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REMARKS ON INTERPOLATION IN CERTAIN LINEAR SPACES (III)

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Abstract. In this paper we study a way of extending the model of interpolation the real functions, with simple nodes, to the case of the functions defined between linear spaces, specially between linear normed spaces.

In order to keep as many characteristics as possible from the case of the interpolation of real functions, in this paper we present a model of construction of the abstract interpolation polynomials, based on the properties of multilinear mappings. In this case we will define the divided differences and establish interpolation formulas with the rest expressed as a divided difference. We give the example of the type of interpolation polynomial built for non-linear mappings between spaces of functions defined on a certain interval.

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1. INTRODUCTION

This paper continues on the extension of the polynomial interpolation with simple nodes of the real function to more general situations.

Let us consider the linear spaces X and Y, the set $D \subseteq X$ and the function $f : D \to Y$. We will note with $\mathcal{L}_n(X,Y)$ the set of the n-linear mappings from X^n to Y, this set being one linear space as well.

If X and Y are linear normed spaces with the norms noted by $\|\cdot\|_X$ and $\|\cdot\|_Y$, it is possible to talk about the set of the linear and continuous mappings from X^n to Y. This set noted $(X^n,Y)^*$ can organized as linear normed space, introducing the norm $\|\cdot\|$ trough:

$$\|T\| = \sup\left\{ \|T(x_1,...,x_n)\|_Y / \, x_i \in X,\ \|x_i\| = 1,\ i=\overline{1,n} \right\}, \tag{1.1}$$

and holding, for any $i=\overline{1,n}$ and for any $x_1,...,x_n \in X$ the inequality:

$$\|T(x_1,...,x_n)\|_Y \leq \|T\| \cdot \|x_1\|_X \cdot ... \cdot \|x_n\|_X. \tag{1.2}$$

In the relations (1.1) and (1.2) the mapping $T \in (X^n,Y)^*$ is arbitrary.

For $n=1$, we use the notation $\mathcal{L}(X,Y)$ replacing $\mathcal{L}_1(X,Y)$ and $(X,Y)^*$ replacing $(X^1,Y)^*$.

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The problem of the determination of one abstract interpolation polynomial, preserving his Newton's form, appears first in the papers of Păvăloiu, I., [8], [9], [10].

In these papers let us give the number $n \in \mathbb{N}$ and the distinct elements $x_0, x_1, ..., x_n \in D$ called interpolation nodes. Let us suppose for any $l \in \{0, ..., n-1\}$ and for any $k \in \{1, ..., n\}$ the existence of the mapping $[x_l, x_{l+1}, ..., x_k] \in \mathcal{L}_{k-l}(X, Y)$. These mapping are denominated generalized divided differences on the nodes $x_l, x_{l+1}, ..., x_k \in D$ and verify the equalities:

$$[x_l, x_{l+1}, ..., x_k; f](x_k - x_l) = [x_{l+1}, ..., x_k; f] - [x_l, ..., x_{k-1}; f]. \tag{1.3}$$

With the help of this mapping let us consider:

(1.4)

$$\mathbf{L}(x_0, x_1, ..., x_n; f) : X \to Y,$$

$$\mathbf{L}(x_0, x_1, ..., x_n; f)(x) = f(x_0) + \sum_{i=0}^{n} [x_0, x_1, ..., x_k; f](x - x_0)...(x - x_{i-1}).$$

This nonlinear mapping is a generalized polynomial in the meaning specified in [6], that for any $k \in \{0, 1, ..., n\}$ verifies the equalities:

$$\mathbf{L}(x_0, x_1, ..., x_n; f)(x_k) = f(x_k), \tag{1.5}$$

corresponding in this way to proposed goal.

From this reason, the introduced nonlinear mapping from (1.4) represent one abstract interpolation polynomial of the function $f : D \to Y$, corresponding to the nodes $x_0, x_1, ..., x_n$.

If for any $x \in D$ there exists the mapping $[x_0, x_1, ..., x_k, x; f]$ representing the generalized divided difference on $n+2$ nodes, that verifies similar equalities with those from (1.3), the next equality holds:

$$f(x) = \mathbf{L}(x_0, x_1, ..., x_n; f)(x) + [x_0, x_1, ..., x_k, x; f](x - x_0)...(x - x_n). \tag{1.6}$$

Another direction to obtain abstract interpolation polynomials is that from the paper of Prenter, M., [11], that has in attention the expression of Lagrange type.

