REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 35 (2006) no. 2, pp. 225–244
ictp.acad.ro/jnaat

ON THE UNIQUENESS OF THE OPTIMAL SOLUTION IN LINEAR PROGRAMMING

P. SZILÁGYI *

Abstract. In this paper numerous necessary *and* sufficient conditions will be given for a vector to be the unique optimal solution of the primal problem, as well as for that of the dual problem, and even for the case when the primal *and* the dual problem have unique optimal solutions at the same time, respectively, by means of using the strict complementarity and the linear independence constraint qualification. Beyond that, the topological structure of the optimal solutions satisfying the strict complementarity will be determined.

MSC 2000. 90C05, 90C33, 90C46.
Keywords. Linear programming, optimality conditions, constraint qualifications, complementarity, duality, theorems of the alternative.

1. INTRODUCTION

Consider the linear programming (LP) problem of the form

(P) $\{\min c^{\mathrm{T}}x \,:\, x \in \mathbb{R}^n,\ Ax \leq b\}$

and its dual

(D) $\{\max(-b)^{\mathrm{T}}y \,:\, x \in \mathbb{R}^m,\ A^{\mathrm{T}}y = -c,\, y \geq 0\}$.

(Here A is an m by n matrix, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $I = \{1, 2, \ldots, m\} \neq \emptyset$ is the set of row indices of A, $J = \{1, 2, \ldots, n\} \neq \emptyset$ is the set of column indices of A.)

There are a number of trivial and non-trivial examples for sufficient conditions as well as for necessary conditions for the uniqueness of the primal optimal solution, and also for those of the dual optimal solution. For example, it is obvious that if (P) has only one feasible point, it is, of course, an optimal solution, too; and this is also true for (D). Hence, the trivial condition that the corresponding feasible set has one element, is a sufficient condition for the uniqueness of the optimal solution for (P) and (D), respectively. From the strict complementary slackness theorem it follows that — if there is an optimal solution either for (P) or for (D) — there are optimal solutions of (P) and (D) that meet the strict complementary condition (SCC). Thus, this last condition is a necessary condition for the uniqueness of the optimal solution for (P) and (D), respectively.

*Szent István University, Ybl M. Faculty, H-1146 Budapest, Hungary, e-mail: Szilagyi.Peter@ymmfk.szie.hu.

Some other examples. It is well-known that if there is an optimal solution x_0 for (P) and the linear independence constraint qualification (LICQ) is fulfilled at x_0 — i.e. the rows of A corresponding to the active indices at x_0 are linearly independent — then the dual has exactly one optimal solution.

(The set of active indices at a primal feasible solution x_0 is defined as

$$I(x_0) = \{i \in I : a_i^{\mathrm{T}} \text{is a row of } A, a_i^{\mathrm{T}} x_0 = b_i\},$$

while at a dual feasible solution y_0 is defined as $I(y_0) = \{i \in I; y_i^0 = 0\}$. y_i^0 denotes the ith element of vector y_0.) Therefore, satisfaction of LICQ at a primal optimal solution x_0 is a sufficient condition for the uniqueness of the dual optimal solution. On the other hand, it is also known that if the primal feasible set is nonempty, but it has no inner point, then there is no unique dual optimal solution. Hence, the Slater constraint qualification is a necessary condition for the uniqueness of the dual optimal solution.

The above-mentioned conditions are either necessary or sufficient conditions. However, it would be important to get necessary *and* sufficient conditions for the uniqueness

- of the primal optimal solution
- of the dual optimal solution
- of both the primal *and* at the same time the dual optimal solution.

In the paper results of that kind will be presented as a primary purpose, with the help of LICQ and SCC. Although, in [3] some necessary *and* sufficient conditions for the uniqueness of the primal optimal solution have been presented, they are too difficult to apply. Thus, the secondary purpose of the paper is to give easy-to-use conditions for the resolution of the question.

First of all, an assumption will be described. Since in case of $c = 0$, every feasible solution is optimal, and since in case of $c \neq 0$ at every primal optimal solution there is at least one active index in $I(x_0)$, from now on it will be assumed that $I(x_0) \neq \emptyset$.

2. BASIC DEFINITIONS AND THEOREMS

In *nonlinear* programming (NLP), in connection with optimality, the so called *Kuhn-Tucker condition (KTC)* plays a crucial role. It is said that the KTC is fulfilled for (P) at a feasible point x_0 iff
KTC:

$$\{\lambda \in \mathbb{R}^m : \nabla f(x_0) + \sum_{i \in I(x_0)} \lambda_i \nabla g_i(x_0) = 0, \lambda_i \geq 0, i \in I(x_0), \lambda_i = 0, i \notin I(x_0)\} \neq \emptyset$$

This equation-inequality system is called the *KT system.*

(The corresponding NLP problem: $\{\min f(x) : x \in \mathbb{R}^n, g_i(x) \leq 0, i \in I(x_0)\}$.)

Its equivalent dual form:
KTC$_{du}$:

$$\{x \in \mathbb{R}^n : \ \nabla f(x_0)^{\mathrm{T}} x < 0, \ \nabla g_i(x_0)^{\mathrm{T}} x \leq 0, i \in I(x_0)\} = \emptyset$$

(One can get the equivalent form by applying Motzkin's theorem of the alternative, see [2])

From now on, the phrases 'for (P)' and 'at x_0' will be omitted everywhere where their use is unambiguous.

A feasible point x_0 at which KTC is fulfilled is called a *KT point (KTP)*, a vector λ satisfying KTC is called a *KT vector (KTV)*. (Sometimes it is called a *multiplier vector*.) It is said that at a feasible point x_0 *the strict complementary condition in connection with KTC (KT-SCC)* is fulfilled iff there is a KTV with $\lambda_i > 0$ for all $i \in I(x_0)$. A feasible point x_0 at which KT-SCC is fulfilled is called a *KT-SC point* (KT-SCP), a vector λ satisfying the KT-SCC is called a *KT-SC vector (KT-SCV)*.

In *linear* programming KTC and KTC$_{du}$ have the following forms.
KTC:

$$\{y \in \mathbb{R}^m : \sum_{i \in I(x_0)} a_i y_i = -c, y_i \geq 0, i \in I(x_0), y_i = 0, i \notin I(x_0)\} \neq \emptyset$$

KTC$_{du}$:

$$\{x \in \mathbb{R}^n : c^{\mathrm{T}} x < 0, \ a_i^{\mathrm{T}} x \leq 0, i \in I(x_0)\} = \emptyset$$

In LP the concept of strict complementarity is defined as follows. *An optimal solution x_0 is said to satisfy the strict complementary condition (OPT-SCC)* iff there is a dual optimal solution y_0 for which:

$$(b - Ax_0)^{\mathrm{T}} y_0 = 0 \text{ and } b - Ax_0 + y_0 > 0.$$

A primal feasible point x_0 at which OPT-SCC is fulfilled is called an *OPT-SC point (OPT-SCP)*, a dual feasible point y_0 satisfying OPT-SCC at some x_0 primal feasible point is called an *OPT-SC vector (OPT-SCV)*.

In case of linear programming the above two kinds of strict complementarity coincide. Namely, the following assertions are true. (They are well-known and can be found in many linear programming textbooks. Therefore, their proofs are omitted from the paper. All of the following propositions pertain to linear programming.)

THEOREM 2.1. *A primal feasible point x_0 is an optimal solution for (P) iff it is a KTP.*

THEOREM 2.2. *A dual feasible point y_0 is an optimal solution for (D) iff it is a KTV for some primal optimal solution x_0.*

THEOREM 2.3. *For every primal optimal solution the set of KTVs is the same.*

Corollary 2.4. *The set of all dual optimal solutions is equal to the set of all KTVs.*

Corollary 2.5. *The set of all OPT-SCPs is equal to the set of all KT-SCPs.*

Corollary 2.6. *The set of all OPT-SCVs is equal to the set of all KT-SCVs.*

Hence, OPT-SCP and KT-SCP are the same concepts, and the same is true for OPT-SCV and KT-SCV. Therefore, from now on it will be written simply SCP and SCV.

