REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 35 (2006) no. 2, pp. 141–146
ictp.acad.ro/jnaat

DOUBLE INEQUALITIES OF NEWTON'S QUADRATURE RULE

MARIUS HELJIU*

Abstract. In this paper double inequalities of Newton's quadrature rule are given.

MSC 2000. 65D32.
Keywords. Newton's inequality, double integral inequalities, numerical integration.

1. INTRODUCTION

In the papers [1], [2], [3], [4], [5], [7] and [8] relatively double inequalities of quadratures rules of the trapezoid and Simpson were given. In this note we will obtain upper and lower error bounds for Newton's quadrature rule.

Let $f:[a,b]\to\mathbb{R},\ f\in C^4([a,b])$ and $x_1,\ x_2\in[a,b]$ so that $x_1=\frac{2a+b}{3},\ x_2=\frac{a+2b}{3}$, then as it is well known [6] the relation is obtained

$$\int_a^b f(x)\mathrm{d}x=\tfrac{b-a}{8}\left[f(a)+3f(x_1)+3f(x_2)+f(b)\right]+R, \tag{1.1}$$

where

$$R=\int_a^b \varphi(x)f^{(4)}(x)\mathrm{d}x. \tag{1.2}$$

The function φ is given by the relation (h denotes $\frac{b-a}{3}$):

$$\varphi(x)=\begin{cases}\frac{(x-a)^4}{4!}-\frac{3h}{8}\frac{(x-a)^3}{3!}, & x\in[a,x_1]\\ \frac{(x-a)^4}{4!}-\frac{3h}{8}\frac{(x-a)^3}{3!}-\frac{9h}{8}\frac{(x-x_1)^3}{3!}, & x\in[x_1,x_2]\\ \frac{(x-a)^4}{4!}-\frac{3h}{8}\frac{(x-a)^3}{3!}-\frac{9h}{8}\frac{(x-x_1)^3}{3!}-\frac{9h}{8}\frac{(x-x_2)^3}{3!}, & x\in[x_2,b].\end{cases} \tag{1.3}$$

2. MAIN RESULT

Under the assumptions of the quadrature formula (1.1) we have the next theorem:

*University of Petroşani, Department of Mathematics and Computer Science, Romania, e-mail: mheljiu@gmail.com.

THEOREM 2.1. *Let* $f \in C^4(a,b)$. *Then:*

$$\frac{23\gamma_4-15S_3}{51840}(b-a)^5 \le \frac{b-a}{8}\left[f(a)+3f(x_1)+3f(x_2)+f(b)\right]-\int_a^b f(x)\mathrm{d}x \le \frac{23\Gamma_4-15S_3}{51840}(b-a)^5, \tag{2.1}$$

where $\gamma_4,\ \Gamma_4 \in R,\ \gamma_4 \le f^{(4)}(x) \le \Gamma_4$, *for any* $x \in [a,b]$ *and* $S_3 = \frac{f^{(3)}(b)-f^{(3)}(a)}{b-a}$. *If* $\gamma_4 = \min\limits_{x\in[a,b]} f^{(4)}(x),\ \Gamma_4 = \max\limits_{x\in[a,b]} f^{(4)}(x)$ *then inequalities are sharp.*

Proof. From (1.2) integrating by parts we get :

$$\int_a^b \varphi(x)f^{(4)}(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x - \frac{b-a}{8}\left[f(a)+3f(x_1)+3f(x_2)+f(b)\right]. \tag{2.2}$$

It is easy to see [6] that we get the equality:

$$\int_a^b \varphi(x)\mathrm{d}x = -\frac{(b-a)^5}{6480}. \tag{2.3}$$

From (2.2) and (2.3) we get the equalities:

$$\begin{aligned}&\int_a^b \left[f^{(4)}(x)-\gamma_4\right]\varphi(x)\mathrm{d}x = \\ &= \int_a^b f(x)\mathrm{d}x - \frac{b-a}{8}\left[f(a)+3f(x_1)+3f(x_2)+f(b)\right] + \frac{\gamma_4}{6480}(b-a)^5\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}&\int_a^b \left[\Gamma_4 - f^{(4)}(x)\right]\varphi(x)\mathrm{d}x = \\ &= -\int_a^b f(x)\mathrm{d}x + \frac{b-a}{8}\left[f(a)+3f(x_1)+3f(x_2)+f(b)\right] - \frac{\Gamma_4}{6480}(b-a)^5.\end{aligned} \tag{2.5}$$

On the other hand:

$$\int_a^b \left[f^{(4)}(x)-\gamma_4\right]\varphi(x)\mathrm{d}x \le \max_{x\in[a,b]}|\varphi(x)| \int_a^b \left|f^{(4)}(x)-\gamma_4\right|\mathrm{d}x. \tag{2.6}$$

From (1.3) we get:

$$\max_{x\in[a,b]}|\varphi(x)| = \frac{(b-a)^4}{3456}. \tag{2.7}$$

On the other hand the equality follows:

$$\begin{aligned}\int_a^b \left|f^{(4)}(x)-\gamma_4\right|\mathrm{d}x &= \int_a^b (f^{(4)}(x)-\gamma_4)\mathrm{d}x \\ &= f^{(3)}(b)-f^{(3)}(a)-\gamma_4(b-a) = (S_3-\gamma_4)(b-a).\end{aligned} \tag{2.8}$$

From the relations (2.4), (2.6), (2.7) and (2.8) it follows :

$$\begin{aligned}&\int_a^b f(x)\mathrm{d}x - \frac{b-a}{8}\left[f(a)+3f(x_1)+3f(x_2)+f(b)\right] \le \\ &\le \frac{15S_3-23\gamma_4}{51840}(b-a)^5,\end{aligned} \tag{2.9}$$

the first inequality of (2.1).