So:

$$\mathbf{L}(x_0, x_1, ..., x_n; f)(x) = \tag{1.7}$$
$$= \sum_{i=1}^{n} v_i(x - x_0, ..., x - x_{i-1}, x - x_{i+1}, ..., x - x_n, f(x_i)),$$

where $v_i \in \mathcal{L}_{n+1}(X^n \times Y, Y)$, for any $i = \overline{1, n}$.

The idea has been taken over and extended in our papers [2], [3], [4]. In the paper [5], we establish that for any $i, j \in \{0, 1, ..., n\}$ with $i \neq j$, because of a consequence of the well-known theorem of Hahn-Banach, there exists the linear linear and continuous functionals $U_{ij} : X \to \mathbb{R}$ that verify:

$$(1.8) \qquad \begin{cases} \|U_{ij}\| = 1, \\ U_{ij}(x_i - x_j) = \|x_i - x_j\|_X \ . \end{cases}$$

With the help of functionals we can construct:

$$(1.9) \quad \mathbf{L}(x_0, x_1, ..., x_n; f) : X \to Y, \ \mathbf{L}(x_0, x_1, ..., x_n; f)(x) = \sum_{i=0}^{n} l_i(x) f(x_i),$$

where:

$$(1.10) \qquad l_i : X \to \mathbb{R}, \ l_i(x) = \prod_{j=\overline{0,n};\ j \neq i} \tfrac{U_{ij}(x - x_j)}{\|x_i - x_j\|_X}.$$

Evidently for any $k = \overline{0,n}$ we have the relations (1.5), so the mapping defined by (1.9)–(1.10) represents an abstract interpolation polynomial.

The drawback of this construction is that we can not estimate the rest of the approximation of the values of the function trough the values of the abstract interpolation polynomial. This drawback justifies the construction presented in this paper.

Significant contributions to the theory of interpolation in abstract spaces are presented by Argyros, I. K., in [1] and by Makarov, V. L., and Hlobistov, V. V., in [7].

2. THE CONSTRUCTION OF ABSTRACT INTERPOLATION POLYNOMIALS. MAIN PROPERTIES

The main notions and results from this paper have been presented in the paper [2], and [6].

Let us consider the linear spaces X and Y over the same body and the mappings $U \in \mathcal{L}(X, Y)$ and $B \in \mathcal{L}_2(Y, Y)$. Using these we will build the sequence $(A_n)_{n \in \mathbb{N}}$ where the mapping $A_n \in \mathcal{L}_n(X, Y)$ is defined through:

$$(2.1) \qquad A_n(u_1, ..., u_n) = B(A_{n-1}(u_1, ..., u_{n-1}), U(u_n)),$$

for any $(u_1, ..., u_n) \in X$ with $n \in \mathbb{N}, \ n \geq 2$ and with the specification that $A_1(u) = U(u)$ for any $u \in X$.

We suppose that the mapping $B \in \mathcal{L}_2(Y, Y)$ verifies the following conditions:

A_1) $B(u, v) = B(v, u)$ for any $u, v \in Y$;

A_2) $B(B(u, v), w) = B(u, B(v, w))$ for any $u, v, w \in Y$, meaning that $B \in \mathcal{L}_2(Y, Y)$ determines on Y a **commutative algebra**.

In the papers [2], [6] we have proved that in this conditions:

a_1) *for any* $n \in \mathbb{N}$, *the mapping* $A_n \in \mathcal{L}_n(X, Y)$ *is symmetrical;*

a_2) *for any* $k, n \in \mathbb{N}, \ k < n$, *we have the following equality:*

$$(2.2) \qquad A_n(u_1, ..., u_n) = B(A_k(u_1, ..., u_k), A_{n-k}(u_{k+1}, ..., u_n)).$$

Let us now suppose the existence of a mapping $U \in \mathcal{L}(X,Y)$ which is clearly injective and of a subset $Y_0 \subseteq U(X)$ so that the bilinear mapping $B \in \mathcal{L}_2(Y,Y)$ can have the following properties:

A3) *$u_0 \in Y_0$ exists so that $B(u,u_0)$ for any $u \in Y$;*
A4) *for any $u \in Y_0$ there exists u' so that $B(u,u') = u_0$;*
A5) *for any $u, v \in Y_0$ we have $B(u,v) \in Y_0$.*

