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ON A COMPOUND APPROXIMATION OPERATOR OF D.D. STANCU TYPE*

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Abstract. In this note we consider a linear and positive compound approximation operator of D.D. Stancu type depending of several parameters; we give the expressions of this operator on the test functions, the conditions under which this operator converges to a given continuous function, an estimate of the order of approximation using the moduli of continuity and an integral representation of the remainder. Also, by using Stancu's method we find an expression for the remainder using divided differences of second order for a special case of this operator.

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1. INTRODUCTION

In [18], [19], [24], [25], [23], [27] D.D. Stancu introduced and studied various classes of compound approximation operators:

$$(S_{m,r,s}f)(x) = \sum_{k=0}^{m-sr} p_{m-sr,k}(x) \sum_{j=0}^{s} p_{s,j}(x) f\left(\tfrac{k+jr}{m}\right), \tag{1}$$

where $f \in C[0,1]$, r and s are nonnegative integer parameters satisfying the condition $2sr < m$, while $p_{n,k}(x) = \binom{n}{k}\frac{p_k(x)p_{n-k}(1-x)}{p_n(1)}$ and $p_k(x) = x^k$ in [19], [18], [25], $p_k(x) = x^{[k,-\alpha]} = x(x+\alpha)\dots(x+(k-1)\alpha)$ in [24], $p_k(x) = x(x+\alpha+k\beta)^{[k-1,-\alpha]}$ in [23].

Cao [2] considered compound operators of Stancu type with $p_k(x) = x^k$ defined on a simplex.

Following the interesting ideas of D.D. Stancu from the previous mentioned papers in [5] we considered a general class of compound operators of D.D. Stancu type:

$$\left(L^{Q}_{m,r_1,\dots,r_s}f\right)(x) = \sum_{k=0}^{m-r_1\dots-r_s} p^{Q}_{m-r_1\dots-r_s,k}(x) \sum_{j=0}^{s} \tfrac{p_j(x)p_{s-j}(1-x)}{p_s(1)} F^{r_1,\dots,r_s}_{m,k,j}(f) \tag{2}$$

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where $p_{n,k}^{Q}(x)=\binom{n}{k}\frac{p_k(x)p_{n-k}(1-x)}{p_n(1)}$, $(p_k)_{k\geq 0}$ is a sequence of binomial type which is the basic sequence for the delta operator Q,

$$\begin{aligned}F_{m,k,j}^{r_1,\ldots,r_s}(f)=&f\left(\tfrac{k+r_1+r_2+\ldots+r_j}{m}\right)+f\left(\tfrac{k+r_2+r_3+\ldots+r_{j+1}}{m}\right)\\&+f\left(\tfrac{k+r_1+r_3+\ldots+r_{j+1}}{m}\right)+\ldots+f\left(\tfrac{k+r_{s-j+1}+\ldots+r_{s-1}+r_s}{m}\right)\end{aligned}$$

and $r_1,\ldots,r_s$ are s non-negative integer parameters, independent of the number m and such that $0\leq r_1\leq\ldots\leq r_s$ and $r_1+\ldots+r_s<m$. We mention that Q is a delta operator if it is a shift invariant operator ($QE^a=E^aQ$, for any a, where E^a is the shift operator) and $Qx=\text{const}\neq 0$, and $(p_k)_{k\geq 0}$ is the basic sequence for the delta operator Q if it satisfies: $p_0(x)=1$, $p_n(0)=0$, $n\geq 1$, $Qp_n=np_{n-1}$.

We have proved the following result:

LEMMA 1. [5] *If $L_{m,r_1,\ldots,r_s}^{Q}$ is the approximation operator defined by* (2) *then*

$$L_{m,r_1,\ldots,r_s}^{Q}e_i=e_i,\ \text{for } i=0,1 \text{ and}$$
$$\left(L_{m,r_1,\ldots,r_s}^{Q}e_2\right)(x)=x^2+x(1-x)A_{m,r_1,\ldots,r_s}^{Q},$$

where

$$A_{m,r_1,\ldots,r_s}^{Q}=\tfrac{1}{m^2}\left[(m-r_1-\ldots-r_s)^2d_{m-r_1\ldots-r_s}^{Q}+r_1^2+\ldots+r_s^2+\tfrac{2}{s-1}\left(sd_s^{Q}-1\right)\sum_{\substack{u,v=1\\u\neq v}}^{s}r_ur_v\right],$$

and $d_m^{Q}=1-\frac{m-1}{m}\frac{(Q')^{-2}p_{m-2}(1)}{p_m(1)}$, *$Q'$ is the Pincherle derivative of Q, $Q'=QX-XQ$.*

