REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 35 (2006) no. 1, pp. 59–63
ictp.acad.ro/jnaat

ON THE LEIBNIZ FORMULA FOR DIVIDED DIFFERENCES

MIRCEA IVAN*

Abstract. We give an identity for the Hermite-Lagrange interpolating polynomial and a short proof of Leibniz-type formula for divided differences in case of coalescing knots.

MSC 2000. 41A05.
Keywords. Interpolation, divided difference, Lagrange polynomial, Hermite interpolating polynomial.

1. INTRODUCTION

Let $x_0, \dots, x_n$, be distinct points and f be a function defined on the set $\{x_0, \dots, x_n\}$. In most books on Numerical Analysis divided differences for distinct points are defined recursively:

$$[x_0]f = f(x_0), \quad \dots,$$

$$[x_0, \dots, x_n]f = \frac{[x_1, \dots, x_n]f - [x_0, \dots, x_{n-1}]f}{x_n - x_0}.$$

Divided differences might fairly be ascribed to Newton [9]. A. M. Ampère [1] called these quantities *"les fonctions interpolaires"* (see also [11, page 1]).

In a personal message, Erik Meijering [8] let us know that the term "divided difference" was used first in 1842 by Augustus de Morgan in Chapter XVIII, "On Interpolation and Summation", of his book [5, p. 550] (see also [16]).

Newton's formula for interpolation by a polynomial was given by Isaac Newton as Lemma 5 of Book III of his Principia Mathematica of 1687 [9] but it was known to him before since he mentioned it in a letter to the German scientist Henry Oldenburg dated October 24, 1676.

Newton proved that the polynomial

$$P_n(X) = f(x_0) + (X - x_0)[x_0, x_1]f + \dots + (X - x_0) \dots (X - x_{n-1})[x_0, \dots, x_n]f.$$

is the unique polynomial of degree at most n such that

$$P_n(x_i) = f(x_i), \qquad i = 0, \dots, n.$$

*Department of Mathematics, Technical University of Cluj-Napoca, Str. C. Daicoviciu 15, 400020 Cluj-Napoca, Romania, e-mail: mircea.ivan@math.utcluj.ro.

In 1795, J. L. Lagrange[1] obtained the following form of Newton's interpolating polynomial:

$$P_n(x) = \sum_{i=0}^{n} f(x_i)\frac{\ell(x)}{(x-x_i)\ell'(x_i)},$$

where $\ell(x) = (x-x_0)(x-x_1)\dots(x-x_n)$.

The polynomial P_n of degree at most n such that

$$P_n(x_i) = f(x_i), \qquad i = 0,\dots,n,$$

is nowadays generally called the *Lagrange interpolation polynomial*, of the function f with respect to $x_0,\dots,x_n$, and is denoted by $L[x_0,\dots,x_n]f$.

However, in 2002, Erik Meijering [8] discovered that early from the year 1779, Edward Waring [15] obtained *"the Lagrange Formula"*.

Let $m, n \in \mathbb{N}$, and $\alpha_0,\dots,\alpha_n \in \mathbb{N}^*$ such that

$$\alpha_0 + \dots + \alpha_n = m+1.$$

Consider the numbers y_{ij}, $i = 0,\dots,n$; $j = 0,\dots,\alpha_i - 1$, and the distinct points $x_0,\dots,x_n$.

In 1878 Charles Hermite [7][2] proved that there exists a unique polynomial H_m of degree at most m, called the *Hermite-Lagrange interpolating polynomial,* such that

$$H_m^{(j)}(x_i) = y_{ij},$$

for $i = 0,\dots,n$; $j = 0,\dots,\alpha_i - 1$.

Suppose that the function $f\colon [a,b] \to \mathbb{R}$ possesses the derivatives $f^{(\alpha_i-1)}(x_i)$, $i = 0,\dots,n$. We denote the *Hermite-Lagrange interpolating polynomial* H_m, satisfying

$$H_m^{(j)}(x_i) = f^{(j)}(x_i), \tag{1.1}$$

$i = 0,\dots,n$; $j = 0,\dots,\alpha_i - 1$, by:

$$H_m[\underbrace{x_0,\dots,x_0}_{\alpha_0\text{times}},\dots,\underbrace{x_n,\dots,x_n}_{\alpha_n\text{times}}]f, \quad \text{or} \quad H_m\begin{bmatrix} x_0 & x_1 & \dots & x_n \\ \alpha_0 & \alpha_1 & \dots & \alpha_n \end{bmatrix} f.$$

2. AN IDENTITY FOR THE HERMITE-LAGRANGE POLYNOMIALS

Rename the knots $(\underbrace{x_0,\dots,x_0}_{\alpha_0 \text{ times}},\dots,\underbrace{x_n,\dots,x_n}_{\alpha_n \text{ times}})$ to $(t_0,\dots,t_m)$.