We can remark:

a$_3$) *If we consider the relations A1)–A5) we notice that the restriction of $B \in L_2(Y,Y)$ to $Y_0 \times Y_0$ determines a structure of the abelian group on Y_0, its neuter element being u_0.*

a$_4$) *It is clear that the restriction of the mapping $U \in \mathcal{L}(X,Y)$ to the set $U^{-1}(Y_0)$, with values in Y_0, is bijective.*

We will now introduce the sequence of mappings $(A_n)_{n\in\mathbb{N}}$ using the relations (2.1), we consider a set $D \subseteq X$ and a sequence $(x_n)_{n\in\mathbb{N}}$. Let $k, n \in \mathbb{N}$, values for which we introduce the non-linear mappings $w_{n,k} : X \to Y$, defined by:

$$w_{n,k}(x) = A_{n+1}(x - x_k, x - x_{k+1}, ..., x - x_{k+n}) \tag{2.3}$$

and for any $x \in X$, the mapping $w'_{n,k}(x) \in \mathcal{L}(X,Y)$ define by:

$$w'_{n,k}(x)\,h = \sum_{i=k}^{k+n} A_{n+1}(x - x_k, ..., x - x_{i-1}, x - x_{i+1,...,} x - x_{k+n}, h)\,. \tag{2.4}$$

In the case when X and Y are normed linear spaces, the mapping defined through (2.4) represents the Fréchet derivative in the point x of the mapping defined by (2.3)

Evidently, for any $n, k \in \mathbb{N}$ and for any $i \in \{k, k+1, ..., k+n\}$ we have:

$$w'_{n,k}(x_i)\,h = A_{n+1}(x_i - x_k, ..., x_i - x_{i-1}, x_i - x_{i+1,...,} x_i - x_{k+n}, h) \tag{2.5}$$

and it is clear that $w'_{n,k}(x_i) \in \mathcal{L}(X,Y)$.

In the paper [2] we have proved:

a$_5$) *If for the mapping $B \in \mathcal{L}_2(Y,Y)$ the properties* A$_3$), A$_4$), A$_5$ are verified on a set $Y_0 \subseteq U(X)$, *the elements of the sequence $(x_n)_{n\in\mathbb{N}}$ are so that for certain values $k, n \in N$ and for any $i, j \in \{k, k+1, ..., k+n\}$ with $i \neq j$ the following is true:*

$$x_i - x_j \in U^{-1}(Y_0)\,, \tag{2.6}$$

so for the same values $k, n \in N$ and for $i \in \{k, k+1, ..., k+n\}$, the restrictions to $U^{-1}(Y_0)$ of the mappings defined by (2.5) *are bijective.*

Now, we can remark:

a$_6$) *If we note by $\left[w'_{k,n}(x_i)\right]_0 : U^{-1}(Y_0) \to Y_0$ the restriction from* a$_5$) *refers to, there exists the mapping $\left[w'_{k,n}(x_i)\right]_0^{-1} : Y_0 \to U^{-1}(Y_0)$*

which represents the inverse of the bijective mapping $\left[w'_{k,n}(x_i)\right]_0 :$ $U^{-1}(Y_0) \to Y_0$.

Considering the set $sp(Y_0)$ *representing the linear cover of the set* Y_0, *the mapping* $\left[w'_{k,n}(x_i)\right]_0^{-1} : Y_0 \to U^{-1}(Y_0)$ *will be prolonged through linearity to* $sp(Y_0)$, *we will thus obtain a mapping* $\left[w'_{k,n}(x_i)\right]_*^{-1} : sp(Y_0) \to X$.

a$_7$) *If the linear space* Y *has also a topological structure (nevertheless the structure of topological vectorial space of* Y_0 *is not compulsory), the mapping* $\left[w'_{k,n}(x_i)\right]_*^{-1} : sp(Y_0) \to X$ *can be prolonged to* $cl(sp(Y_0))$ *thus obtaining the mapping* $\left[w'_{k,n}(x_i)\right]_{**}^{-1} \in \mathcal{L}(cl(sp(Y_0)), X)$, *by* $cl(A)$ *we denote the closure of the set* A *in the considered topology.*

Let us now consider a natural number n, $D \subseteq X$ and the elements $x_0, \dots, x_n \in D$. We suppose that the following hypotheses are true:

I) The mapping $B \in \mathcal{L}_2(Y, Y)$ verifies the hypotheses A$_3$), A$_4$), A$_5$), referring to a set $Y_0 \subseteq U(X) \subseteq Y$.