2. DEFINITION AND CONVERGENCE OF THE OPERATOR $L_{m,r_1,\ldots,r_s}^{\frac{\nabla_\alpha}{\alpha}}f$

One remarkable operator from the class mentioned above is the operator obtained by taking the Stancu polynomials $p_{n,k}^{\alpha}(x)=\binom{n}{k}\frac{x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}}{1^{[n,-\alpha]}}$ ($x^{[k,-\alpha]}$ is the basic sequence for the delta operator $\frac{\nabla_\alpha}{\alpha}$) in relation (2):

$$\begin{aligned}(3)\quad &\left(L_{m,r_1,\ldots,r_s}^{\frac{\nabla_\alpha}{\alpha}}f\right)(x)=\\&=\frac{1}{1^{[m-r_1-\ldots-r_s,-\alpha]}}\sum_{k=0}^{m-r_1\ldots-r_s}\binom{m-r_1-\ldots-r_s}{k}x^{[k,-\alpha]}(1-x)^{[m-r_1-\ldots-r_s-k,-\alpha]}\\&\cdot\sum_{j=0}^{s}\frac{x^{[j,-\alpha]}(1-x)^{[s-j,-\alpha]}}{1^{[s,-\alpha]}}F_{m,k,j}^{r_1,\ldots,r_s}(f).\end{aligned}$$

It is easy to see that this operator is linear and if $\alpha>0$ it is positive.

It interpolates the function f at both ends of the interval $[0,1]$.

In order to prove the convergence of this operator we need the values of this operator on the test functions $e_i(x)=x^i$, $i=0,1,2$. According to the results from [5] we can state

THEOREM 2. *The values of the operator $L_{m,r_1,\ldots,r_s}^{\frac{\nabla_\alpha}{\alpha}}f$ on the test functions are:*

$$\left(L_{m,r_1,\ldots,r_s}^{\frac{\nabla_\alpha}{\alpha}}e_i\right)(x)=e_i(x),\ \text{for } i=0,1 \text{ and}$$
$$\left(L_{m,r_1,\ldots,r_s}^{\frac{\nabla_\alpha}{\alpha}}e_2\right)(x)=x^2+x(1-x)A_{m,r_1,\ldots,r_s}^{\frac{\nabla_\alpha}{\alpha}},$$

where
(4)
$$A_{m,r_1,\dots,r_s}^{\frac{\nabla\alpha}{\alpha}} = \frac{1}{m^2}\left[(m-r_1-\dots-r_s)\,\frac{1+\alpha(m-r_1-\dots-r_s)}{1+\alpha} + r_1^2+\dots+r_s^2+\frac{2\alpha}{1+\alpha}\sum_{\substack{u,v=1\\u\neq v}}^{s} r_u r_v\right].$$

If $r_1 = \dots = r_s = r$ in the relation (3) then this operator reduces to the operator studied by D.D. Stancu and J.W. Drane in [24] and the expression (4) in this case is
$$A_{m,r,s}^{\frac{\nabla\alpha}{\alpha}} = \frac{sr^2(1+\alpha s)+(m-sr)(1+\alpha(m-sr))}{m^2(1+\alpha)}.$$
In the case $s=0$ we obtain the well-known classical Stancu operator introduced in [14] and studied by many mathematicians.

When $\alpha = 0$ the corresponding operator was introduced by D.D. Stancu in [18]; the same author found for the remainder of the corresponding approximation formula a convex combination of second order divided differences in [19].

By applying the Bohman-Korovkin convergence criterion we can state the following result:

Theorem 3. *If $f \in C[0,1]$ and the parameter α depend on m such that $\alpha = \alpha(m) \to 0$ as $m \to \infty$, then the sequence $\left(L_{m,r_1,\dots,r_s}^{\frac{\nabla\alpha}{\alpha}} f\right)$ converges uniformly to f on $[0,1]$.*

In the following we obtain some estimates of the order of approximation of a function $f \in C[0,1]$ by means of the operator $L_{m,r_1,\dots,r_s}^{\frac{\nabla\alpha}{\alpha}} f$.