We also have:

$$\int_a^b \left[\Gamma_4 - f^{(4)}(x)\right] \varphi(x)\mathrm{d}x \le \max_{x\in[a,b]} |\varphi(x)| \int_a^b \left|\Gamma_4 - f^{(4)}(x)\right| \mathrm{d}x \tag{2.10}$$

and

$$\int_a^b \left|\Gamma_4 - f^{(4)}(x)\right| \mathrm{d}x = \int_a^b (\Gamma_4 - f^{(4)}(x))\mathrm{d}x = \tag{2.11}$$

$$= \Gamma_4(b-a) + f^{(3)}(a) - f^{(3)}(b) = (\Gamma_4 - S_3)(b-a).$$

By analogy from (2.5), (2.7), (2.10) and (2.11) we get:

$$\begin{aligned} &\int_a^b f(x)\mathrm{d}x - \tfrac{b-a}{8}\left[f(a) + 3f(x_1) + 3f(x_2) + f(b)\right] \ge \\ &\ge \tfrac{15S_3 - 23\Gamma_4}{51840}(b-a)^5. \end{aligned} \tag{2.12}$$

The last relation and (2.9) lead us to the inequality (2.1).

To show that inequality (2.1) is sharp we consider the function f given by the relation $f(x) = (x-a)^4$. It is easy to see that the equalities $f^{(4)}(x) = 24$ and $\gamma_4 = \Gamma_4 = 24, S_3 = 24$ are obtained.

Calculating the three members of the inequality (2.1) under the given circumstances, we notice that these have the common value given by the expression $\frac{1}{270}(b-a)^5$. Hence, we deduce that the inequality (2.1) is sharp. □

Another relation is given by the next theorem:

THEOREM 2.2. *Under the assumptions of Theorem* 2.1 *we have:*

$$\begin{aligned} \tfrac{7\gamma_4 - 15S_3}{51840}(b-a)^5 \le &\int_a^b f(x)\mathrm{d}x - \tfrac{b-a}{8}\left[f(a) + 3f(x_1) + 3f(x_2) + f(b)\right] \\ &\le \tfrac{7\Gamma_4 - 15S_3}{51840}(b-a)^5. \end{aligned} \tag{2.13}$$

If $\gamma_4 = \min_{x\in[a,b]} f^{(4)}(x), \Gamma_4 = \max_{x\in[a,b]} f^{(4)}(x)$ *then the inequalities* (2.13) *are sharp.*

Proof. From (2.4), (2.6), (2.7)and (2.8) we have:

$$\begin{aligned} &-\int_a^b f(x)\mathrm{d}x + \tfrac{b-a}{8}\left[f(a) + 3f(x_1) + 3f(x_2) + f(b)\right] \le \\ &\le \tfrac{15S_3 - 7\gamma_4}{51840}(b-a)^5. \end{aligned} \tag{2.14}$$

By analogy from (2.5), (2.10), (2.11) and (2.12) we have:

$$\begin{aligned} &\int_a^b f(x)\mathrm{d}x - \tfrac{b-a}{8}\left[f(a) + 3f(x_1) + 3f(x_2) + f(b)\right] \le \\ &\le \tfrac{7\Gamma_4 - 15S_3}{51840}(b-a)^5. \end{aligned} \tag{2.15}$$

From (2.14) and (2.15) we will have immediately the inequalities (2.13).

To show that the inequalities are sharp we choose $f(x) = (x-a)^4$ and we follow the steps of the proof for Theorem 2.1. □

The next theorem offers us inequalities which do not depend upon S_3.

Theorem 2.3. *Under the assumptions of Theorem* 2.1 *we have:*

(2.16)
$$\frac{7\gamma_4 - 23\Gamma_4}{103680}(b-a)^5 \le \int_a^b f(x)\mathrm{d}x - \frac{b-a}{8}\left[f(a) + 3f(x_1) + 3f(x_2) + f(b)\right] \le \le \frac{7\Gamma_4 - 23\gamma_4}{103680}(b-a)^5.$$

If $\gamma_4 = \min_{x\in[a,b]} f^{(4)}(x)$, $\Gamma_4 = \max_{x\in[a,b]} f^{(4)}(x)$ *then the inequalities* (2.16) *are sharp.*

Proof. The inequalities (2.16) are easily deduced by (2.9) and (2.14) , respectively (2.12) and (2.15). To show that the inequalities are sharp we follow the steps of the proof for Theorem 2.1. □

In use the next theorem is important:

Theorem 2.4. *Under the assumptions of Theorem* 2.1 *we have:*

(2.17)
$$\frac{23\gamma_4 - 15S_3}{51840n^4}(b-a)^5 \le \le \frac{b-a}{8n}\left[f(a) + f(b) + 2\sum_{i=1}^{n-1} f(x_i) + 3\sum_{i=1}^{n} f(x_i') + 3\sum_{i=1}^{n} f(x_i'')\right] - \int_a^b f(x)\mathrm{d}x \le \frac{23\Gamma_4 - 15S_3}{51840n^4}(b-a)^5,$$

where $x_i = a + ih$, $h = \frac{b-a}{n}, i = 0, 1, ..., n$ *and* x_i', x_i'' *divide every interval* $[x_i, x_{i+1}]$ *in three equal parts.*

Proof. We divide each interval $[x_i, x_{i+1}]$ in the equal parts by points x_i', x_i'', then we use Theorem 2.1 on the interval $[x_i, x_{i+1}]$:

$$\frac{23\gamma_4 - 15S_3^i}{51840}(x_{i+1} - x_i)^5 \le \le \frac{x_{i+1} - x_i}{8}\left[f(x_i) + 3f(x_i') + 3f(x_i'') + f(x_{i+1})\right] - \int_{x_i}^{x_{i+1}} f(x)\mathrm{d}x \le \frac{23\Gamma_4 - 15S_3^i}{51840}(x_{i+1} - x_i)^5, \tag{2.18}$$

where $S_3^i = \frac{f^{(3)}(x_{i+1}) - f^{(3)}(x_i)}{h}, i = 0, 1, ..., n-1$. By adding the formula we have got so far for $i = 0, 1, ..., n-1$ and by noticing that $\sum_{i=0}^{n-1} S_3^i = \frac{f^{(3)}(b) - f^{(3)}(a)}{h}$ we get the relation we wanted. □

From Theorem 2.2, following the steps done, we get:

Theorem 2.5. *Under the assumptions of Theorem* 2.1 *we have:*

$$\begin{aligned}(2.19)\quad &\tfrac{7\gamma_4-15S_3}{51840n^4}(b-a)^5 \leq \\ &\leq \int_a^b f(x)\mathrm{d}x - \tfrac{b-a}{8n}\left[f(a)+f(b)+2\sum_{i=1}^{n-1}f(x_i)+3\sum_{i=1}^{n}f(x_i')+3\sum_{i=1}^{n}f(x_i'')\right] \\ &\leq \tfrac{7\Gamma_4-15S_3}{51840n^4}(b-a)^5.\end{aligned}$$

Proof. Proving this theorem is similarly to proving Theorem 2.4. Then we use Theorem 2.2. □

Also Theorem 2.3 leads us to:

Theorem 2.6. *Under the assumptions of Theorem* 2.1 *we have:*

$$\begin{aligned}(2.20)\quad &\tfrac{7\gamma_4-23\Gamma_4}{103680n^4}(b-a)^5 \leq \\ &\leq \int_a^b f(x)\mathrm{d}x - \tfrac{b-a}{8n}\left[f(a)+f(b)+2\sum_{i=1}^{n-1}f(x_i)+3\sum_{i=1}^{n}f(x_i')+3\sum_{i=1}^{n}f(x_i'')\right] \\ &\leq \tfrac{7\Gamma_4-23\gamma_4}{103680n^4}(b-a)^5.\end{aligned}$$

Proof. Proving this theorem is similarly to proving Theorem 2.4. Then we use Theorem 2.3. □

REFERENCES

[1] Cerone, P., *Three points rules in numerical integration*, Non. Anal. Theor. Meth. Appl., **47** (4), pp. 2341–2352, 2001.

[2] Cerone, P. and Dragomir, S. S., *Midpoint-type Rules from an Inequalities Point of View*, Handbook of Analytic-Computational Methods in Applied Mathematics, Editor: G. Anastassiou, CRC Press, New York, pp. 135–200, 2000.

[3] Cerone, P. and Dragomir, S. S., *Trapezoidal-type Rules from an Inequalities Point of View*, Handbook of Analytic-Computational Methods in Applied Mathematics, Editor: G. Anastassiou, CRC Press, New York, pp. 65–134, 2000.

[4] Dragomir, S. S., Agarwal, R. P. and Cerone, P., *On Simpson's inequality and applications*, J. Inequal. Appl., **5**, pp. 533–579, 2000.

[5] Dragomir, S. S., Pecarić, J. and Wang, S., *The unified treatment of trapezoid, Simpson and Ostrowski type inequalities for monotonic mappings and applications*, Math. Comput. Modelling, **31**, pp. 61–70, 2000.

[6] Ionescu, D. V., *Cuadraturi numerice*, Editura Tehnică, Bucureşti, 1957.

[7] Ujević, N., *Some double integral inequalities and applications*, Acta Math. Univ. Comenianae, **71** (2), pp. 187, 2002.

[8] Ujević, N., *Double integral inequalities of simpson type and applications*, J. Appl. Math. & Computing, **14**, (1–2), pp. 213–223, 2004.

Received by the editors: January 12, 2006.