II) For any $i, j \in \{0, 1, ..., n\}$ with $i \neq j$ we have $x_i - x_j \in U^{-1}(Y_0)$.

III) $f(x_0), f(x_1), ..., f(x_n) \in sp(Y_0)$.

It is clear that the mapping:

$$\left[w'_{0,n}(x_i)\right]_*^{-1} \in \mathcal{L}(sp(Y_0), X)$$

exists, for $i \in \{0, 1, ..., n\}$. So there exists the mapping:

$$\mathbf{L}(x_0, x_1, ..., x_n; f) : X \to Y$$

defined by:

$$\begin{aligned} \text{(2.7)} \qquad & \mathbf{L}(x_0, x_1, ..., x_n; f)(x) = \\ & = \sum_{i=0}^{n} A_{n+1}(x - x_0, ..., x - x_{i-1}, x - x_{i+1}, ..., x - x_n, y_{n,i}), \end{aligned}$$

where:

$$y_{n,i} = \left[w'_{0,n}(x_i)\right]_*^{-1} f(x_i).$$

We have the following:

Theorem 1. *If the hypothesis* I)–III) *are verified, then the mapping defined by* (2.7) *verifies for any* $i \in \{0, 1, ..., n\}$ *the equalities:*

$$\mathbf{L}(x_0, x_1, ..., x_n; f)(x_i) = f(x_i).$$

For the proof, see [6].

Because of theorem 1 we have the following:

Definition 2. *The non-linear mapping defined through the relation* (2.7) *under the conditions* I)–III) *is called* $(\mathbf{U}-\mathbf{B})$ ***abstract interpolation polynomial of the function*** f ***on the nodes*** $x_0, x_1, ..., x_n$.

We have the next remark:

Remark 2.1. If we take into account the linearity of the mapping A_{n+1} we deduce that:

$$\mathbf{L}\left(x_0, x_1, ..., x_n; f\right)(x) = D_n x^n + D_{n-1}x^{n-1} + ... + D_1 x + D_0, \tag{2.8}$$

where $D_k \in \mathcal{L}_k\left(X, Y\right)$ for any $k = \overline{1, n}$ and for $D_0 \in Y$ we have:

$$D_0 = (-1)^n \sum_{i=0}^{n} A_{n+1}(x_0, ..., x_{i-1}, x_{i+1}, ..., x_n; \left[w'_{0,n}\left(x_i\right)\right]_*^{-1} f\left(x_i\right)). \tag{2.9}$$

From here results the character of abstract polynomial of the mapping $\mathbf{L}\left(x_0, x_1, ..., x_n; f\right)(x) : X \to Y$.

As in the previous paragraph, a special role is held by the mapping $D_n \in \mathcal{L}_n\left(X, Y\right)$ from the relation (2.8).

In this respect we will have:

Definition 3. *Let us consider the conditions* I)–III) *fulfilled.*

The mapping $[x_0, x_1, ..., x_n; f] \in \mathcal{L}_n\left(X, Y\right)$ *defined by*

$$[x_0, x_1, ..., x_n; f]\, h_1...h_n = \sum_{i=0}^{n} A_{n+1}\left(h_1, ..., h_n; \left[w'_{0,n}\left(x_i\right)\right]_*^{-1} f\left(x_i\right)\right) \tag{2.10}$$

is called ***the divided difference of the order*** n ***of the function*** f ***on the nodes*** $x_0, x_1, ..., x_n$.

In the paper [6] we have proved certain properties of the $(\mathbf{U}-\mathbf{B})$ interpolation abstract polynomials and of the divided differences.

We will now establish the following theorem:

Theorem 4. *If the hypotheses of Theorem* 1 *are true and furthermore the hypotheses* $\mathrm{A}_1)$ *and* $\mathrm{A}_2)$ *referring to the mapping* $B \in \mathcal{L}_2\left(Y, Y\right)$ *are verified, then:*

j) *the following equality is true:*

$$[x_0, x_1, ..., x_n; f]\left(x_n - x_0\right) = [x_1, ..., x_n; f] - [x_0, ..., x_{n-1}; f], \tag{2.11}$$

the equality being understood as between the elements of the space $\mathcal{L}\left(X, Y\right)$.

