* of the set Z is the dimension of this subspace Y of X.

A subspace Y of a space with asymmetric norm (X, p) is called *p-pseudo-Chebyshev* if it is p-proximinal and the set $P_Y(x)$ has finite affine dimension for every $x \in X$.

The following theorem extends a result proved by Mohebi [28] in normed spaces.

THEOREM 2.5. *Let Y be a subspace of an asymmetric normed space (X, p) such that $p(x) > 0$ for every $x \neq 0$. The following assertions are equivalent.*

(1) *The subspace Y is p-pseudo-Chebyshev.*

(2) *There do not exist $\varphi \in Y_p^{\perp},\ x_0 \in X,$ and infinitely many linearly independent elements $x_n \in X,\ n \in \mathbb{N},$ with $x_0 - x_n \in Y,\ n \in \mathbb{N},$ such that*

$$\text{(i) } \|\varphi|_p = 1 \quad \textit{and} \quad \text{(ii) } \varphi(-x_n) = p(-x_n),\ n = 0, 1, \dots\ .$$

(3) *There do not exist $\psi \in Y_p^{\perp},\ x_0 \in X,$ and infinitely many linearly independent elements $y_n \in Y,\ n \in \mathbb{N},$ such that*

$$\text{(j) } \|\psi|_p = 1 \quad \textit{and} \quad \text{(jj) } \psi(-x_0) = p(-x_0) = p(y_n - x_0),\ n = 1, 2, \dots\ .$$

Proof. (1) $\Rightarrow$ (2) Suppose that (2) does not hold. Then there exist $\varphi \in Y_p^{\perp},\ x_0 \in X,$ and infinitely many linearly independent elements $x_n \in X,$ with $x_0 - x_n \in Y,\ n \in \mathbb{N},$ satisfying the conditions (i) and (ii). The elements $y_n := x_0 - x_n,\ n \in \mathbb{N},$ all belong to Y, are linearly independent, and

$$\varphi(y_n - x_0) = \varphi(-x_n) = p(-x_n) = p(y_n - x_0),$$

so that, by Corollary 2.2, they are all contained in $P_Y(x_0)$, showing that Y is not p-pseudo-Chebyshev.

(2) $\Rightarrow$ (3) Suppose again that (3) does not hold, and let $\psi \in Y_p^{\perp},\ x_0 \in X,$ and the linearly independent elements $\{y_n : n = 1, 2, ...\} \subset Y$ fulfilling the conditions (j) and (jj).

Then $x_n := x_0 - y_n,\ n = 1, 2, ...,$ are linearly independent elements in X and

$$\psi(-x_n) = \psi(y_n - x_0) = p(-x_n),\ n = 0, 1, 2, ...,$$

showing that (2) does not hold.

(3) $\Rightarrow$ (1) Supposing that (1) does not hold, there exist an element $z \in X$ and an infinite set $\{y_n : n = 1, 2, ...\}$ of linearly independent elements contained in $P_Y(z)$.

By Corollary 2.2, there exists $\varphi \in Y_p^{\perp},\ \|\varphi|_p = 1,$ such that

$$\varphi(y_n - z) = d_p(z, Y) = p(y_n - z),\ n = 1, 2, \dots\ .$$

Putting $x := z - y_1$ we have

$$d_p(x,Y) = \inf\{p(y+y_1-z) : y \in Y\} = \inf\{p(y'-z) : y' \in Y\} = $$
$$= d_p(z,Y) = p(y_n - z) = p(y_n - y_1 - x),\ n = 2,3,...,$$

showing that $\{y_n - y_1 : n = 2,3,...\} \subset P_Y(x)$. By Corollary 2.2, there exists $\psi \in Y_p^{\perp}$ with $\|\psi|_p = 1$ such that

$$\psi(y_n - y_y - x_0) = p(y_n - y_1 - x_0),\ n = 2,3,...,$$

showing that (3) does not hold. □

Phelps [37] emphasized for the first time some close connections existing between the approximation properties of the annihilator $Y^{\perp}$ of a subspace Y of a normed space X and the extension properties of the subspace Y. A presentation of various situations in which such a connection occurs is done in [8]. The case of spaces with asymmetric norms was considered in [34, 35].