THEOREM 2.1. *The following formula*

$$\begin{aligned} &H_m[t_0,\dots,t_m]\big((X-t_0)\dots(X-t_i)\,f\big) \\ = \;& (X-t_0)\dots(X-t_i)\,H_m[t_{i+1},\dots,t_m]f, \end{aligned} \tag{2.1}$$

[1] cf. J. L. Lagrange, *"Leçons élémentaires sur les mathématiques données a l'École Normale"*, in Oeuvres de Lagrange, J.-A. Serret (ed.), vol. 7, Gauthier-Villars, Paris, 1877, pp. 183–287. Lecture notes first published in 1795.

[2] Also in Oeuvres de Charles Hermite, vol. III, Gauthier-Villars, Paris, 1912, pp. 432–443.

is satisfied for $i = 0, 1, \ldots, m-1$.

Proof. Use the notations:

$$\begin{aligned} P &= H_m \begin{bmatrix} x_0 & x_1 & \ldots & x_n \\ \alpha_0 & \alpha_1 & \ldots & \alpha_n \end{bmatrix} ((X - x_0) f), \\ Q &= (X - x_0) H_m \begin{bmatrix} x_0 & x_1 & \ldots & x_n \\ \alpha_0 - 1 & \alpha_1 & \ldots & \alpha_n \end{bmatrix} f. \end{aligned}$$

It is sufficient to prove that $P = Q$.
For $i = 0$, we have:

$$Q^{(j)}(x_0) = j \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{j-1} \left(H_m \begin{bmatrix} x_0 & x_1 & \ldots & x_n \\ \alpha_0 - 1 & \alpha_1 & \ldots & \alpha_n \end{bmatrix} f \right)\Big|_{x=x_0}$$

$(1 \le j \le \alpha_0 - 1)$ hence

$$\begin{aligned} Q^{(j)}(x_0) &= \begin{cases} 0, & j = 0 \\ j f^{(j-1)}(x_0), & 1 \le j \le \alpha_0 - 1 \end{cases} \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{j} ((x - x_0) f(x))\Big|_{x=x_0} \\ &= P^{(j)}(x_0) \quad (0 \le j \le \alpha_0 - 1). \end{aligned}$$

For $1 \le i \le n$, we obtain:

$$\begin{aligned} Q^{(j)}(x_i) &= (x_i - x_0) \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{j} \left(H_m \begin{bmatrix} x_0 & x_1 & \ldots & x_n \\ \alpha_0 - 1 & \alpha_1 & \ldots & \alpha_n \end{bmatrix} f \right)\Big|_{x=x_i} \\ &\quad + j \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{j-1} \left(H_m \begin{bmatrix} x_0 & x_1 & \ldots & x_n \\ \alpha_0 - 1 & \alpha_1 & \ldots & \alpha_n \end{bmatrix} f \right)\Big|_{x=x_i} \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{j} ((x - x_0) f(x))\Big|_{x=x_i} \\ &= P^{(j)}(x_i) \quad (0 \le j \le \alpha_i - 1). \end{aligned}$$

The degrees of the polynomials P and Q are at most m, and $Q^{(j)}(x_i) = P^{(j)}(x_i)$, $0 \le i \le n$, $0 \le j \le \alpha_i - 1$. Hence $P = Q$, and the proof is completed. □

Identifying the coefficients of X^m in Eq. (2.1) we reobtain the following reduction formula for the divided difference in the case of multiple knots

(2.2) $[t_0, \ldots t_m]\big((X - t_0) \ldots (X - t_i)\, f\big) = [t_{i+1}, \ldots, t_m] f, \quad i = 0, \ldots, m-1.$

3. THE LEIBNIZ-TYPE FORMULA FOR DIVIDED DIFFERENCES

Let f and g be functions defined on an interval containing the points $t_0, \dots, t_n$ not necessarily distinct. The following Leibniz-type formula for divided differences is satisfied

$$[t_0, \dots, t_m](fg) = \sum_{k=0}^{m} [t_0, \dots, t_k]f \; [t_k, \dots, t_n]g. \tag{3.1}$$

Formula (3.1) is generally credited to Steffensen, because of his paper [14]. However our earlist reference for Eq. (3.1) is that of Tiberiu Popoviciu [12, p. 12, Eq. (18)]. In 1940 Tiberiu Popoviciu [13, p. 65] protested: *"Je me suis décidé de revenir sur ces questions en remarquant l'apparition de quelques travaux où mes résultats ne sont pas cités; voir, par exemple,* [14]*"* ("I decided to come back upon these problems, because I have noticed some works, e.g., [14], in which results from my dissertation [12] are not quoted").

On the occasion of Tiberiu Popoviciu's 50^{th} birthday, Academician Miron Nicolescu, G. Pic, D. V. Ionescu, E. Gergely, L. Németi, L. Bal and F. Radó published an overview of Popoviciu's mathematical activity [10]. In this paper the authors reiterated the priority of Tiberiu Popoviciu in what concerns the Leibniz formula for divided differences.