Using an inequality established by O. Shisha and B. Mond in [13] we can write:
$$\left|f(x) - \left(L_{m,r_1,\dots,r_s}^{\frac{\nabla\alpha}{\alpha}} f\right)(x)\right| \leq \left[1+\frac{1}{\delta^2}L_{m,r_1,\dots,r_s}^{\frac{\nabla\alpha}{\alpha}}\left((t-x)^2;x\right)\right]\omega_1(f;\delta).$$
Using theorem 2 we obtain that $L_{m,r_1,\dots,r_s}^{\frac{\nabla\alpha}{\alpha}}\left((t-x)^2;x\right) = x(1-x)\,A_{m,r_1,\dots,r_s}^{\frac{\nabla\alpha}{\alpha}}$. Taking into account that $x(1-x) \leq \frac{1}{4}$, $\forall x \in [0,1]$ and replacing δ with $\sqrt{A_{m,r_1,\dots,r_s}^{\frac{\nabla\alpha}{\alpha}}}$, we obtain that
$$\left\|f - L_{m,r_1,\dots,r_s}^{\frac{\nabla\alpha}{\alpha}} f\right\| \leq \frac{5}{4}\omega_1\left(f;\sqrt{A_{m,r_1,\dots,r_s}^{\frac{\nabla\alpha}{\alpha}}}\right).$$
Using a result of H.H. Gonska and R.K. Kovacheva [6] we can give the following evaluation with the second order modulus of continuity, for every $x \in [0,1]$ and $\delta \in [0,1/2]$:
$$\left|f(x) - \left(L_{m,r_1,\dots,r_s}^{\frac{\nabla\alpha}{\alpha}} f\right)(x)\right| \leq \frac{3}{2}\left[1+\frac{x(1-x)}{2\delta^2}A_{m,r_1,\dots,r_s}^{\frac{\nabla\alpha}{\alpha}}\right]\omega_2(f;\delta).$$
This relation implies:
$$\left\|f - L_{m,r_1,\dots,r_s}^{\frac{\nabla\alpha}{\alpha}} f\right\| \leq \frac{27}{16}\omega_2\left(f;\sqrt{A_{m,r_1,\dots,r_s}^{\frac{\nabla\alpha}{\alpha}}}\right).$$

3. REPRESENTATIONS FOR THE REMAINDER

We consider the following approximation formula
$$f(x) = \left(L_{m,r_1,\dots,r_s}^{\frac{\nabla\alpha}{\alpha}} f\right)(x) + \left(R_{m,r_1,\dots,r_s}^{\frac{\nabla\alpha}{\alpha}} f\right)(x). \tag{5}$$

From Theorem 2 it results that the degree of exactness of this formula is 1, so for every function $f \in C^2[0,1]$ we can apply the Peano's theorem and we obtain the following representation:

$$\left(R_{m,r_1,\ldots,r_s}^{\overset{\nabla\alpha}{\alpha}} f\right)(x) = \int_0^1 G_{m,r_1,\ldots,r_s}^{\alpha}(t;x) f''(t)\,\mathrm{d}t,$$

where $G_{m,r_1,\ldots,r_s}^{\alpha}(t;x)$ is the Peano kernel defined by:

$$G_{m,r_1,\ldots,r_s}^{\alpha}(t;x) = \left(R_{m,r_1,\ldots,r_s}^{\overset{\nabla\alpha}{\alpha}} \varphi_x\right)(t), \qquad \varphi_x(t) = (x-t)_+ = \tfrac{1}{2}\left[x - t + |x-t|\right].$$

Since the expression $G_{m,r_1,\ldots,r_s}^{\alpha}(t;x)$ is negative, we can apply the mean value theorem to the integral and we obtain that there exists $\xi \in [0,1]$ such that

$$\left(R_{m,r_1,\ldots,r_s}^{\overset{\nabla\alpha}{\alpha}} f\right)(x) = f''(\xi) \int_0^1 G_{m,r_1,\ldots,r_s}^{\alpha}(t;x)\,\mathrm{d}t.$$

Taking $f(x) = x^2$ in the previous relation we obtain that

$$\int_0^1 G_{m,r_1,\ldots,r_s}^{\alpha}(t;x)\,\mathrm{d}t = \tfrac{1}{2}\left(R_{m,r_1,\ldots,r_s}^{\overset{\nabla\alpha}{\alpha}} e_2\right)(x) = \tfrac{x(x-1)}{2} A_{m,r_1,\ldots,r_s}^{\overset{\nabla\alpha}{\alpha}}.$$

So, for every function $f \in C^2[0,1]$, we obtain a Cauchy-type form for the remainder in the approximation formula (5)

$$\left(R_{m,r_1,\ldots,r_s}^{\overset{\nabla\alpha}{\alpha}} f\right)(x) = \tfrac{x(x-1)}{2} A_{m,r_1,\ldots,r_s}^{\overset{\nabla\alpha}{\alpha}} f''(\xi),$$

where $\xi \in [0,1]$ and $A_{m,r_1,\ldots,r_s}^{\overset{\nabla\alpha}{\alpha}}$ is defined by (4).