Corollary 2.7. *The set of all SCPs for (P) is equal to the set of all SCVs for (D) and vice versa.*

The first one of the following two theorems is called the *weak complementary slackness theorem*, while the second one is called the *strict complementary slackness theorem*.

Theorem 2.8. *A primal feasible point x_0 and a dual feasible point y_0 are optimal solutions for (P) and (D), respectively iff they satisfy the condition $(b - Ax_0)^{\mathrm{T}} y_0 = 0$.*

Theorem 2.9. *Let (P) or (D) has an optimal solution. Then the other has an optimal solution, too, and there are a primal optimal point x_0 and a dual optimal point y_0 with $b - Ax_0 + y_0 > 0$. (Hence, x_0 is a SCP and y_0 is a SCV.)*

3. TOPOLOGICAL STRUCTURE OF THE POINTS SATISFYING STRICT COMPLEMENTARITY

In this section the topological structure of SCPs and SCVs will be investigated within the optimal solution set of (P) and (D), respectively. First, the support of a feasible solution will be defined.

The *support of a primal feasible solution* x_0: $\operatorname{supp}(x_0) = \{i \in I : i \notin I(x_0)\}$, while the *support of a dual feasible solution* y_0: $\operatorname{supp}(y_0) = \{i \in I : i \notin I(y_0)\}$.

The next two theorems immediately follow from the strict complementary slackness theorem.

Theorem 3.1. *An optimal solution of (P) is a SCP iff it is an optimal point for (P) of maximal support. An optimal solution of (D) is a SCV iff it is an optimal point for (D) of maximal support.*

(Maximality means that the given support is a set of maximal number of elements among the supports in question.)

The following propositions show the structure of SCPs and SCVs.

Theorem 3.2. *Let (P) or (D) has an optimal solution. Then there is exactly one maximal support among the supports belong to the primal optimal solutions, and there is exactly one maximal support among the supports belong to the dual optimal solutions.*

Proof. The theorem follows from the fact that the optimal sets of (P) and (D) are convex sets, and if e.g. x_1 and x_2 are two different primal optimal solutions of maximal support, then every inner point x_0 of the line segment between x_1 and x_2 is an optimal solution with $\text{supp}(x_0) = \text{supp}(x_1) \cup \text{supp}(x_2)$. □

COROLLARY 3.3. *SCPs are identical with the relative interior points of the primal optimal solution set (if there is any primal optimal solution) and SCVs are identical with the relative interior points of the dual optimal solution set (if there is any dual optimal solution).*

COROLLARY 3.4. *If there is only one optimal solution of (P), it is a SCP. If there is only one optimal solution of (D), it is a SCV.*

COROLLARY 3.5. *Every optimal solution of (P) is a SCP iff the set of optimal solutions of (P) is nonempty and relatively open. Every optimal solution of (D) is a SCV iff the set of optimal solutions of (D) is nonempty and relatively open.*

4. UNIQUENESS OF THE PRIMAL OPTIMAL SOLUTION

First, two conditions will be defined that are modifications of KTC. The first one is called the *strict KTC (STKTC)*, the second one is called the *weak strict KTC (WSTKTC)*. (Here and from now on in the paper only the equivalent dual forms of the conditions will be described. The index "du" will be omitted.)

STKTC:

$$\{x \in \mathbb{R}^n : c^{\mathrm{T}}x \leq 0,\ a_i^{\mathrm{T}}x \leq 0, i \in I(x_0)\} = \{0\}$$

WSTKTC:

$$\{x \in \mathbb{R}^n : c^{\mathrm{T}}x = 0,\ a_i^{\mathrm{T}}x \leq 0, i \in I(x_0)\} = \{0\}$$

(The name WSTKTC can be explained by the dual form. It can be thought that first KTC_{du} had been strengthened to STKTC, and then, by a weakening of STKTC, namely, substituting $c^{\mathrm{T}}x \leq 0$ by $c^{\mathrm{T}}x = 0$ was done to get WSTKTC.)

By means of these two conditions it is possible to give necessary *and* sufficient conditions for the uniqueness of the primal optimal solution.

THEOREM 4.1. *A primal feasible point x_0 is the only optimal solution of (P) iff STKTC is satisfied at x_0.*

Proof. 1) *Necessity.* Let x_0 be the only optimal solution of (P). Then: $\{x \in \mathbb{R}^n : c^{\mathrm{T}}x \leq c^{\mathrm{T}}x_0, a_i^{\mathrm{T}}x \leq b_i, i \in I\} = \{x_0\}$, $a_i^{\mathrm{T}}x_0 = b_i, i \in I(x_0)$ and $a_i^{\mathrm{T}}x_0 < b_i, i \notin I(x_0)$. On the contrary, assume that STKTC is not fulfilled at x_0. Then there exists a vector $z \neq 0$ for which $c^{\mathrm{T}}z \leq 0, a_i^{\mathrm{T}}z \leq 0, i \in I(x_0)$. One can choose z in such a way that $a_i^{\mathrm{T}}z < b_i - a_i^{\mathrm{T}}x_0$ could also be satisfied for all $i \notin I(x_0)$. Then the vector $x_0 + z$ is also an optimal solution for (P) different from x_0, what is a contradiction to the assumption.

2) *Sufficiency.* Let x_0 be a primal feasible point and let STKTC be satisfied at x_0. On the contrary, assume that it is not true that x_0 is the only optimal solution for (P). Then there exists a primal feasible solution $x \neq x_0$ for which $c^{\mathrm{T}}x \leq c^{\mathrm{T}}x_0$, $a_i^{\mathrm{T}}x \leq b_i, i \in I$. Let $z = x - x_0$. Then $z \neq 0$ and $c^{\mathrm{T}}z \leq 0$, $a_i^{\mathrm{T}}z \leq 0, i \in I(x_0)$, and this is a contradiction to the satisfaction of STKTC at x_0. □

Remark 4.2. Theorem 4.1 is slightly different from the equivalence of (i) and (iii) of Theorem 2 in [3], since from Theorem 2 — writing the O matrix instead of A and −C instead of C in Theorem 2 — one can have that if x_0 is a primal *optimal* solution, then x_0 is the unique optimal solution of (P) iff STKTC is fulfilled. Since the preassumption in Theorem 4.1 is weaker than the one in Theorem 2, Theorem 4.1 had to be proved.

Theorem 4.3. *A primal optimal solution x_0 is the only optimal solution of (P) iff WSTKTC is satisfied at x_0.*

Proof. It is similar to the proof of Theorem 4.1. □

The following two theorems are a little bit stronger than the equivalence of (i) and (iv), as well as that of (i) and (x) of Theorem 2 in [3]. Denote by $I(x_0, y_0)$ the set $I(x_0, y_0) = I(x_0) \cap I(y_0)$.

Theorem 4.4. *Let x_0 be the unique primal optimal solution. Then for every dual optimal solution y_0:*
$\{x \in \mathbb{R}^n : a_i^{\mathrm{T}}x = 0, i \in \mathrm{supp}(y_0),\ a_i^{\mathrm{T}}x \leq 0, i \in I(x_0, y_0)\} = \{0\}$.