jj) *the* $(U - B)$ *interpolation abstract polynomials verify the relation of recurrence:*

$$\begin{aligned} &\mathbf{L}\left(x_0, x_1, ..., x_n; f\right)(x) = \\ &= \mathbf{L}\left(x_0, x_1, ..., x_{n-1}; f\right)(x) \\ &\quad + [x_0, x_1, ..., x_n; f]\left(x - x_0\right)\left(x - x_1\right)...\left(x - x_{n-1}\right); \end{aligned} \tag{2.12}$$

jjj) *the* $(U-B)$ *interpolation abstract polynomial of the function* $f:X\to Y$ *on the nodes* $x_0,x_1,...,x_n$ *can be written under Newton's form:*

$$\mathbf{L}\left(x_0,x_1,...,x_n;f\right)(x)= \tag{2.13}$$
$$=f\left(x_0\right)+\sum_{i=1}^{n}\left[x_0,x_1,...,x_i;f\right]\left(x-x_0\right)\left(x-x_1\right)...\left(x-x_{i-1}\right);$$

jv) *the* $(U-B)$ *abstract interpolation polynomials verify the following relation of the Aitken-Steffenson type:*

$$B\left(\mathbf{L}\left(x_0,x_1,...,x_n;f\right)(x),U\left(x_n-x_0\right)\right)= \tag{2.14}$$
$$=B\left(\mathbf{L}\left(x_1,...,x_n;f\right)(x),U\left(x-x_0\right)\right)-$$
$$-B\left(\mathbf{L}\left(x_0,...,x_{n-1};f\right)(x),U\left(x-x_n\right)\right);$$

v) *the following equality is verified:*

$$f(x)=\mathbf{L}\left(x_0,x_1,...,x_n;f\right)(x)+ \tag{2.15}$$
$$+\left[x_0,x_1,...,x_n,x;f\right]\left(x-x_0\right)\left(x-x_1\right)...\left(x-x_{n-1}\right)\left(x-x_n\right),$$

for any $n\in\mathbb{N}$ *and* $x\in X$.

For the proof, see [6].

3. ESTIMATED OF THE APPROXIMATIONS REST

The equality (2.15) allows for the evaluation of the error through which the $(U-B)$ interpolation abstract polynomial approximates the function $f:D\to Y$. Let us suppose that X and Y are normed linear spaces with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ and that $\|\cdot\|$ is the norm of the space $(X,Y)^*$. We will also suppose that $U\in(X,Y)^*$ and $B\in\left(X^2,Y\right)^*$, then $A_n\in\left(X^n,Y\right)^*$ for any $n\in\mathbb{N}$ and for any $x_0,x_1,...,x_n,x\in D$ we have:

$$\left\|f(x)-\mathbf{L}\left(x_0,x_1,...,x_n;f\right)(x)\right\|_Y\leq \tag{3.1}$$
$$\leq\left\|\left[x_0,x_1,...,x_n,x;f\right]\right\|\cdot\left\|x-x_0\right\|_X\cdot\left\|x-x_1\right\|_X\cdot...\cdot\left\|x-x_n\right\|_X.$$

It is evident that:

$$\left\|\left[x_0,x_1,...,x_n,x;f\right]\right\|=$$
$$=\sup_{h_i\in X,\ \|h_i\|_X\leq1,\ i=\overline{1,n+1}}\left\|\left[x_0,x_1,...,x_n,x;f\right]h_1h_2...h_nh_{n+1}\right\|_Y.$$

But:

$$\left\|\left[x_0,x_1,...,x_n,x;f\right]h_1h_2...h_nh_{n+1}\right\|_Y=$$
$$=\left\|A_{n+2}\left(h_1,h_2,...,h_{n+1},z_{n+1}\right)\right\|_Y$$
$$\leq\left\|A_{n+1}\right\|\cdot\left\|h_1\right\|_X\cdot...\cdot\left\|h_{n+1}\right\|_X\cdot\left\|z_{n+1}\right\|_X$$
$$\leq\left\|A_{n+1}\right\|\cdot\left\|z_{n+1}\right\|_X,$$

here:

$$z_{n+1}=\sum_{i=0}^{n}\left[\mathbf{P}_{ni}\left(x\right)\right]^{-1}f\left(x_i\right)+\left[\mathbf{S}_{n}\left(x\right)\right]^{-1}f\left(x\right),$$

where:

$$\mathbf{P}_{ni}\left(x\right),\mathbf{S}_{n}\left(x\right)\in\left(X,Y\right)^{*},$$

$$\mathbf{P}_{ni}\left(x\right)h=A_{n+2}\left(x_i-x_0,...,x_i-x_{i-1},x_i-x_{i+1},...,x_i-x_n,x_i-x,h\right)$$

and

$$\mathbf{S}_{n}\left(x\right)h=A_{n+2}\left(x-x_0,...,x-x_n,h\right)$$

for any $i=\overline{0,n},\ n\in\mathbb{N}$.