Let (X,p) be a space with asymmetric seminorm and Y a subspace of X. For a p-bounded linear functional $\varphi : Y \to \mathbb{R}$ denote by

$$E_p(\varphi) = \{\psi \in X_p^{\flat} : \psi|_Y = \varphi,\ \|\psi|_p = \|\varphi|_p\},$$

the set of all norm-preserving extensions of the functional φ. By the Hahn-Banach theorem (Theorem 1.6) the set $E_p(\varphi)$ is always nonempty.

For $\varphi \in X_p^{\flat}$ consider the following minimization problem

$$\gamma(\varphi, Y_p^{\perp}) := \inf\{\|\varphi + \psi|_p : \psi \in Y_p^{\perp}\}. \tag{2.2}$$

A solution to this problem is an element $\psi_0 \in Y_p^{\perp}$ such that $\|\varphi + \psi_0|_p = \gamma(\varphi, Y_p^{\perp})$. Denote by $\Pi_{Y_p^{\perp}}(\varphi)$ the set of all these solutions.

THEOREM 2.6. *If the linear functional $\varphi : X \to \mathbb{R}$ is $(p,\bar{p})$-bounded then the minimization problem* (2.2) *has a solution and the following formulae hold*

$$\gamma(\varphi, Y_p^{\perp}) = \|\varphi|_Y|_p \quad \textit{and} \quad \Pi_{Y_p^{\perp}}(\varphi) = E_p(\varphi|_Y) - \varphi. \tag{2.3}$$

Proof. Let $\varphi \in X_p^{\flat} \cap X_{\bar{p}}^{\flat}$ and $\psi \in Y_p^{\perp}$. Then

$$\|\varphi + \psi|_p \geq \|(\varphi + \psi)|_Y|_p = \|\varphi|_Y|_p,$$

implying $\gamma(\varphi, Y_p^{\perp}) \geq \|\varphi|_Y|_p$.

If $\Phi \in E_p(\varphi|_Y)$ then, because φ is $(p,\bar{p})$-bounded, $-\varphi \in X_p^{\flat}$ (see Proposition 1.5, $\psi := \Phi - \varphi \in Y_p^{\perp}$, and $\gamma(\varphi, Y_p^{\perp}) \leq \|\varphi + \psi|_p = \|\Phi|_p$. Therefore $\gamma(\varphi, Y_p^{\perp}) = \|\varphi|_Y|_p$ and

$$E_p(\varphi|_Y) - \varphi \subset \Pi_{Y_p^{\perp}}(\varphi).$$

Conversely, if $\psi \in \Pi_{Y_p^{\perp}}(\varphi)$, then $\Phi := \varphi + \psi$ satisfies $\Phi|_Y = \varphi|_Y$ and $\|\Phi|_p = \|\varphi + \psi|_p = \gamma(\varphi, Y_p^{\perp}) = \|\varphi|_Y|_p$, i.e. $\Phi \in E_p(\varphi|_Y)$ and

$$\varphi + \Pi_{Y_p^{\perp}}(\varphi) \subset E_p(\varphi|_Y) \iff \Pi_{Y_p^{\perp}}(\varphi) \subset E_p(\varphi|_Y) - \varphi.$$

□

Denoting by

$$Y^{\perp} = \{\psi \in X^* : \psi|_Y = 0\}, \tag{2.4}$$

the annihilator $Y^{\perp}$ of a subspace Y of X in the symmetric dual X^* of the seminormed space $(X, p^s)^*$, it follows that $Y^{\perp}$ is a subspace of X^*. Consider on X^* the asymmetric extended norm $\| |_p^* : X^* \to [0;\infty]$ defined by

$$\|\varphi|_p^* = \sup \varphi(B_p).$$

We have for any $\varphi \in X^*$

$$\varphi \in X_p^\flat \iff \|\varphi|_p^* < \infty,$$

and $\|\varphi|_p^* = \|\varphi|_p$ for $\varphi \in X_p^\flat$ (see Proposition 1.5).