Let x_0 be a primal optimal solution. Assume that there exists a dual optimal solution y_0 for which:
$\{x \in \mathbb{R}^n : a_i^{\mathrm{T}}x = 0, i \in \mathrm{supp}(y_0),\ a_i^{\mathrm{T}}x \leq 0, i \in I(x_0, y_0)\} = \{0\}$. *Then x_0 is the unique primal optimal solution.*

Proof. It can be proved similarly as it was done in the proof in [3]. □

Introduce the following *notation*. Let A_K be the sub-matrix of A consisting of those rows of A indices of which belong to the index set K, where $K \subseteq I$. Denote by p the cardinality of $I(x_0)$, i.e. let $p = \mathrm{card}(I(x_0))$.

Theorem 4.5. *Let x_0 be the unique primal optimal solution. Then the columns of the matrix $A_{I(x_0)}$ are linearly independent, and for every dual optimal solution y_0: $\{v \in \mathbb{R}^p : \sum_{i \in I(x_0)} v_i a_i = 0, v_i > 0, i \in I(x_0, y_0)\} \neq \emptyset$.*

Let x_0 be a primal optimal solution. Assume that the columns of the matrix $A_{I(x_0)}$ are linearly independent, and there exists a dual optimal solution y_0 for which: $\{v \in \mathbb{R}^p : \sum_{i \in I(x_0)} v_i a_i = 0, v_i > 0, i \in I(x_0, y_0)\} \neq \emptyset$. Then x_0 is the unique primal optimal solution.

Proof. For the condition in Theorem 4.4:
$\{x \in \mathbb{R}^n : a_i^{\mathrm{T}}x = 0, i \in \mathrm{supp}(y_0),\ a_i^{\mathrm{T}}x \leq 0, i \in I(x_0, y_0)\} = \{0\} \quad \Leftrightarrow$

$\Leftrightarrow$ $(\{x \in \mathbb{R}^n : a_i^{\mathrm{T}} x = 0, i \in I(x_0)\} = \{0\}$ and

$\{x \in \mathbb{R}^n : a_i^{\mathrm{T}} x = 0, i \in \operatorname{supp}(y_0),\ a_i^{\mathrm{T}} x \leq 0, i \in I(x_0, y_0), \sum_{i \in I(x_0, y_0)} a_i^{\mathrm{T}} x < 0\} = \emptyset)$

$\Leftrightarrow$ the columns of the matrix $A_{I(x_0)}$are linearly independent, and

$$\{v \in \mathbb{R}^p : \sum_{i \in I(x_0)} v_i a_i = 0, v_i > 0, i \in I(x_0, y_0)\} \neq \emptyset$$

(The last equivalence has been got by Tucker's theorem of the alternative, see [2].) □

Now, it will be assumed that x_0 is a vertex of the primal feasible set. A primal feasible point x_0 is said to be a *vertex of the primal feasible set* iff there is no line segment belonging to the feasible set and which contains x_0 as an inner point. It is well-known that x_0 is a vertex of the primal feasible set iff x_0 is a primal feasible point and there are $i_1, i_2, ..., i_n$ different active indices in $I(x_0)$ for which vectors $a_{i_1}, a_{i_2}, ..., a_{i_n}$ are linearly independent. (Vector $a_{i_j}^{\mathrm{T}}$ is the row of A that belongs to the index $i_j, j = 1, 2, ..., n$.) It is also widely known that if there is an optimal solution and the set of feasible solutions has a vertex, then there is an optimal solution that is a vertex.

First, three conditions, the *weak KTC (WKTC)*, the *strict weak KTC (STWKTC)* and the *weak strict weak KTC (WSTWKTC)* will be introduced:

WKTC:

$$\{x \in \mathbb{R}^n : c^{\mathrm{T}} x < 0,\ a_i^{\mathrm{T}} x \leq 0, i \in I(x_0), \sum_{i \in I(x_0)} a_i^{\mathrm{T}} x < 0\} = \emptyset$$

STWKTC:

$$\{x \in \mathbb{R}^n : c^{\mathrm{T}} x \leq 0,\ a_i^{\mathrm{T}} x \leq 0, i \in I(x_0), \sum_{i \in I(x_0)} a_i^{\mathrm{T}} x < 0\} = \emptyset$$

WSTKTC:

$$\{x \in \mathbb{R}^n : c^{\mathrm{T}} x = 0,\ a_i^{\mathrm{T}} x \leq 0, i \in I(x_0), \sum_{i \in I(x_0)} a_i^{\mathrm{T}} x < 0\} = \emptyset$$

(The names WKTC, STWKTC and WSTWKTC can be explained in the following way. It can be thought that first KTC was weakened to the condition WKTC, then by a strengthening of WKTC, namely, substituting $c^{\mathrm{T}} x < 0$ by $c^{\mathrm{T}} x \leq 0$ was done to get STWKTC. Finally, by weakening this last condition by substituting $c^{\mathrm{T}} x \leq 0$ by $c^{\mathrm{T}} x = 0$, one can get WSTWKTC.)

Theorem 4.6. *Let x_0 be a vertex of the set of primal feasible solutions. Then it is an optimal solution of (P) iff WKTC is satisfied at x_0.*

Proof. A feasible point x_0 is an optimal solution of (P) iff it is a KTP, i.e.: x_0 is an optimal solution of (P) $\Leftrightarrow$

$$\Leftrightarrow \ \{x \in \mathbb{R}^n : c^{\mathrm{T}}x < 0,\ a_i^{\mathrm{T}}x \le 0, i \in I(x_0)\} = \emptyset$$

$$\Leftrightarrow \ \{x \in \mathbb{R}^n : c^{\mathrm{T}}x < 0,\ a_i^{\mathrm{T}}x = 0, i \in I(x_0)\} = \emptyset \text{ and}$$

$$\{x \in \mathbb{R}^n : c^{\mathrm{T}}x < 0,\ a_i^{\mathrm{T}}x \le 0, i \in I(x_0), \sum_{i \in I(x_0)} a_i^{\mathrm{T}}x < 0\} = \emptyset$$

Since x_0 is a vertex of the primal feasible set, there are $i_1, i_2, ..., i_n$ different active indices in $I(x_0)$ for which the vectors $a_{i_1}, a_{i_2}, ..., a_{i_n}$ are linearly independent, and hence $\{x \in \mathbb{R}^n : a_{i_1}^{\mathrm{T}}x = 0, a_{i_2}^{\mathrm{T}}x = 0, ..., a_{i_n}^{\mathrm{T}}x = 0\} = \{0\}$.

Thus $\{x \in \mathbb{R}^n :\ a_i^{\mathrm{T}}x = 0, i \in I(x_0)\} = \{0\}$.

Therefore $\{x \in \mathbb{R}^n : c^{\mathrm{T}}x < 0,\ a_i^{\mathrm{T}}x = 0, i \in I(x_0)\} = \emptyset$, and hence the optimality of x_0 is equivalent to the satisfaction of WKTC. □

Theorem 4.7. *A primal feasible point x_0 is the only optimal solution of (P) iff it is a vertex of the primal feasible set and STWKTC is satisfied at x_0.*

Proof. By Theorem 4.1, a primal feasible point x_0 is the only optimal solution of (P) iff $\{x \in \mathbb{R}^n : c^{\mathrm{T}}x \le 0,\ a_i^{\mathrm{T}}x \le 0, i \in I(x_0)\} = \{0\}$.

But the last condition is equivalent to
$\{x \in \mathbb{R}^n : c^{\mathrm{T}}x \le 0,\ a_i^{\mathrm{T}}x = 0, i \in I(x_0)\} = \{0\}$ and

$$\{x \in \mathbb{R}^n : c^{\mathrm{T}}x \le 0,\ a_i^{\mathrm{T}}x \le 0, i \in I(x_0), \sum_{i \in I(x_0)} a_i^{\mathrm{T}}x < 0\} = \emptyset$$

1)*Sufficiency.* Let x_0 be a vertex of the primal feasible set. Then — as it was proved in the proof of Theorem 4.6 — $\{x \in \mathbb{R}^n :\ a_i^{\mathrm{T}}x = 0, i \in I(x_0)\} = \{0\}$.