So

$$\left\|z_{n+1}\right\|_X\leq\sum_{i=0}^{n}\left\|\left[\mathbf{P}_{ni}\left(x\right)\right]^{-1}\right\|\cdot\left\|f\left(x_i\right)\right\|_Y+\left\|\left[\mathbf{S}_{n}\left(x\right)\right]^{-1}\right\|\cdot\left\|f\left(x\right)\right\|_Y.$$

Through induction it is easy to prove that$\left\|A_{n+1}\right\|\leq\left(\left\|B\right\|\cdot\left\|U\right\|\right)^n$, and from (3.1) we deduce that:

(3.2)

$$\left\|f\left(x\right)-\mathbf{L}\left(x_0,x_1,...,x_n;f\right)\left(x\right)\right\|_Y\leq$$
$$\leq\left(\left\|B\right\|\cdot\left\|U\right\|\right)^n\left(\sum_{i=0}^{n}\left\|\left[\mathbf{P}_{ni}\left(x\right)\right]^{-1}\right\|\cdot\left\|f\left(x_i\right)\right\|_Y+\left\|\left[\mathbf{S}_{n}\left(x\right)\right]^{-1}\right\|\cdot\left\|f\left(x\right)\right\|_Y\right).$$

2) The previous construction is conditioned by the relations $f\left(x_i\right)\in sp\left(Y_0\right)$ ($f\left(x_i\right)\in cl\left(sp\left(Y_0\right)\right)$, respectively) for any $i=\overline{0,n}$. If $sp\left(Y_0\right)=Y$ ($cl\left(sp\left(Y_0\right)\right)=Y$, respectively) these conditions are evidently verified.

4. A CONCRETE EXAMPLE FOR THE CONSTRUCTION OF THE GENERALIZED INTERPOLATION POLYNOMIALS

In this paragraph we will give an example for the construction of an interpolation polynomial which we call generalized because we are referring to a case which is different from the interpolation of real functions of a real variable.

We still mention for the beginning that if $X=Y=\mathbb{K}$, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, taking

$$B\left(u,v\right)=u\cdot v$$

for every $u,v\in\mathbb{K}$, as well $U:\mathbb{K}\to\mathbb{K}$ is the identical mapping and $Y_0=\mathbb{K}\setminus\{0\}$ (0 is the null real or complex number), it is obvious that $sp\left(Y_0\right)=\mathbb{K}$, so we will obtain again the theory of the interpolation of real function with a real variable.

Let now $\mathbb{K}$ be the set of real or complex numbers and we consider:

$$\mathcal{F}_{\mathbb{K}}\left[a,b\right]=\left\{x/\,x:\left[a,b\right]\to\mathbb{K}\right\},$$

meaning that $\mathcal{F}_{\mathbb{K}}\left[a,b\right]$ represents the set of all functions defined on the interval $\left[a,b\right]$ taking values in $\mathbb{K}$.

We will take $U = I$ the identical mapping from $\mathcal{F}_{\mathbb{K}}[a,b]$ to itself and evidently $U \in \mathcal{L}(\mathcal{F}_{\mathbb{K}}[a,b], \mathcal{F}_{\mathbb{K}}[a,b])$. Then $B : \mathcal{F}_{\mathbb{K}}[a,b] \times \mathcal{F}_{\mathbb{K}}[a,b] \to \mathcal{F}_{\mathbb{K}}[a,b]$ through $[B(u,v)](t) = u(t) \cdot v(t)$ with $u, v \in \mathcal{F}_{\mathbb{K}}[a,b]$ and $t \in [a,b]$. One can easily notice that the hypotheses A_1) and A_2) are verified.

We now define:

$$Y_0 = \{x \in \mathcal{F}_{\mathbb{K}}[a,b] / \, x(t) \neq 0 \text{ for any } t \in [a,b]\}$$

and we notice that the hypotheses A_3)–A_5) are verified if we consider $u_0 : [a,b] \to K$ through $u_0(t) = 1$ for any $t \in [a,b]$. It is also evident that $sp(Y_0) = \mathcal{F}_{\mathbb{K}}[a,b]$.