For $\varphi \in X_p^\flat$ consider the distance from φ to $Y^{\perp}$ defined by

$$d_p(Y^{\perp}, \varphi) = \inf\{\|\varphi - \psi|_p^* : \psi \in Y^{\perp}\}.$$

Because $\|\varphi - 0|_p^* = \|\varphi|_p < \infty$ this distance is always finite. Put

$$P_{Y^{\perp}}(\varphi) = \{\psi \in Y^{\perp} : \|\varphi - \psi|_p = d_p(Y^{\perp}, \varphi)\}.$$

Theorem 2.7. *Every $\varphi \in X_p^\flat$ has a $\bar{p}$-nearest point in $Y^{\perp}$ and the following formulae hold*

$$d_p(Y^{\perp}, \varphi) = \|\varphi|_Y|_p \quad \text{and} \quad P_{Y^{\perp}}(\varphi) = \varphi - E_p(\varphi|_Y).$$

Proof. For $\psi \in Y^{\perp}$ we have

$$\|\varphi - \psi|_p^* \geq \|(\varphi - \psi)|_Y|_p^* = \|\varphi|_Y|_p,$$

implying $d_p(Y^{\perp}, \varphi) \geq \|\varphi|_Y|_p$. If $\Phi \in E_p(\varphi|_Y)$, then $\psi := \varphi - \Phi \in Y^{\perp}$ and

$$d_p(Y^{\perp}, \varphi) \leq \|\varphi - \psi|_p^* = \|\Phi|_p = \|\varphi|_Y|_p.$$

Therefore $d_p(Y^{\perp}, \varphi) = \|\varphi|_Y|_p$ and $\varphi - E_p(\varphi|_Y) \subset P_{Y^{\perp}}(\varphi)$.

If $\psi \in P_{Y^{\perp}}(\varphi)$ and $\Phi := \varphi - \psi$, then $\Phi|_Y = \varphi|_Y$ and $\|\Phi|_p = \|\varphi - \psi|_p = d_p(Y^{\perp}, \varphi) = \|\varphi|_Y|_p$, i.e. $\Phi \in E_p(\varphi|_Y)$, showing that $\varphi - P_{Y^{\perp}}(\varphi) \subset E_p(\varphi|_Y)$, or equivalently, $P_{Y^{\perp}}(\varphi) \subset \varphi - E_p(\varphi|_Y)$. □

From these theorems we obtain some uniqueness conditions for the minimization problems we have considered, in terms of the uniqueness of norm-preserving extensions.

Corollary 2.8. *Let (X, p) be a space with asymmetric seminorm and Y a subspace of X.*

(1) *If every $f \in Y_p^\flat$ has a unique norm preserving extension $F \in X_p^\flat$, then the minimization problem* (2.2) *has a unique solution for every $\varphi \in X_p^\flat$.*

(2) *Every point* $\varphi \in X_p^\flat$ *has a unique* $\bar{p}$*-nearest point in* $Y^\perp$ *if and only if every* $f \in Y_p^\flat$ *has a unique norm-preserving extension* $F \in X_p^\flat$.

Proof. (1) If every $f \in Y_p^\flat$ has a unique norm-preserving extension $F \in X_p^\flat$, then for every $\varphi \in X_p^\flat$ the set $\Pi_{Y_p^\perp}(\varphi) = \varphi + E_p(\varphi|_Y)$ contains exactly one element.

(2) Similarly, $P_{Y^\perp}(\varphi) = \varphi - E_p(\varphi|_Y)$ contains exactly one element, provided every $f \in Y_p^\flat$ has exactly one norm-preserving extension $F \in X_p^\flat$.

