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THE EQUIVALENCE BETWEEN THE T-STABILITIES OF MODIFIED MANN-ISHIKAWA AND MANN-ISHIKAWA ITERATIONS*

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Abstract. We show that all T-stabilities of Mann-Ishikawa iterations and modified Mann-Ishikawa iterations are equivalent.

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1. INTRODUCTION

Let X be a normed space and T a selfmap of X. Let x_0 be a point of X, and assume that $x_{n+1} = f(T, x_n)$ is an iteration procedure, involving T, which yields a sequence $\{x_n\}$ of point from X. Suppose $\{x_n\}$ converges to a fixed point x^* of T. Let $\{\xi_n\}$ be an arbitrary sequence in X, and set $\epsilon_n = \|\xi_{n+1} - f(T, \xi_n)\|$ for all $n \in \mathbb{N}$.

Definition 1. [2] *If* $((\lim_{n\to\infty} \epsilon_n = 0) \Rightarrow (\lim_{n\to\infty} \xi_n = p))$, *then the iteration procedure* $x_{n+1} = f(T, x_n)$ *is said to be T-stable with respect to T.*

Remark 1. [2] *In practice, such a sequence $\{\xi_n\}$ could arise in the following way. Let x_0 be a point in X. Set $x_{n+1} = f(T, x_n)$. Let $\xi_0 = x_0$. Now $x_1 = f(T, x_0)$. Because of rounding or discretization in the function T, a new value ξ_1 approximately equal to x_1 might be obtained instead of the true value of $f(T, x_0)$. Then to approximate ξ_2, the value $f(T, \xi_1)$ is computed to yields ξ_2, an approximation of $f(T, \xi_1)$. This computation is continued to obtain $\{\xi_n\}$ an approximate sequence of $\{x_n\}$.*

Consider $e_0 = s_0 = t_0 = g_0 = h_0$, the Mann iteration [3], is defined by

$$e_{n+1} = (1 - \alpha_n)e_n + \alpha_n T e_n, \tag{1}$$

and Ishikawa iteration [1], is defined by

$$\begin{aligned} s_{n+1} &= (1 - \alpha_n)s_n + \alpha_n T t_n, \\ t_n &= (1 - \beta_n)s_n + \beta_n T s_n. \end{aligned} \tag{2}$$

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The modified Ishikawa iteration is given by

$$(3)\qquad \begin{aligned} g_{n+1} &= (1-\alpha_n)g_n + \alpha_n T^n h_n,\\ h_n &= (1-\beta_n)g_n + \beta_n T^n g_n. \end{aligned}$$

Set $\beta_n = 0$ for all $n \in \mathbb{N}$, to obtain the modified Mann iteration.

The sequence $\{\alpha_n\}$ is in $(0,1)$ and $\{\beta_n\} \subset [0,1)$, moreover $\{\alpha_n\}$ satisfies additionally

$$(4)\qquad \lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Recently, the equivalences between the T-stabilities of Mann and Ishikawa iterations respectively for modified Mann-Ishikawa iterations were shown in [4]. We shall prove the equivalence between T-stabilities of modified Mann-Ishikawa and Mann-Ishikawa iterations, unifying all results concerning the equivalence between T-stabilities for such iterations.

2. THE EQUIVALENCE BETWEEN T-STABILITIES

Let X be a normed space and $T: X \to X$ a map. Let $\{c_n\}, \{w_n\}, \{x_n\}, \{u_n\} \subset X$ be such that $c_0 = w_0 = x_0 = u_0$, and

$$\begin{aligned} \xi_n &:= \|c_{n+1} - (1-\alpha_n)c_n - \alpha_n T c_n\|,\\ \theta_n &:= \|w_{n+1} - (1-\alpha_n)w_n - \alpha_n T^n w_n\|. \end{aligned}$$

Consider $\{x_n\}$ and $\{u_n\}$ two given sequences and $\{\varepsilon_n\}$, $\{\delta_n\}$ defined below

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T x_n,\\ v_n &= (1-\beta_n)u_n + \beta_n T^n u_n, \end{aligned}$$

and

$$(5)\qquad \varepsilon_n := \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T y_n\|,$$

$$(6)\qquad \delta_n := \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n v_n\|.$$

Definition 2. *Definition* 1 *for* (5) *and* (6) *gives:*

(i) *The Ishikawa iteration* (2), *is said to be T-stable if and only if for all $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ satisfying* (4), $\forall \{x_n\} \subset X$ *given, we have*

$$(7)\qquad \lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T y_n\| = 0 \Rightarrow \lim_{n\to\infty} x_n = x^*.$$