We mention that D.D. Stancu found also representations of the remainder by convex combinations of second-order divided differences in the case $r_1 = r_2 = \ldots = r_s$. For $\alpha = 0$ he obtained the following representation for the remainder:

$$(R_{m,r,s}f)(x) = \tfrac{x(x-1)}{m^2}(C_{m,r,s}f)(x),$$

where

$$\begin{aligned}(C_{m,r,s}f)(x) = {}&(m - sr)\sum_{k=0}^{m-sr-1} p_{m-sr-1,k}(x) \sum_{j=0}^{s} p_{s,j}(x)\left[x, \tfrac{k+jr}{m}, \tfrac{k+1+jr}{m}; f\right]\\ &+ sr^2 \sum_{k=0}^{m-sr} p_{m-sr-1,k}(x) \sum_{j=0}^{s-1} p_{s-1,j}(x)\left[x, \tfrac{k+jr}{m}, \tfrac{k+r+jr}{m}; f\right].\end{aligned}$$

Using Stancu's method in the following we intend to find an expression for the remainder using divided differences in the case $\alpha = 0$ and $s = 2$, namely for the operator

$$\begin{aligned}(L_{m,r_1,r_2}f)(x) = &\sum_{k=0}^{m-r_1-r_2} p_{m-r_1-r_2,k}(x)\cdot\Big\{(1-x)^2 f\left(\tfrac{k}{m}\right)\\ &+ x(1-x)\left[f\left(\tfrac{k+r_1}{m}\right) + f\left(\tfrac{k+r_2}{m}\right)\right] + x^2 f\left(\tfrac{k+r_1+r_2}{m}\right)\Big\},\end{aligned}$$

where $p_{m,k}(x) = \binom{m}{k} x^k (1-x)^k$.

Because $(L_{m,r_1,r_2}e_0)(x) = 1$ we have

$$
\begin{aligned}
&(R_{m,r_1,r_2}f)(x) = \\
&= f(x) - (L_{m,r_1,r_2}f)(x) \\
&= \sum_{k=0}^{m-r_1-r_2} p_{m-r_1-r_2,k}(x) \cdot \Big\{ (1-x)^2 \left(f(x) - f\left(\tfrac{k}{m}\right)\right) + x(1-x) \cdot \\
&\qquad \cdot \left[\left(f(x) - f\left(\tfrac{k+r_1}{m}\right)\right) + \left(f(x) - f\left(\tfrac{k+r_2}{m}\right)\right)\right] + x^2 \left(f(x) - f\left(\tfrac{k+r_1+r_2}{m}\right)\right) \Big\}.
\end{aligned}
$$

In case x coincides with one of the nodes, the definition of the divided differences requires the differentiability of f at x; this will be implicitly assumed further on.

If we use the following relations

$$
\begin{aligned}
f(x) - f\left(\tfrac{k}{m}\right) &= \left(x - \tfrac{k}{m}\right)\left[x, \tfrac{k}{m}; f\right], \\
f(x) - f\left(\tfrac{k+r_i}{m}\right) &= \left(x - \tfrac{k+r_1}{m}\right)\left[x, \tfrac{k+r_i}{m}; f\right],\ i = 1,2, \\
f(x) - f\left(\tfrac{k+r_1+r_2}{m}\right) &= \left(x - \tfrac{k+r_1+r_2}{m}\right)\left[x, \tfrac{k+r_1+r_2}{m}; f\right], \\
mx - k &= (m - r_1 - r_2 - k)\,x - k(1-x) + (r_1 + r_2)\,x, \\
mx - k - r_1 &= (m - r_1 - r_2 - k)\,x - k(1-x) + r_2 x - r_1(1-x), \\
mx - k - r_2 &= (m - r_1 - r_2 - k)\,x - k(1-x) + r_1 x - r_2(1-x), \\
mx - k - r_1 - r_2 &= (m - r_1 - r_2 - k)\,x - k(1-x) + (r_1 + r_2)(1-x),
\end{aligned}
$$