We easily notice that:

$$(A_n(u_1, ..., u_n))(t) = u_1(t) \cdot u_2(t) \cdot ... \cdot u_n(t) \tag{4.1}$$

for any $u_i \in \mathcal{F}_{\mathbb{K}}[a,b]$, $i = \overline{1,n}$, and $t \in [a,b]$;

$$(w_{0,n}(x))(t) = \prod_{i=0}^{n} (x(t) - x_i(t)) \tag{4.2}$$

for any $x_0, x_1, ..., x_n, x \in \mathcal{F}_{\mathbb{K}}[a,b]$ and $t \in [a,b]$;

$$\left(w'_{0,n}(x_i)\, h\right)(t) = h(t) \prod_{j=0,\, j\neq i}^{n} (x_i(t) - x_j(t)) \tag{4.3}$$

for any $i = \overline{0,n}$; $x_0, x_1, ..., x_n, h \in \mathcal{F}_K[a,b]$ and $t \in [a,b]$.

Among the hypotheses of the theorems from the previous paragraph we have $x_i - x_j \in U^{-1}(Y_0) = Y_0$, which is reduced to

$$x_i(t) \neq x_j(t) \tag{4.4}$$

for any $t \in [a,b]$ and $i, j \in \{0, 1...., n\}$, $i \neq j$.

If these conditions are fulfilled it is evident that:

$$\left(\left[w'_{0,n}(x_i)\right]^{-1} u\right)(t) = \frac{u(t)}{\prod\limits_{j=0,\, j\neq i}^{n} (x_i(t) - x_j(t))}$$

for any $i = \overline{0,n}$; $x_0, x_1, ..., x_n, u \in \mathcal{F}_K[a,b]$ if the relations (4.4) are verified and $t \in [a,b]$.

Thus it is clear that for $f : \mathcal{F}_K[a,b] \to \mathcal{F}_K[a,b]$ we have:

$$\begin{aligned}
&\left(A_{n+1}\left(x - x_0, ..., x - x_{i-1}, x - x_{i+1}, ..., x - x_n, \left[w'_{0,n}(x_i)\right]^{-1} f(x_i)\right)\right)(t) \\
&= \prod_{j=0,\, j\neq i}^{n} [x(t) - x_j(t)] \cdot \left(\left[w'_{0,n}(x_i)\right]^{-1} f(x_i)\right)(t) \\
&= (f(x_i))(t) \cdot \prod_{j=0,\, j\neq i}^{n} \frac{x(t) - x_j(t)}{x_i(t) - x_j(t)},
\end{aligned}$$

for any $i = \overline{0,n}$.

We finally have:

$$\mathbf{L}\left(x_0, x_1, ..., x_n; f\right) : \mathcal{F}_{\mathbb{K}}\left[a, b\right] \to \mathcal{F}_{\mathbb{K}}\left[a, b\right]$$

through:

$$\left[\, \mathbf{L}\left(x_0, x_1, ..., x_n; f\right)(x)\right](t) = \sum_{i=0}^{n} \left[\left(f\left(x_i\right)\right)(t) \cdot \prod_{j=0,\ j\neq i}^{n} \tfrac{x(t)-x_j(t)}{x_i(t)-x_j(t)} \right]$$

and

$$\left[x_0, x_1, ..., x_n; f\right] \in \mathcal{L}\left(\left(\mathcal{F}_{\mathbb{K}}\left[a, b\right]\right)^n, \mathcal{F}_{\mathbb{K}}\left[a, b\right]\right)$$

through:

$$\left(\left[x_0, x_1, ..., x_n; f\right] h_1 ... h_n\right)(t) =$$
$$= h_1(t) \cdot ... \cdot h_n(t) \sum_{i=0}^{n} \left[\left(f\left(x_i\right)\right)(t) \cdot \prod_{j=0,\ j\neq i}^{n} \tfrac{1}{x_i(t)-x_j(t)} \right],$$

for any $n \in \mathbb{N}$, $x_0, x_1, ..., x_n \in \mathcal{F}_K\left[a, b\right]$ with the relations (4.4) verified, $h_1, ..., h_n \in \mathcal{F}_K\left[a, b\right]$ and $t \in \left[a, b\right]$.

For other example one can consult the paper [2].

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