Therefore $\{x \in \mathbb{R}^n : c^{\mathrm{T}}x \le 0,\ a_i^{\mathrm{T}}x = 0, i \in I(x_0)\} = \{0\}$. This is the first condition described in the beginning of the proof; and since STWKTC is just the second condition, x_0 is the only optimal solution of (P) and sufficiency has been proved.

2)*Necessity.* Assume that x_0 is the only optimal solution of (P). By Corollary 3.4 it is a SCP. By Corollary 3.5 the set of optimal solutions of (P) is relatively open. It could be in the only possible way that x_0 is a vertex. The satisfaction of STWKTC follows from the equivalent relations described in the beginning of the proof. □

Theorem 4.8. *A primal optimal solution x_0 is the only optimal solution of (P) iff it is a vertex of the primal feasible set and WSTWKTC is satisfied at x_0.*

Proof. By Theorem 4.3, a primal optimal solution x_0 is the only optimal solution of (P) iff $\{x \in \mathbb{R}^n : c^{\mathrm{T}}x = 0,\ a_i^{\mathrm{T}}x \le 0, i \in I(x_0)\} = \{0\}$. The remaining part of the proof is the same as that of the proof of Theorem 4.7, with the exception that $c^{\mathrm{T}}x = 0$ has to be written instead of $c^{\mathrm{T}}x \le 0$ and Theorem 4.3 has to be applied in the proof instead of Theorem 4.1. □

The following theorem shows the connection between strict complementarity and WSTWKTC.

THEOREM 4.9. *Let x_0 be a primal optimal solution. Then x_0 is a SCP iff WSTWKTC is satisfied at x_0.*

Proof. Since x_0 is a primal optimal solution, it is a KTP, too. x_0 is a SCP

$$\Leftrightarrow \{\lambda \in \mathbb{R}^m : c + \sum_{i\in I(x_0)} \lambda_i a_i = 0, \lambda_i > 0, i \in I(x_0), \lambda_i = 0, i \notin I(x_0)\} \neq \emptyset$$

(by Tucker's theorem of the alternative, [2])

$$\Leftrightarrow \{x \in \mathbb{R}^n : c^{\mathrm{T}} x \leq 0,\ a_i^{\mathrm{T}} x \leq 0, i \in I(x_0), c^{\mathrm{T}} x + \sum_{i\in I(x_0)} a_i^{\mathrm{T}} x < 0\} = \emptyset$$

$$\Leftrightarrow \{x \in \mathbb{R}^n : c^{\mathrm{T}} x < 0,\ a_i^{\mathrm{T}} x \leq 0, i \in I(x_0)\} = \emptyset \text{ and}$$
$$\{x \in \mathbb{R}^n : c^{\mathrm{T}} x = 0,\ a_i^{\mathrm{T}} x \leq 0, i \in I(x_0), \sum_{i\in I(x_0)} a_i^{\mathrm{T}} x < 0\} = \emptyset$$

$\Leftrightarrow$KTC and WSTWKTC.

Since x_0 is a KTP, KTC is fulfilled. Hence, x_0 is a SCP iff WSTWKTC is satisfied at x_0. □

COROLLARY 4.10. *A primal optimal solution x_0 is the only optimal solution of (P) iff it is a vertex of the primal feasible set and x_0 is a SCP.*

Proof. It comes immediately from Theorems 4.8 and 4.9. □

5. UNIQUENESS OF THE DUAL OPTIMAL SOLUTION

First of all, the *Kyparisis' regularity condition (KYPRC) for NLP* will be introduced (see [1]).

KYPRC: At x_0 there is a KTV λ for which – denoting by $I_{NN}(\lambda(x_0))$ the set of indices of its zero components within $I(x_0)$ – it is true that

$$\{v \in \mathbb{R}^m : \sum_{i\in I(x_0)} v_i \nabla g_i(x_0) = 0, v_i \geq 0, i \in I_{NN}(\lambda(x_0))\} = \{0\}.$$

In nonlinear programming the following well-known assertion is true: There exists a unique KTV at x_0 iff KYPRC is fulfilled at x_0 (see [1]). The result can be applied also for LP.

Since in case of LP, by Theorem 2.1 the primal optimal solutions and KTPs are identical, and by Corollary 2.4 the dual optimal solutions and KTVs are identical, and, furthermore, by Theorem 2.3 for every primal optimal solution the set of KTVs is the same, *in case of LP KYPRC has the following form*, and the above assertion about the uniqueness of the KTV can be reformulated in a slightly stronger form in Theorem 5.1 as follows.

KYPRC: At a given primal optimal solution x_0 there is a dual optimal solution y_0 such that

$$\{v \in \mathbb{R}^m : \sum_{i \in I(x_0)} v_i a_i = 0, v_i \geq 0, i \in I(x_0, y_0)\} = \{0\}$$

or equivalently, doing the same as in the proof of Theorem 4.5, but making use of Motzkin's theorem of the alternative instead of Tucker's theorem of the alternative:

KYPRC$_{du}$: At a given primal optimal solution x_0 there is a dual optimal solution y_0 such that

$$\{x \in \mathbb{R}^n : a_i^{\mathrm{T}} x = 0, i \in \operatorname{supp}(y_0), a_i^{\mathrm{T}} x > 0, i \in I(x_0, y_0)\} \neq \emptyset$$

is satisfied and vectors $a_i, i \in \operatorname{supp}(y_0)$ are linearly independent.

Theorem 5.1. *If there are a primal optimal solution x_0 and a dual optimal solution y_0 such that KYPRC is satisfied, then y_0 is the only dual optimal solution.*

If y_0 is the only dual optimal solution, then for y_0 and every primal optimal solution x_0 KYPRC is satisfied.

Corollary 5.2. *If at some SCP x_0 LICQ is satisfied, then there exists a unique dual optimal solution y_0.*

If y_0 is the only dual optimal solution, then at every SCP x_0 LICQ is satisfied.

Proof. In the case when x_0 is a SCP, applying the strict complementary slackness theorem, there exists a dual optimal solution y_0 that is a SCV at x_0. $I(x_0, y_0) = \emptyset$, and hence $I(x_0) = \operatorname{supp}(y_0)$. Therefore, condition $\{x \in \mathbb{R}^n : a_i^{\mathrm{T}} x = 0, i \in \operatorname{supp}(y_0), a_i^{\mathrm{T}} x > 0, i \in I(x_0, y_0)\} \neq \emptyset$ in KYPRC becomes the condition $\{x \in \mathbb{R}^n : a_i^{\mathrm{T}} x = 0, i \in I(x_0)\} \neq \emptyset$. But, this is always true, since $0 \in \{x \in \mathbb{R}^n : a_i^{\mathrm{T}} x = 0, i \in I(x_0)\}$. Hence, in the case when x_0 is a SCP KYPRC is satisfied at x_0 iff LICQ is satisfied at x_0. □

Theorem 5.3. *If there is a unique dual optimal solution, then LICQ is satisfied at every primal optimal solution.*

If there is a primal optimal solution in which LICQ is satisfied, then there is a unique dual optimal solution.

Proof. 1) Assume first that there is a unique dual optimal solution. According to the duality theorem (or to Theorem 2.9) there exists a primal optimal solution. Take an arbitrary primal optimal solution x_0. By Theorem 2.1, it is a KTP, and the KT system belonging to x_0 has a unique solution, since there is a unique dual optimal solution. From this, it follows that system $\{\sum_{i \in I(x_0)} \lambda_i a_i = -c\}$ has a unique solution. Hence vectors $a_i, i \in I(x_0)$ are linearly independent.