The Mann iteration is said to be T-stable if and only if for all $\{\alpha_n\} \subset (0,1)$ satisfying (4), $\forall \{c_n\} \subset X$ *given, we have*

$$(8)\qquad \lim_{n\to\infty} \xi_n = \lim_{n\to\infty} \|c_{n+1} - (1-\alpha_n)c_n - \alpha_n T c_n\| = 0 \Rightarrow \lim_{n\to\infty} c_n = x^*.$$

(ii) *The modified Ishikawa iteration* (2), *is said to be T-stable if and only if for all $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ satisfying* (4), $\forall \{u_n\} \subset X$ *given, we have*

$$(9)\qquad \lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n v_n\| = 0 \Rightarrow \lim_{n\to\infty} u_n = x^*.$$

The modified Mann iteration is said to be T-stable if and only if for all $\{\alpha_n\} \subset (0,1)$ satisfying (4) , $\forall \{w_n\} \subset X$ *given, we have*

$$\lim_{n\to\infty} \theta_n = \lim_{n\to\infty} \|w_{n+1} - (1-\alpha_n)w_n - \alpha_n T^n w_n\| = 0 \Rightarrow \lim_{n\to\infty} w_n = x^*. \tag{10}$$

THEOREM 3. *Let X be a normed space and $T : X \to X$ a map with bounded range. Then the following are equivalent:*

(i) *for all $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ satisfying* (4) , *the Ishikawa iteration is T-stable,*

(ii) *for all $\{\alpha_n\} \subset (0,1)$, satisfying* (4) , *the modified Ishikawa iteration is T-stable.*

Proof. Let

$$M := \max\left\{\sup_{x\in X}\{\|T(x)\|\}, \|x_0\|\right\}.$$

Since T has bounded range, one gets $M < \infty$.

We prove that $(i) \Rightarrow (ii)$. Take $\lim_{n\to\infty} \delta_n = 0$ to aim $\lim_{n\to\infty} u_n = x^*$. Then,

$$\begin{aligned}
\varepsilon_n &= \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T y_n\| \\
&\le \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n v_n\| + \|\alpha_n T^n v_n - \alpha_n T y_n\| \\
&\le \delta_n + 2\alpha_n M \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Condition (i) assures that if $\lim_{n\to\infty} \varepsilon_n = 0$, then $\lim_{n\to\infty} u_n = x^*$. Thus, for a $\{u_n\}$ satisfying

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n v_n\| = 0,$$

we have shown that $\lim_{n\to\infty} u_n = x^*$.

Conversely, we prove $(ii) \Rightarrow (i)$. Take $\lim_{n\to\infty} \varepsilon_n = 0$. Eventually, our aim becomes $\lim_{n\to\infty} x_n = x^*$. Observe that

$$\begin{aligned}
\delta_n &= \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n v_n\| \\
&\le \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T y_n\| + \|\alpha_n T y_n - \alpha_n T^n v_n\| \\
&\le \varepsilon_n + 2\alpha_n M \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Condition (ii) assures that if $\lim_{n\to\infty} \delta_n = 0$, then $\lim_{n\to\infty} x_n = x^*$. Thus, for a $\{x_n\}$ satisfying

$$\lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T y_n\| = 0,$$

we have shown that $\lim_{n\to\infty} x_n = x^*$. □

THEOREM 4. [4] *Let X be a normed space and $T : X \to X$ a map with $T(X)$ bounded. Then the following are equivalent:*

(i) *for all $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ satisfying* (4) , *the Ishikawa iteration* (2) *is T-stable,*

(ii) *for all $\{\alpha_n\} \subset (0,1)$, satisfying* (4) , *the Mann iteration* (1) *is T-stable.*

Corollary 5. [4] *Let* X *be a normed space and* $T : X \to X$ *a map with* $T(X)$ *bounded. Then the following are equivalent:*

(i) *for all* $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ *satisfying* (4), *the modified Ishikawa iteration is* T*-stable*,

(ii) *for all* $\{\alpha_n\} \subset (0,1)$, *satisfying* (4), *the modified Mann iteration is* T*-stable.*

Theorem 3, Theorem 4 and Corollary 5 lead to the following result.

Corollary 6. *Let* X *be a normed space and* $T : X \to X$ *a map with* $T(X)$ *bounded. Then the following are equivalent:*

(i) *for all* $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ *satisfying* (4), *the Ishikawa iteration is* T*-stable*,

(i') *for all* $\{\alpha_n\} \subset (0,1)$, *satisfying* (4), *the Mann iteration is* T*-stable*

(ii) *for all* $\{\alpha_n\} \subset (0,1)$, *satisfying* (4), *the modified Ishikawa iteration is* T*-stable*,

(ii') *for all* $\{\alpha_n\} \subset (0,1)$, *satisfying* (4), *the modified Mann iteration is* T*-stable.*

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