ON THE LEIBNIZ FORMULA FOR DIVIDED DIFFERENCES

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Abstract. We give an identity for the Hermite-Lagrange interpolating polynomial and a short proof of Leibniz-type formula for divided differences in case of coalescing knots.

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1. INTRODUCTION

Let $x_0, \ldots, x_n$ be distinct points and $f$ be a function defined on the set \{x_0, \ldots, x_n\}. In most books on Numerical Analysis divided differences for distinct points are defined recursively:

$$[x_0]f = f(x_0), \quad \ldots,$$

$$[x_0, \ldots, x_n]f = \frac{[x_1, \ldots, x_n]f - [x_0, \ldots, x_{n-1}]f}{x_n - x_0}.$$

Divided differences might fairly be ascribed to Newton [9]. A. M. Ampère [1] called these quantities “les fonctions interpolaires” (see also [11, page 1]).

In a personal message, Erik Meijering [8] let us know that the term “divided difference” was used first in 1842 by Augustus de Morgan in Chapter XVIII, “On Interpolation and Summation”, of his book [5, p. 550] (see also [16]).

Newton’s formula for interpolation by a polynomial was given by Isaac Newton as Lemma 5 of Book III of his Principia Mathematica of 1687 [9] but it was known to him before since he mentioned it in a letter to the German scientist Henry Oldenburg dated October 24, 1676.

Newton proved that the polynomial

$$P_n(X) = f(x_0) + (X - x_0)[x_0, x_1]f + \cdots + (X - x_n)\cdots(X - x_{n-1})[x_0, \ldots, x_n]f,$$

is the unique polynomial of degree at most $n$ such that

$$P_n(x_i) = f(x_i), \quad i = 0, \ldots, n.$$
In 1795, J. L. Lagrange\[1\] obtained the following form of Newton’s interpolating polynomial:

\[ P_n(x) = \sum_{i=0}^{n} f(x_i) \frac{\ell(x)}{(x-x_i)\ell'(x_i)}, \]

where \( \ell(x) = (x-x_0)(x-x_1)\ldots(x-x_n) \).

The polynomial \( P_n \) of degree at most \( n \) such that

\[ P_n(x_i) = f(x_i), \quad i = 0, \ldots, n, \]

is nowadays generally called the Lagrange interpolation polynomial, of the function \( f \) with respect to \( x_0, \ldots, x_n \), and is denoted by \( L[x_0, \ldots, x_n]f \).

However, in 2002, Erik Meijering [8] discovered that early from the year 1779, Edward Waring [15] obtained “the Lagrange Formula”.

Let \( m, n \in \mathbb{N} \), and \( \alpha_0, \ldots, \alpha_n \in \mathbb{N}^\ast \) such that

\[ \alpha_0 + \cdots + \alpha_n = m + 1. \]

Consider the numbers \( y_{ij} \), \( i = 0, \ldots, n; \ j = 0, \ldots, \alpha_i - 1 \), and the distinct points \( x_0, \ldots, x_n \).

In 1878 Charles Hermite [7] proved that there exists a unique polynomial \( H_m \) of degree at most \( m \), called the Hermite-Lagrange interpolating polynomial, such that

\[ H_m^{(j)}(x_i) = y_{ij}, \]

for \( i = 0, \ldots, n; \ j = 0, \ldots, \alpha_i - 1 \).

Suppose that the function \( f: [a, b] \to \mathbb{R} \) possesses the derivatives \( f^{(\alpha_i - 1)}(x_i) \), \( i = 0, \ldots, n \). We denote the Hermite-Lagrange interpolating polynomial \( H_m \), satisfying

\[ H_m^{(j)}(x_i) = f^{(j)}(x_i), \]

\( i = 0, \ldots, n; \ j = 0, \ldots, \alpha_i - 1 \), by:

\[
H_m[x_0, \ldots, x_0, \ldots, x_n, \ldots, x_n]_{\alpha_0 \text{ times}} f, \quad \text{or} \quad H_m \begin{bmatrix} x_0 & x_1 & \ldots & x_n \end{bmatrix}_{\alpha_0 \text{ times}} \begin{bmatrix} \alpha_0 & \alpha_1 & \ldots & \alpha_n \end{bmatrix}_{\alpha_n \text{ times}} f.
\]

2. AN IDENTITY FOR THE HERMITE-LAGRANGE POLYNOMIALS

Rename the knots \((x_0, \ldots, x_0, \ldots, x_n, \ldots, x_n)\) to \((t_0, \ldots, t_m)\).

**Theorem 2.1.** The following formula

\[ H_m[t_0, \ldots, t_m]((X - t_0) \ldots (X - t_i) f) = (X - t_0) \ldots (X - t_i) H_m[t_{i+1}, \ldots, t_m] f, \]


\[ 2 \text{Also in Oeuvres de Charles Hermite, vol. III, Gauthier-Villars, Paris, 1912, pp. 432–443.} \]
is satisfied for \( i = 0, 1, \ldots, m - 1 \).

**Proof.** Use the notations:

\[
P = H_m \begin{bmatrix} x_0 & x_1 & \cdots & x_n \\ \alpha_0 & \alpha_1 & \cdots & \alpha_n \end{bmatrix} ((X - x_0)f),
\]

\[
Q = (X - x_0)H_m \begin{bmatrix} x_0 & x_1 & \cdots & x_n \\ \alpha_0 - 1 & \alpha_1 & \cdots & \alpha_n \end{bmatrix} f.
\]

It is sufficient to prove that \( P = Q \).