we can write

$$
(R_{m,r_1,r_2}f)(x) = (P_{m,r_1,r_2}f)(x) + (Q_{m,r_1,r_2}f)(x),
$$

where

$$
\begin{aligned}
(P_{m,r_1,r_2}f)(x) =& \tfrac{1}{m} \sum_{k=0}^{m-r_1-r_2} p_{m-r_1-r_2,k}(x) \cdot \left[(m - r_1 - r_2 - k)\,x - k(1-x)\right] \\
& \cdot \Big\{ (1-x)^2 \left[x, \tfrac{k}{m}; f\right] + x(1-x) \sum_{i=1}^{2} \left[x, \tfrac{k+r_i}{m}; f\right] + x^2 \left[x, \tfrac{k+r_1+r_2}{m}; f\right] \Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
(Q_{m,r_1,r_2}f)(x) =& \tfrac{x(1-x)}{m} \sum_{k=0}^{m-r_1-r_2} p_{m-r_1-r_2,k}(x) \cdot \Big\{ (1-x)(r_1 + r_2)\left[x, \tfrac{k}{m}; f\right] \\
& + \Big((r_2 x - r_1(1-x))\left[x, \tfrac{k+r_1}{m}; f\right] + (r_1 x - r_2(1-x))\left[x, \tfrac{k+r_2}{m}; f\right] \Big) \\
& + x(r_1 + r_2)\left[x, \tfrac{k+r_1+r_2}{m}; f\right] \Big\}.
\end{aligned}
$$

Using the combinatorial identities

$(m - r_1 - r_2 - k)\binom{m-r_1-r_2}{k} = (m - r_1 - r_2)\binom{m-r_1-r_2-1}{k}$ and

$k\binom{m-r_1-r_2}{k} = (m-r_1-r_2)\binom{m-r_1-r_2-1}{k-1}$ we can write

$$
\begin{aligned}
(P_{m,r_1,r_2}f)(x) = &\tfrac{(m-r_1-r_2)}{m} \sum_{k=0}^{m-r_1-r_2-1} \tbinom{m-r_1-r_2-1}{k} x^{k+1}(1-x)^{m-r_1-r_2-k} \\
&\cdot\Big\{(1-x)^2\big[x,\tfrac{k}{m};f\big] + x(1-x)\sum_{i=1}^{2}\big[x,\tfrac{k+r_i}{m};f\big] + x^2\big[x,\tfrac{k+r_1+r_2}{m};f\big]\Big\} \\
&- \sum_{k=1}^{m-r_1-r_2} \tbinom{m-r_1-r_2-1}{k-1} x^k (1-x)^{m-r_1-r_2-k+1} \\
&\cdot\Big\{(1-x)^2\big[x,\tfrac{k}{m};f\big] + x(1-x)\sum_{i=1}^{2}\big[x,\tfrac{k+r_i}{m};f\big] + x^2\big[x,\tfrac{k+r_1+r_2}{m};f\big]\Big\}.
\end{aligned}
$$

By changing the summation index in the second sum and using the recurrence relation of divided differences

$$[x,a;f] - [x,b;f] = (a-b)[x,a,b;f]$$

we obtain

$$
\begin{aligned}
(P_{m,r_1,r_2}f)(x) = &\tfrac{x(x-1)(m-r_1-r_2)}{m^2} \sum_{k=0}^{m-r_1-r_2-1} p_{m-r_1-r_2-1,k}(x)\cdot\Big\{(1-x)^2\big[x,\tfrac{k}{m},\tfrac{k+1}{m};f\big] \\
&+ x(1-x)\sum_{i=1}^{2}\big[x,\tfrac{k+r_i}{m},\tfrac{k+r_i+1}{m};f\big] + x^2\big[x,\tfrac{k+r_1+r_2}{m},\tfrac{k+r_1+r_2+1}{m};f\big]\Big\}.
\end{aligned}
$$

Combining the terms that contains $(1-x)\,r_i$ and xr_i, respectively, in Q_{m,r_1,r_2}, we have:

$$
\begin{aligned}
&(Q_{m,r_1,r_2}f)(x) = \\
&= \tfrac{x(1-x)}{m} \sum_{k=0}^{m-r_1-r_2} p_{m-r_1-r_2,k}(x) \cdot \Big\{ (1-x)\sum_{i=1}^{2} r_i\left(\big[x,\tfrac{k}{m};f\big] - \big[x,\tfrac{k+r_i}{m};f\big]\right) \\
&\quad + x\sum_{i=1}^{2} r_i\left(\big[x,\tfrac{k+r_i}{m};f\big] - \big[x,\tfrac{k+r_1+r_2}{m};f\big]\right)\Big\}.
\end{aligned}
$$