2) Assume now that there is a primal optimal solution in which LICQ is satisfied. By Theorem 2.1, it is a KTP, and the KT system that belongs to x_0

has a solution, i.e. system $\{\sum_{i\in I(x_0)} \lambda_i a_i = -c, \lambda_i \geq 0, i \in I(x_0)\}$ has a solution. But, as LICQ is satisfied at x_0, it has a unique solution, i.e. there is a unique KTV belonging to x_0. Hence, by Theorems 2.3 and 2.2 there is a unique dual optimal solution. □

COROLLARY 5.4. *If there is a primal optimal solution at which LICQ is satisfied, then LICQ is satisfied at every primal optimal solution.*

Proof. It follows from Theorem 5.3. □

COROLLARY 5.5. *In case of linear programming, KYPRC and LICQ are equivalent, assuming that they are investigated at a primal optimal solution.*

Proof. Making use of Theorem 5.3 and Corollary 2.4 and taking into account that there is a unique KTV at x_0 iff KYPRC is satisfied at x_0, the proof will be complete. □

THEOREM 5.6. *Let x_0 be a primal optimal solution. If x_0 is a SCP for which SCC is satisfied with every dual optimal solution, then LICQ is fulfilled at x_0.*

Proof. According to the assumption, there is a dual optimal solution, and every dual optimal solution is a SCV. Hence, by Corollary 3.5, the set of dual optimal solutions is nonempty and relatively open. But it is well-known that an LP problem in the form of $\{\max(-b)^{\mathrm{T}}y : y \in \mathbb{R}^m, A^{\mathrm{T}}y = -c, y \geq 0\}$ always has a feasible solution that is a vertex, provided it has a feasible solution. Therefore (see in Section 4), there is a dual optimal solution that is a vertex. Thus, the set of dual optimal solutions can not be relatively open, except for the case when it consists only of one point. In such a way, there is a unique dual optimal solution, and applying Corollary 5.2, it follows that LICQ is satisfied a x_0. □

THEOREM 5.7. *The conversion of Theorem* 5.3 *is not true, namely from the fact that at a primal optimal solution x_0 LICQ is satisfied, it does not follow that every dual optimal solution y_0 is a SCV that belongs to x_0.*

Proof. Consider the following counterexample.
(P) $\{\min(-x_1 - x_2) :\ x_1 \leq 0, x_2 \leq 0, x_3 \leq 0\}$.

Of course, $x_0 = 0 \in \mathbb{R}^3$ is an optimal solution for (P). $I(x_0) = \{1, 2, 3\}$ and LICQ is satisfied at x_0. The only dual feasible point, and hence the only dual optimal solution is $y_0^{\mathrm{T}} = (1, 1, 0)$. But with x_0 and y_0 SCC is not fulfilled, since $3 \in I(x_0)$ and at the same time the third component of y_0 is zero. (Nevertheless, $z_0^{\mathrm{T}} = (0, 0, 1)$ is another primal optimal solution with which y_0 satisfies SCC, and LICQ is satisfied at z_0.) □

6. UNIQUENESS OF THE PRIMAL *AND* THE DUAL OPTIMAL SOLUTIONS

In the preceding two sections necessary *and* sufficient conditions have been shown for the uniqueness of the primal optimal solution, as well as for the dual optimal solution. In this section the two questions will be investigated simultaneously.

The first theorem uses the concepts of the vertex, SCP and LICQ.

Theorem 6.1. *If there is a unique primal optimal solution* x_0 *and at the same time there is a unique dual optimal solution, then for* x_0 *the following conditions are satisfied:*

- x_0 *is a vertex*
- x_0 *is a SCP*
- *LICQ is satisfied at* x_0.

If there is a primal optimal solution x_0 *for which the above three properties are satisfied, then* x_0 *is the only primal optimal solution and at the same time there is a unique dual optimal solution.*

Proof. Applying Corollary 4.10 and Theorem 5.3, Theorem 6.1 will be proved. □

The following theorem gives a connection between LICQ and the cardinality of $I(x_0)$.

Theorem 6.2. *Let* x_0 *be a vertex of the set of primal feasible solutions. Then LICQ is satisfied at* x_0 *iff the cardinality of* $I(x_0)$ *is equal to* n, *i.e.* $\text{card}(I(x_0)) = n$.

Proof. Since x_0 is a vertex, there are $i_1, i_2, ..., i_n$ different active indices in $I(x_0)$ for which vectors $a_{i_1}, a_{i_2}, ..., a_{i_n}$ are linearly independent (see Section 4). Thus $\{i_j, j = 1, 2, \ldots, n\} \subseteq I(x_0)$.

1) *Necessity.* On the contrary, assume that $\text{card}(I(x_0)) \neq n$. Then $\text{card}(I(x_0)) > n$. But, by LICQ it would be more than n linear independent vector in $\mathbb{R}^n$. It is a contradiction.

2) *Sufficiency.* If $\text{card}(I(x_0)) = n$, then $\{i_j, j = 1, 2, \ldots, n\} = I(x_0)$, and since vectors $\{a_{i_j}, j = 1, 2, \ldots, n\}$ are linearly independent, LICQ is fulfilled at x_0. □

In the following two propositions it is assumed that there is a primal optimal solution.

Theorem 6.3. *If there is a unique primal optimal solution* x_0 *and at the same time there is a unique dual optimal solution, then for* x_0 *the following conditions are satisfied:*

- x_0 *is a vertex*
- $(A^{\mathrm{T}}_{I(x_0)})^{-1} c < 0$
- *LICQ is satisfied at* x_0

If there is a primal optimal solution x_0 *for which the above three properties are satisfied, then* x_0 *is the only primal optimal solution and at the same time there is a unique dual optimal solution.*

(For notation $A^{\mathrm{T}}_{I(x_0)}$ see the definition of A_K before Theorem 4.5.)

Proof. 1) *Sufficiency.* By Theorem 6.1, it is enough to prove only that x_0 is a SCP.

As x_0 is a vertex and LICQ is satisfied at x_0, by Theorem 6.2, $\operatorname{card}(I(x_0)) = n$. Thus, matrix $A^{\mathrm{T}}_{I(x_0)}$ is a quadratic one and has an inverse. Let vector $y_0 \in \mathbb{R}^m$ be defined as follows: let y^0_i be the ith component of the vector $-(A^{\mathrm{T}}_{I(x_0)})^{-1}c$ if $i \in I(x_0)$ and let $y^0_i = 0$ if $i \notin I(x_0)$. Then, by the assumption: $-(A^{\mathrm{T}}_{I(x_0)})^{-1}c > 0$. Thus $y_0 \geq 0$. On the other hand: $A^{\mathrm{T}}y_0 = A^{\mathrm{T}}_{I(x_0)}y^0_{I(x_0)} + A^{\mathrm{T}}_{\mathrm{supp}(x_0)}y^0_{\mathrm{supp}(x_0)} = -c$, as $y^0_{\mathrm{supp}(x_0)} = 0$. Hence, y_0 is a dual feasible solution. Then, by the construction of y_0, for x_0 and y_0 SCC is fulfilled. Therefore, by the complementary slackness theorem, y_0 is a dual optimal solution and x_0 is a SCP.

2) *Necessity.* By Theorem 6.1, x_0 is a vertex and LICQ is satisfied at x_0. Then, by Theorem 6.2, matrix $A^{\mathrm{T}}_{I(x_0)}$ is a quadratic one and has an inverse. Then, since y_0 is a SCV with the SCP x_0: $-(A^{\mathrm{T}}_{I(x_0)})^{-1}c > 0$ is fulfilled. □

The following two corollaries follow immediately from Theorems 6.2 and 6.3.