In a recent paper [4], de Boor wrote: *"My first reference for the Leibniz formula is* [12]*, though Steffensen later devotes an entire paper,* [14]*, to it and this has become the standard reference for it despite the fact that Popoviciu, in response, wrote his own overview of divided differences,* [13]*, trying, in vain, to correct the record".*

A new proof of Leibniz's formula is obtained by Tiberiu Popoviciu as an application of his Fundamental Formula for Divided Differences [13, p. 70].

In the case of distinct knots, an interesting proof of (3.1) is given by de Boor [2, p. 5]. De Boor's idea is used by DeVore and Lorentz [6, p. 121] to prove (3.1) in the case of coalescing points, but assuming that the derivatives of f and g, required for the divided difference $[t_0, \dots, t_m]$, are continuous at the corresponding points. A generalization of formula (3.1) is given by de Boor in [3].

4. THE PROOF

We give a proof of the Leibniz-type formula (3.1) in the case of coalescing points. Using the Newton form of the interpolating polynomial (1.1) and the reduction formula (2.2) we have:

$$\begin{aligned}
[t_0, \dots, t_m](fg) =& [t_0, \dots, t_m]\big(H[t_0, \dots, t_m](f)\; g\big) \\
=& [t_0, \dots, t_m]_t\Big(\sum_{k=0}^{m} (t-t_0)\dots(t-t_{k-1})[t_0, \dots, t_k](f)\; g(t)\Big)
\end{aligned}$$

$$= \sum_{k=0}^{m} [t_0, \ldots, t_k] f \; [t_0, \ldots, t_m]_t \big((t - t_0) \ldots (t - t_{k-1}) \, g(t) \big)$$

$$= \sum_{k=0}^{m} [t_0, \ldots, t_k] f \; [t_k, \ldots, t_m] g,$$

and the proof is completed.

REFERENCES

[1] Ampère, A.-M., *Essai sur un nouveau mode d'exposition des principes du calcul différentiel, du calcul aux différences et de l'interpolation des suites, considérées comme dérivant d'une source commune*, Ann. Math. Pures Appl. (Gergonne), **16**, pp. 329–349, 1825.

[2] de Boor, C., *A practical guide to splines*, Springer-Verlag, New York Heidelberg Berlin, 1978.

[3] de Boor, C., *A Leibniz formula for multivariate divided differences*, SIAM J. Numer. Anal., **41** (3), pp. 856–868, 2003.

[4] de Boor, C., *Divided differences*, Surveys in Approximation Theory, **1**, pp. 46–69, 2005.

[5] de Morgan, A., *The Differential and Integral Calculus*, Baldwin & Cradock, London, 1842.

[6] DeVore, R. A. and Lorentz, G. G., *Constructive Approximation*, Springer-Verlag, Berlin Heidelberg New York, 1993.

[7] Hermite, C., *Sur la formule d'interpolation de Lagrange*, J. Reine Angew. Math., **84**, pp. 70–79, 1878.

[8] Meijering, E., *A Chronology of Interpolation: From Ancient Astronomy to Modern Signal and Image Processing.*, Proceedings of the IEEE, **90** (3), pp. 319–342, 2002.

[9] Newton, I., *Philosophiae Naturalis Principia Mathematica*, Printed by Joseph Streater by order of the Royal Society, London, 1687.

[10] Nicolescu, M., Pic, G., Ionescu, D., Gergely, E., Németi, L., Bal, L. and Radó, F., *The mathematical activity of Professor Tiberiu Popoviciu*, Studii şi cerc. de matematică (Cluj), **8** (1–2), pp. 7–19, 1957.

[11] Nörlund, N. E., *Leçons sur les séries d'interpolation*, Gauthier-Villars et C$^{\text{ie}}$, Paris, 1926.

[12] Popoviciu, T., *Sur quelques propriétés des fonctions dŠune ou de deux variables réellesŤ*, Ph.D. thesis, Faculté des Sciences de Paris, 1933, published by Institutul de Arte Grafice "Ardealul" (Cluj, Romania).

[13] Popoviciu, T., *Introduction à la théorie des différences divisées*, Bull. Math. Soc. Roumaine Sci., **42** (1), pp. 65–78, 1940.

[14] Steffensen, J., *Note on divided differences*, Danske Vid. Selsk. Math.-Fys. Medd., **17** (3), pp. 1–12, 1939.

[15] Waring, E., *Problems concerning interpolations*, Philosophical Transactions of the Royal Society of London, **69**, pp. 59–67, 1779.

[16] Whittaker, E. T. and Robinson, G., *The Calculus of Observations*, Blackie & Son, Limited, London; Glasgow; Bombay, 1924.

Received by the editors: March 1, 2006.