If we use again the recurrence relation for divided differences we can write

$$
\begin{aligned}
(Q_{m,r_1,r_2}f)(x) = &\tfrac{x(x-1)}{m^2} \sum_{k=0}^{m-r_1-r_2} p_{m-r_1-r_2,k}(x)\cdot \\
&\cdot\left\{(1-x)\sum_{i=1}^{2} r_i^2\left[x,\tfrac{k}{m},\tfrac{k+r_i}{m};f\right] + x\sum_{i=1}^{2} r_i^2\left[x,\tfrac{k+r_i}{m},\tfrac{k+r_1+r_2}{m};f\right]\right\}.
\end{aligned}
$$

So, using divided differences we obtained the following form for the remainder (6)

$$
\begin{aligned}
&\left(R_{m,r_1,r_2}f\right)(x)=\\
&=\tfrac{x(x-1)}{m^2}\Bigg\{(m-r_1-r_2)\sum_{k=0}^{m-r_1-r_2-1}p_{m-r_1-r_2-1,k}(x)\cdot\Bigg((1-x)^2\left[x,\tfrac{k}{m},\tfrac{k+1}{m};f\right]\\
&\quad+x(1-x)\sum_{i=1}^{2}\left[x,\tfrac{k+r_i}{m},\tfrac{k+r_i+1}{m};f\right]+x^2\left[x,\tfrac{k+r_1+r_2}{m},\tfrac{k+r_1+r_2+1}{m};f\right]\Bigg)\\
&\quad+\sum_{k=0}^{m-r_1-r_2}p_{m-r_1-r_2,k}(x)\sum_{i=1}^{2}r_i^2\Big((1-x)\left[x,\tfrac{k}{m},\tfrac{k+r_i}{m};f\right]+x\left[x,\tfrac{k+r_i}{m},\tfrac{k+r_1+r_2}{m};f\right]\Big)\Bigg\}.
\end{aligned}
$$

Taking into account that $\left(R_{m,r_1,r_2}e_2\right)(x)=\frac{x(x-1)}{m^2}\left(m-r_1-r_2+r_1^2+r_2^2\right)$ one observes that the previous expression can be represented in the following form:

$$\left(R_{m,r_1,r_2}f\right)(x)=\left(R_{m,r_1,r_2}e_2\right)(x)\left(D_{m,r_1,r_2}f\right)(x),$$

where

$$
\begin{aligned}
&\left(D_{m,r_1,r_2}f\right)(x)=\\
&=\tfrac{1}{m-r_1-r_2+r_1^2+r_2^2}\Bigg\{(m-r_1-r_2)\sum_{k=0}^{m-r_1-r_2-1}p_{m-r_1-r_2-1,k}(x)\cdot\Bigg((1-x)^2\left[x,\tfrac{k}{m},\tfrac{k+1}{m};f\right]\\
&\quad+x(1-x)\sum_{i=1}^{2}\left[x,\tfrac{k+r_i}{m},\tfrac{k+r_i+1}{m};f\right]+x^2\left[x,\tfrac{k+r_1+r_2}{m},\tfrac{k+r_1+r_2+1}{m};f\right]\Bigg)\\
&\quad+\sum_{k=0}^{m-r_1-r_2}p_{m-r_1-r_2,k}(x)\sum_{i=1}^{2}r_i^2\Big((1-x)\left[x,\tfrac{k}{m},\tfrac{k+r_i}{m};f\right]+x\left[x,\tfrac{k+r_i}{m},\tfrac{k+r_1+r_2}{m};f\right]\Big)\Bigg\}.
\end{aligned}
$$

It is easy to see that all the coefficients of this linear functional are positive and that their sum is $\left(D_{m,r_1,r_2}e_2\right)(x)=1$ for any x in $[0,1]$, so it is a convex combination of the second-order divided differences.

From the above representation it results that if f is a nonlinear convex (resp. concave) function on $[0,1]$ then we have $L_{m,r_1,r_2}f>f$ (resp. $L_{m,r_1,r_2}f<f$).

Because the degree of the corresponding approximation formula is one and $R_{m,r_1,r_2}f\neq 0$ whenever f is a convex function, by using a theorem of T. Popoviciu [10] we can state that there exist three distinct points $\xi_1,\ \xi_2,\ \xi_3$ on $[0,1]$ such that

$$\left(R_{m,r_1,r_2}f\right)(x)=\left(R_{m,r_1,r_2}e_2\right)(x)\left[\xi_1,\xi_2,\xi_3;f\right].$$

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