Corollary 6.4. *If there is a unique primal optimal solution x_0 and at the same time a unique dual optimal solution, then for x_0 the following conditions are satisfied:*

- *x_0 is a vertex*
- $(A^{\mathrm{T}}_{I(x_0)})^{-1}c < 0$
- *$I(x_0)$ has exactly n elements*

If there is a primal optimal solution x_0 for which the above three properties are satisfied, then x_0 is the only primal optimal solution and at the same time there is a unique dual optimal solution.

Corollary 6.5. *If there is a unique primal optimal solution x_0 and at the same time a unique dual optimal solution, then for x_0 the following conditions are satisfied:*

- *x_0 is a vertex*
- *x_0 is a SCP*
- *$I(x_0)$ has exactly n elements*

If there is a primal optimal solution x_0 for which the above three properties are satisfied, then x_0 is the only primal optimal solution and at the same time there is a unique dual optimal solution.

Remark 6.6. There is apparently a contradiction in the case when both (P) and (D) have exactly one optimal solution each. (It is easy to create such an LP problem.) Namely, the conditions mentioned in Theorems 4.5 and 5.1, i.e. $\{v \in \mathbb{R}^p : \sum_{i\in I(x_0)} v_i a_i = 0, v_i > 0, i \in I(x_0, y_0)\} \neq \emptyset$ and

$\{v \in \mathbb{R}^m : \sum_{i \in I(x_0)} v_i a_i = 0, v_i \geq 0, i \in I(x_0, y_0)\} = \{0\}$ seem to be contradictory, since an apparently wider set is empty, while its subset is nonempty. However, if $I(x_0, y_0) = \emptyset$, there will be no contradiction. And now that is the case.

7. UNIQUENESS OF THE OPTIMAL SOLUTIONS IN THE CASE OF THE SIMPLEX METHOD

In this section conditions and theorems that are pendants of the ones occurring in the preceding sections will be given for the optimal solution(s) to be unique when the primal problem is given in the form for which the primal simplex method can be applied. Thus, in this case, the LP problem is considered to be in the form:

(P') $\{\max\ c^{\mathrm{T}}x\ :\ x \in \mathbb{R}^n,\ Ax = b, x \geq 0\}$
and its dual
(D') $\{\min\ b^{\mathrm{T}}y\ :\ y \in \mathbb{R}^m,\ A^{\mathrm{T}}y \geq c\}$.

It is well-known that if there is an optimal solution for (P'), then (a proper variant of) the simplex method results an optimal solution for (P') that is a vertex of the feasible set of (P').

First, the pendants of the results got in the preceding sections for the pair of (P) and (D) will be given for the pair of (P') and (D'), and after that these new results will be specialized for the special optimal solutions got by the simplex method. (These last results are mostly widely known.)

First, the pendants of some definitions occurred in the preceding sections will be described. Let x_0 be a feasible solution for (P') and y_0 for (D'). For problem pair (P') and (D'):

$J(x_0) = \{j \in J : x_j^0 = 0\}, J(y_0) = \{j \in J : (A^{\mathrm{T}}y_0)_j = c_j\}$, where x_j^0 and $(A^{\mathrm{T}}y_0)_j$ denote the jth elements of vectors x_0 and $A^{\mathrm{T}}y_0$, respectively.

LICQ: vectors $a_i, i \in I$ and $e_j, j \in J(x_0)$ are (together) linearly independent
KTC:

$$\{(\lambda^{\mathrm{T}}, \mu^{\mathrm{T}}) \in \mathbb{R}^{n+m} : c + \sum_{j \in J(x_0)} \lambda_j e_j + \sum_{i \in I} \mu_i a_i = 0, \lambda_j \geq 0, j \in J(x_0)\} \neq \emptyset,$$

or equivalently:
$\{y \in \mathbb{R}^m : (A^{\mathrm{T}}y)_j \geq c_j, j \in J(x_0), (A^{\mathrm{T}}y)_j = c_j, j \notin J(x_0)\} \neq \emptyset$
KTC$_{du}$:

$$\{x \in \mathbb{R}^n : c^{\mathrm{T}}x < 0, x_j \leq 0, j \in J(x_0), a_i^{\mathrm{T}}x = 0, i \in I\} = \emptyset$$

(By Motzkin's theorem of the alternative.)

The definition of a KTP is the same as it was earlier. A vector $(\lambda^{\mathrm{T}}, \mu^{\mathrm{T}})$ satisfying KTC is called a *KT vector (KTV)*. It is said that at a feasible point x_0 the *strict complementary condition in connection with KTC (KT-SCC)* is fulfilled iff there is a KTV with $\lambda_j > 0$ for all $j \in J(x_0)$. A feasible point

x_0 at which KT-SCC is fulfilled is called a *KT-SC point (KT-SCP)*, a vector $(\lambda^{\mathrm{T}}, \mu^{\mathrm{T}})$ satisfying the KT-SCC is called a *KT-SC vector (KT-SCV)*.

OPT-SCC at x_0: There is a dual optimal solution y_0 for which $x_0^{\mathrm{T}}(A^{\mathrm{T}}y_0 - c) = 0$ and $x_0 + A^{\mathrm{T}}y_0 - c > 0$.

The definitions *OPT-SCP* and *OPT-SCV* are the same as earlier.

$\mathrm{supp}(x_0) = \{j \in J : j \notin J(x_0)\}$ and $\mathrm{supp}(y_0) = \{j \in J : j \notin J(y_0)\}$ and $J(x_0, y_0) = J(x_0) \cap J(y_0)$.

Theorem 7.1. *All the propositions of Sections* 2 *and* 3 *remain valid for the case of the pair (P') and (D'). The only difference is that in case of a KTV* $(\lambda^{\mathrm{T}}, \mu^{\mathrm{T}})$ *one has to consider its last m components as a dual optimal solution* y_0.

The definition of a *vertex for (P')* is the same as it was earlier for (P). It is well-known that a feasible solution for (P') is a vertex iff it is a feasible basic solution.

The pendants of the conditions introduced in Sections 4 and 5 are the following. STKTC:

$$\{x \in \mathbb{R}^n : c^{\mathrm{T}}x \le 0, x_j \le 0, j \in J(x_0), a_i^{\mathrm{T}}x = 0, i \in I\} = \{0\}$$

WSTKTC:

$$\{x \in \mathbb{R}^n : c^{\mathrm{T}}x = 0, x_j \le 0, j \in J(x_0), a_i^{\mathrm{T}}x = 0, i \in I\} = \{0\}$$

WKTC:

$$\{x \in \mathbb{R}^n : c^{\mathrm{T}}x < 0, x_j \le 0, j \in J(x_0), \sum_{j \in J(x_0)} x_j < 0, a_i^{\mathrm{T}}x = 0, i \in I\} = \emptyset$$

STWKTC:

$$\{x \in \mathbb{R}^n : c^{\mathrm{T}}x \le 0, x_j \le 0, j \in J(x_0), \sum_{j \in J(x_0)} x_j < 0, a_i^{\mathrm{T}}x = 0, i \in I\} = \emptyset$$

WSTWKTC:

$$\{x \in \mathbb{R}^n : c^{\mathrm{T}}x = 0, x_j \le 0, j \in J(x_0), \sum_{j \in J(x_0)} x_j < 0, a_i^{\mathrm{T}}x = 0, i \in I\} = \emptyset$$

KYPRC: there is such an optimal solution y_0 of (D') that

$$\{(\lambda^{\mathrm{T}}, \mu^{\mathrm{T}}) \in \mathbb{R}^{n+m} : \sum_{j \in J(x_0)} \lambda_j e_j + \sum_{i \in I} \mu_i a_i = 0, \lambda_j \ge 0, j \in J(x_0, y_0)\} = \{0\},$$

or equivalently (by Motzkin's theorem of the alternative):

$$\{x \in \mathbb{R}^n : a_i^{\mathrm{T}}x = 0, i \in I, x_j < 0, j \in J(x_0, y_0), x_j = 0, j \in \mathrm{supp}(y_0)\} \ne \emptyset$$

and the rows of the matrix $\binom{A}{E_{\mathrm{supp}(y_0)}}$ are linearly independent.