Conversely, suppose that there exists $f \in Y_p^\flat$ having two distinct norm-preserving extensions $F_1, F_2 \in X_p^\flat$. Then

$$P_{Y^\perp}(F_1) = F_1 - E_p(F_1|_Y) = F_1 - E_p(f) \supset \{0, F_1 - F_2\}.$$

□

Remark 2.9. We can not prove the reverse implication in the assertion (1) of the above corollary. To do this we would need an extension theorem for $(p, \bar{p})$-bounded linear functionals, preserving both p- and $\bar{p}$-norm, and we are not aware of such a result. □

Some results connecting the ϵ-approximations and ϵ-extensions were obtained by Rezapour [38]. In the next proposition we transpose these results to the asymmetric case.

Let (X, p) be a space with asymmetric seminorm and Y a subspace of X. For $x \in X$ and $\epsilon > 0$ let

$$P_Y^\epsilon(x) = \{y \in Y : p(y - x) \le d_p(x, Y) + \epsilon\}$$

and

$$\bar{P}_Y^\epsilon(x) = \{y \in Y : p(x - y) \le d_p(Y, x) + \epsilon\}$$

denote the nonempty sets of ϵ-p-, respectively ϵ-$\bar{p}$-nearest points to x in Y. For $\varphi \in X_p^\flat$ consider the set of ϵ-solutions of the minimization problem (2.2)

$$\Pi_{Y_p^\perp}^\epsilon(\varphi) = \{\psi \in Y_p^\perp : \|\varphi + \psi|_p \le \gamma(\varphi, Y_p^\perp) + \epsilon\}$$

and, finally, denote by

$$E_p^\epsilon(f) = \{F \in X_p^\flat : F|_Y = f \text{ and } \|F|_p \le \|f|_p + \epsilon\},$$

the set of ϵ-extensions of a functional $f \in Y_p^\flat$.

These two sets are related in the following way.

Proposition 2.10. *Let* (X, p) *be a space with asymmetric seminorm,* Y *a subspace of* X *and* $\varphi \in X_p^\flat$. *Then*

$$\Pi_{Y_p^\perp}^\epsilon(\varphi) = E_p^\epsilon(\varphi|_Y) - \varphi.$$

Proof. Indeed, by Theorem 2.6,

$$\psi \in \Pi^{\epsilon}_{Y_p^{\perp}}(\varphi) \iff \psi \in Y_p^{\perp} \text{ and } \|\varphi + \psi|_p \le \gamma(\varphi, Y_p^{\perp}) + \epsilon = \|\varphi|_Y|_p + \epsilon$$
$$\iff \varphi + \psi \in E_p^{\epsilon}(\varphi|_Y).$$

□

Working with the annihilator $Y^{\perp}$ of the subspace Y in the symmetric dual $X^* = (X, p^s)^*$ given by (2.4) and putting

$$\bar{P}^{\epsilon}_{Y^{\perp}}(\varphi) = \{\psi \in Y^{\perp} : \|\varphi - \psi|_p \le d_p(\varphi, Y^{\perp}) + \epsilon\},$$

we have

Proposition 2.11. *Let Y be a subspace of a space with asymmetric seminorm (X, p), $\epsilon > 0$, and $\varphi \in X_p^{\flat}$. Then*

$$\bar{P}^{\epsilon}_{Y}(\varphi) = \varphi - E_p^{\epsilon}(\varphi|_Y).$$

Proof. Indeed, by Theorem 2.7,

$$\psi \in \bar{P}^{\epsilon}_{Y^{\perp}}(\varphi) \iff \psi \in Y^{\perp} \text{ and } \|\varphi - \psi|_p \le d_p(Y^{\perp}, \varphi) + \epsilon = \|\varphi|_Y|_p + \epsilon$$
$$\iff \varphi - \psi \in E_p^{\epsilon}(\varphi|_Y).$$

□

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