For \( i = 0 \), we have:

\[
Q^{(j)}(x_0) = j \left( \frac{d}{dx} \right)^{j-1} \left( H_m \begin{bmatrix} x_0 & x_1 & \cdots & x_n \\ \alpha_0 - 1 & \alpha_1 & \cdots & \alpha_n \end{bmatrix} f \right) \bigg|_{x=x_0}
\]

\((1 \leq j \leq \alpha_0 - 1)\) hence

\[
Q^{(j)}(x_0) = \begin{cases} 
0, & j = 0 \\
 j f^{(j-1)}(x_0), & 1 \leq j \leq \alpha_0 - 1 
\end{cases}
\]

\[
= \left( \frac{d}{dx} \right)^{j-1} ((x - x_0)f(x)) \bigg|_{x=x_0}
\]

\[
= P^{(j)}(x_0) \quad (0 \leq j \leq \alpha_0 - 1).
\]

For \( 1 \leq i \leq n \), we obtain:

\[
Q^{(j)}(x_i) = (x_i - x_0) \left( \frac{d}{dx} \right)^{j} \left( H_m \begin{bmatrix} x_0 & x_1 & \cdots & x_n \\ \alpha_0 - 1 & \alpha_1 & \cdots & \alpha_n \end{bmatrix} f \right) \bigg|_{x=x_i}
\]

\[
+ j \left( \frac{d}{dx} \right)^{j-1} \left( H_m \begin{bmatrix} x_0 & x_1 & \cdots & x_n \\ \alpha_0 - 1 & \alpha_1 & \cdots & \alpha_n \end{bmatrix} f \right) \bigg|_{x=x_i}
\]

\[
= \left( \frac{d}{dx} \right)^{j} ((x - x_0)f(x)) \bigg|_{x=x_i}
\]

\[
= P^{(j)}(x_i) \quad (0 \leq j \leq \alpha_i - 1).
\]

The degrees of the polynomials \( P \) and \( Q \) are at most \( m \), and \( Q^{(j)}(x_i) = P^{(j)}(x_i) \), \( 0 \leq i \leq n \), \( 0 \leq j \leq \alpha_i - 1 \). Hence \( P = Q \), and the proof is completed. \( \square \)

Identifying the coefficients of \( X^m \) in Eq. (2.1) we reobtain the following reduction formula for the divided difference in the case of multiple knots

\[
(2.2) \ [t_0, \ldots, t_m]((X - t_0) \ldots (X - t_i)f) = [t_{i+1}, \ldots, t_m]f, \quad i = 0, \ldots, m - 1.
\]
3. THE LEIBNIZ-TYPE FORMULA FOR DIVIDED DIFFERENCES

Let \( f \) and \( g \) be functions defined on an interval containing the points \( t_0, \ldots, t_n \) not necessarily distinct. The following Leibniz-type formula for divided differences is satisfied

\[
[t_0, \ldots, t_m](fg) = \sum_{k=0}^{m} [t_0, \ldots, t_k]f \ [t_k, \ldots, t_n]g.
\]

Formula (3.1) is generally credited to Steffensen, because of his paper [14]. However our earliest reference for Eq. (3.1) is that of Tiberiu Popoviciu [12, p. 12, Eq. (18)]. In 1940 Tiberiu Popoviciu [13, p. 65] protested: “Je me suis décidé de revenir sur ces questions en remarquant l’apparition de quelques travaux où mes résultats ne sont pas cités; voir, par exemple, [14]” (“I decided to come back upon these problems, because I have noticed some works, e.g., [14], in which results from my dissertation [12] are not quoted”).

On the occasion of Tiberiu Popoviciu’s 50th birthday, Academician Miron Nicolescu, G. Pic, D. V. Ionescu, E. Gergely, L. Németi, L. Bal and F. Rado published an overview of Popoviciu’s mathematical activity [10]. In this paper the authors reiterated the priority of Tiberiu Popoviciu in what concerns the Leibniz formula for divided differences.

In a recent paper [4], de Boor wrote: “My first reference for the Leibniz formula is [12], though Steffensen later devotes an entire paper, [14], to it and this has become the standard reference for it despite the fact that Popoviciu, in response, wrote his own overview of divided differences, [13], trying, in vain, to correct the record”.

A new proof of Leibniz’s formula is obtained by Tiberiu Popoviciu as an application of his Fundamental Formula for Divided Differences [13, p. 70].

In the case of distinct knots, an interesting proof of (3.1) is given by de Boor [2, p. 5]. De Boor’s idea is used by DeVore and Lorentz [6, p. 121] to prove (3.1) in the case of coalescing points, but assuming that the derivatives of \( f \) and \( g \), required for the divided difference \([t_0, \ldots, t_m]\), are continuous at the corresponding points. A generalization of formula (3.1) is given by de Boor in [3].

4. THE PROOF

We give a proof of the Leibniz-type formula (3.1) in the case of coalescing points. Using the Newton form of the interpolating polynomial (1.1) and the reduction formula (2.2) we have:

\[
[t_0, \ldots, t_m](fg) = [t_0, \ldots, t_m](H[t_0, \ldots, t_m](f) \ g)
\]

\[
= [t_0, \ldots, t_m]\left( \sum_{k=0}^{m} (t - t_0) \ldots (t - t_{k-1})[t_0, \ldots, t_k](f) \ g(t) \right)
\]
\[
= \sum_{k=0}^{m} [t_0, \ldots, t_k] f [t_0, \ldots, t_m]_r ((t - t_0) \ldots (t - t_{k-1}) g(t)) \\
= \sum_{k=0}^{m} [t_0, \ldots, t_k] f [t_k, \ldots, t_m] g,
\]
and the proof is completed.

REFERENCES


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