Having redefined all the former concepts, it is possible to give the propositions corresponding to the ones in Sections 4, 5 and 6. Denote them by writing a letter 'p' after the name of the corresponding theorem or corollary, e.g. Theorem 4.1p ('p' means pendant.) First, theorems pertaining the uniqueness of the primal optimal solution will be given.

THEOREM 7.2. *All the propositions of Section* 4 *remain valid for the case of the pair (P') and (D'). The only difference is that in the case of Theorem* 4.4p *one must write*
$\{x \in \mathbb{R}^n : a_i^{\mathrm{T}} x = 0, i \in I, x_j \leq 0, j \in J(x_0, y_0), x_j = 0, j \in \operatorname{supp}(y_0)\} = \{0\}$
and in the case of Theorem 4.5p

$$\text{`}\{(\lambda^{\mathrm{T}}, \mu^{\mathrm{T}}) \in \mathbb{R}^{n+m} : \sum_{j \in J(x_0)} \lambda_j e_j + \sum_{i \in I} \mu_i a_i = 0, \lambda_j > 0, j \in J(x_0, y_0)\} \neq \emptyset,$$

and the columns of matrix A *belonging to the indices* $j \in J - J(x_0)$ *are linearly independent'*
instead of the corresponding formulas in Section 4.

Proof. The proofs are similar to the ones in Section 4. Only the proofs of Theorems 4.6p, 4.7p and 4.9p have to be written down.

- *Proof of Theorem* 4.6p. A feasible point x_0 is an optimal solution of (P') iff it is a KTP, i.e.: x_0 is an optimal solution of (P) $\Leftrightarrow$

$$\Leftrightarrow \{x \in \mathbb{R}^n : c^{\mathrm{T}} x < 0, x_j \leq 0, j \in J(x_0), a_i^{\mathrm{T}} x = 0, i \in I\} = \emptyset$$
$$\Leftrightarrow \{x \in \mathbb{R}^n : c^{\mathrm{T}} x < 0, x_j = 0, j \in J(x_0), a_i^{\mathrm{T}} x = 0, i \in I\} = \emptyset \text{ and}$$
$$\{x \in \mathbb{R}^n : c^{\mathrm{T}} x < 0, x_j \leq 0, j \in J(x_0), \sum_{j \in J(x_0)} x_j < 0, a_i x = 0, i \in I\} = \emptyset$$

  The last condition is WKTC.

  Since x_0 is a vertex, it is a basic solution, hence the columns of matrix A corresponding to $j \notin J(x_0)$ are linearly independent. Taking into account that $x_j = 0, j \in J(x_0)$:

$$\{x \in \mathbb{R}^n : x_j = 0, j \in J(x_0), a_i^{\mathrm{T}} x = 0, i \in I\} =$$
$$= \{x \in \mathbb{R}^n : x_j = 0, j \in J(x_0), \sum_{j \in J} a^j x_j = 0\}$$
$$= \{x \in \mathbb{R}^n : x_j = 0, j \in J(x_0), \sum_{j \notin J(x_0)} a^j x_j = 0\} = \{0\}.$$

  Thus:
  $\{x \in \mathbb{R}^n : c^{\mathrm{T}} x < 0, x_j = 0, j \in J(x_0), a_i^{\mathrm{T}} x = 0, i \in I\} = \emptyset$, and hence x_0 is an optimal solution of (P') iff WKTC is fulfilled at x_0.

- *Proof of Theorem* 4.7p. By Theorem 4.1p, a primal feasible point x_0 is the only optimal solution of (P') $\Leftrightarrow$
  $\{x \in \mathbb{R}^n : c^{\mathrm{T}} x \leq 0, x_j \leq 0, j \in J(x_0), a_i^{\mathrm{T}} x = 0, i \in I\} = \{0\} \Leftrightarrow$
  $\{x \in \mathbb{R}^n : c^{\mathrm{T}} x \leq 0, x_j = 0, j \in J(x_0), a_i^{\mathrm{T}} x = 0, i \in I\} = \{0\}$
  and

$\{x \in \mathbb{R}^n : c^{\mathrm{T}}x \leq 0, x_j \leq 0, j \in J(x_0), \sum_{j \in J(x_0)} x_j < 0, a_i x = 0, i \in I\} = \emptyset$
The last condition is the STWKTC.

1)*Sufficiency.* Similarly to the proof of Theorem 4.6p, from the fact that x_0 is a vertex,
$\{x \in \mathbb{R}^n : x_j = 0, j \in J(x_0), a_i^{\mathrm{T}}x = 0, i \in I\} = \{0\}$.

Thus:
$\{x \in \mathbb{R}^n : c^{\mathrm{T}}x \leq 0, x_j = 0, j \in J(x_0), a_i^{\mathrm{T}}x = 0, i \in I\} = \{0\}$,
and hence x_0 is the only optimal solution of (P') iff STWKTC is fulfilled at x_0.

2)The proof of *necessity* is the same as it is in Theorem 4.7.

- *Proof of Theorem* 4.9p. A feasible point x_0 is an optimal solution of (P') iff it is a KTP, i.e.: x_0 is an optimal solution of (P') $\Leftrightarrow$
$\{x \in \mathbb{R}^n : c^{\mathrm{T}}x < 0, x_j \leq 0, j \in J(x_0), a_i^{\mathrm{T}}x = 0, i \in I\} = \emptyset$.
x_0 is SCP iff
$\{y \in \mathbb{R}^m : (A^{\mathrm{T}}y_0)_j > c_j, j \in J(x_0), (A^{\mathrm{T}}y_0)_j = c_j, j \notin J(x_0)\} \neq \emptyset$.
According to the nonhomogeneous generalization of Motzkin's theorem of the alternative (see [4]) the last condition is equivalent to:
$\{x \in \mathbb{R}^n : c^{\mathrm{T}}x < 0, x_j \leq 0, j \in J(x_0), a_i^{\mathrm{T}}x = 0, i \in I\} = \emptyset$
and
$\{x \in \mathbb{R}^n : c^{\mathrm{T}}x = 0, x_j \leq 0, j \in J(x_0), \sum_{j \in J(x_0)} x_j < 0, a_i x = 0, i \in I\} = \emptyset$.

  The first condition is fulfilled, since x_0 is an optimal solution of (P'), the second is WSTWKTC. Hence, in this case SCP is equivalent to WSTWKTC.

In such a way, Theorem 7.2 has been proved. □

In the following theorem a proof using the propositions of the paper will be given to prove the widely known result for the uniqueness of the primal optimal solution obtained by the simplex method.

THEOREM 7.3. *(P') has a unique optimal solution iff the final basis B obtained by the primal simplex method as a primal optimal basis, is dual nondegenerate (i.e. all the negative reduced costs for the non-basic variables are positive).*

Proof. 1)*Necessity.* Let x_0 be the primal optimal solution obtained by the simplex method, and assume that the sufficient optimality condition of the simplex method is fulfilled in the final simplex tableau, i.e. all the negative reduced costs are non-negative. Since x_0 is a feasible basic solution for (P'), it is a vertex, and according to Corollary 4.9p, from the fact that x_0 is the only optimal solution of (P'), it follows that x_0 is a SCP. Therefore — since for all $j \notin J_B$ $x_j = 0$ (where J_B is the set of indices of the basic variables) —, it is necessary that for all $j \notin J_B$ the negative reduced cost be positive. Thus, necessity has been proved.

2)*Sufficiency.* Now, it will be shown that the dual non-degeneracy condition is a sufficient condition for x_0 to be the only primal optimal solution. Since (P') has an optimal solution, by Theorem 2.9p, (D') has an optimal solution, too.

Let y_0 be an arbitrary vector for which $(A^{\mathrm{T}} y_0)_j = c_j, j \in J_B$. (If $r(A) = m$, y_0 is unique; if $r(A) < m$, then there are infinitely many y_0 satisfying the preceding condition.) According to the simplex tableau, $(A^{\mathrm{T}} y_0)_j > c_j, j \notin J_B$. Hence, y_0 is a feasible solution of (D') and satisfies the (weak) complementary condition with x_0. Therefore, by Theorem 2.8p, y_0 is an optimal solution of (D').

Consider the dual problem as a primal problem. It has the same form as (P) has in Sections 1-6, apart from the fact that the inequalities are $\geq$ type instead of $\leq$ type. Nevertheless, it does not have any effect on the proof. Hence, the results of Sections 1-6 can be applied in the proof.

By Theorem 5.3, if it could be shown that there is an optimal solution of (D') at which LICQ is fulfilled, then the dual of the original dual problem (D'), i.e. (P') would have a unique optimal solution. Let y_0 be a solution of the system $(A^{\mathrm{T}} y_0)_j = c_j, j \in J_B$. (It has been shown that it has a solution, and every solution y_1 of it is an optimal solution of (D') for which $(A^{\mathrm{T}} y_1)_j > c_j, j \notin J_B$.) Therefore $J(y_0) = J_B$. Since x_0 is a basic solution,the rows of A^{T} corresponding to the indices $j \in J_B$ are linearly independent, and hence LICQ is satisfied at y_0. Thus, (P') has a unique optimal solution.

Theorem 7.3 has been proved. □

Now, the corresponding results for the uniqueness of the dual optimal solution will be shown.

THEOREM 7.4. *Theorems* 5.1 *and* 5.3*, as well as Corollaries* 5.2, 5.4 *and* 5.5 *are valid for the case of the pair (P') and (D').*

Proof. The proofs are the same as they were earlier. □

The following theorem shows what LICQ means in case of problem (P').

THEOREM 7.5. *Let x_0 be a vertex of the set of primal feasible solutions. Then LICQ is satisfied at x_0 iff*

- *matrix A has full rank, i.e. $r(A) = m$, and*
- *x_0 is a non-degenerate basic solution, i.e. $x_j > 0, j \in J_B$, where $J = J_B \cup J_N$ a disjoint partition of the index set J for the basic and the non-basic indices, respectively.*

Proof. 1)*Necessity.* From the fact that x_0 is a vertex and from the definition of LICQ it follows that $r(A) = m$. Hence, there is a basis B of the columns of A consisting of m vectors. For the sake of simplicity, assume that

$$A = (B\ N). \text{ Then: } \begin{pmatrix} Bx_B & + & Nx_N & = b \\ x_B & & & \geq 0 \\ & & x_N & \geq 0 \end{pmatrix}$$

Hence, rank of matrix $\begin{pmatrix} B & N \\ E_B & O \\ O & E_N \end{pmatrix}$ is equal to n. Since x_0 is a basic solution, $x_j = 0,\ j \in J_N$. Hence, $J_N \subseteq J(x_0)$. But it can not be $J_N \subset J(x_0), J_N \neq J(x_0)$, as $\text{card}(J_N) = n-m$ and if it could be, it would be $\text{card}(J(x_0)) > n-m$, and this is a contradiction to the assumption that LICQ is fulfilled at x_0, since in this case more than n vectors – namely $m + \text{card}(J(x_0))$ – would be linearly independent in $\mathbb{R}^n$. Thus, it follows that $J_N = J(x_0)$, i.e. $x_j > 0, j \in J_B$. Necessity has been proved.

2)*Sufficiency.* By the assumption, $r(A) = m$ and the basis is non-degenerate. Hence, $J_N = J(x_0)$ and matrix $\begin{pmatrix} B & N \\ O & E_N \end{pmatrix}$ has full rank, i.e. LICQ is fulfilled at x_0.

Theorem 7.5 is proved. □

Making use of the preceding theorem, one can get the well-known result for the optimal solution obtained by the primal simplex method.

Corollary 7.6. *(D') has a unique optimal solution iff $r(A) = m$ and the optimal basic solution of (P') obtained by the simplex method is non-degenerate, i.e. $x_j > 0, j \in J_B$.*

Proof. The simplex method results an optimal solution that is a vertex. Applying Theorems 5.3p and 7.5, the corollary will be proved. □

Finally, a necessary *and* sufficient condition will be given for the uniqueness of the primal and at the same time for that of the dual optimal solution. First, it will be described for the pair of (P') and (D'), and then, giving a specialization of this result, for the case when the primal optimal solution has been got by the primal simplex method.

Theorem 7.7. *Theorem* 6.1 *is valid also for the pair of (P') and (D').*

Proof. The proof is the same as the proof of Theorem 6.1. □

Theorem 7.8. *If there is a unique primal optimal solution x_0 and at the same time a unique dual optimal solution, then*

- $r(A) = m$ *(i.e. matrix A has full rank)*

and for x_0 the following conditions are satisfied:

- x_0 *is a vertex corresponding to a basis B for which:*
- $x_j > 0, j \in J_B$ *(i.e. the basis B is primal non-degenerate)*
- $z_j - c_j = c_B^{\mathrm{T}} B^{-1} a^j - c_j > 0, j \in J_N$, *(i.e. the basis B is dual non-degenerate); where c_B is the vector that contains the components $c_k, k \in J_B$, a^j is the jth column of A (i.e. the negative reduced costs for the non-basic vectors are all positive)*

If there is a primal optimal solution x_0 for which the above four properties are satisfied, then x_0 is the only primal optimal solution and at the same time there is a unique dual optimal solution.

Proof. 1)*Necessity.* Since x_0 is the only primal optimal solution, it is a vertex. As there is a unique dual optimal solution, by Theorem 5.3p, at x_0 LICQ is satisfied. According to Theorem 7.6, $r(A) = m$ and $x_j > 0, j \in J_B$. Making use of Corollary 4.10p, it follows that x_0 is a SCP. From this, it follows that, since the dual optimal solution is unique, too, the dual optimal solution y_0 is a SCV. Hence $B^{\mathrm{T}} y_0 = c_B$, i.e. $y_0^{\mathrm{T}} = c_B^{\mathrm{T}} B^{-1}$, and $(a^j)^{\mathrm{T}} y_0 > c_j, j \in J_N$. Hence the last property is satisfied and necessity is proved.

2)*Sufficiency.* Since x_0 is a vertex, according to $z_j - c_j > 0$ and $x_j > 0, j \in J_B$, x_0 is a SCP. By Corollary 4.10p, x_0 is the only primal optimal solution. From $r(A) = m$ and $x_j > 0, j \in J_B$, it follows that LICQ is fulfilled at x_0. Hence, by Theorem 5.3p, the dual optimal solution is unique.

Theorem 7.8 has been proved. □

COROLLARY 7.9. *Let x_0 be the optimal solution got by the primal simplex method. Then it is the only primal optimal solution and at the same time the dual problem has a unique optimal solution iff the following properties are satisfied:*

- *$r(A) = m$ (i.e. matrix A has full rank)*

and for the optimal basis B corresponding to x_0:

- *$x_j > 0, j \in J_B$ (i.e. the basis B is primal non-degenerate)*
- *$z_j - c_j = c_B^{\mathrm{T}} B^{-1} a^j - c_j > 0, j \in J_N$, (i.e. the basis B is dual non-degenerate)*

Proof. x_0 is a vertex, and the corollary follows immediately from the preceding theorem. □

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Received by the editors: October 19, 2005.