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A-STATISTICAL CONVERGENCE FOR A CLASS OF POSITIVE LINEAR OPERATORS

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Abstract. In this paper, we introduce a sequence of positive linear operators defined on the space $C[0,a]$ $(0<a<1)$, and provide an approximation theorem for these operators via the concept of A-statistical convergence. We also compute the rates of convergence of these approximation operators by means of the first and second order modulus of continuity and the elements of the Lipschitz class. Furthermore, by defining the generalization of r-th order of these operators we show that the similar approximation properties are preserved on $C[0,a]$.

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1. INTRODUCTION

The concept of a limit of a sequence has been extended to statistical limit ([15], [17], [18]) by using the natural density δ of a set K of positive integers: $\delta(K) := \lim_n \frac{1}{n}\{\text{the number } k \leq n \text{ such that } k \in K\}$ whenever the limit exists (for the natural density, see [27]). A sequence $x=(x_k)$ is said to be statistically convergent to a number L if for every $\varepsilon>0$, $\delta\{k : |x_k - L| \geq \varepsilon\} = 0$ and it is denoted by $st-\lim_k x_k = L$. Let $A=(a_{jn})$, $j,n=1,2,...,$ be an infinite summability matrix. The A-transform of the sequence x, denoted by $Ax := \{(Ax)_j\}$, is given by $(Ax)_j := \sum_{n=1}^{\infty} a_{jn}x_n$ provided the series converges for each j. A is said to be regular if $\lim_j (Ax)_j = L$ whenever $\lim_j x_j = L$ [4]. Assume that A is a nonnegative regular summability matrix. The A-density of K, denoted by $\delta_A(K)$, is defined by $\delta_A(K) := \lim_j \sum_{n=1}^{\infty} a_{jn}\chi_K(n)$ provided the limit exists, where χ_K is the characteristic function of K. Then, $x=(x_n)$ is said to be A-statistically convergent to a number L if, for every $\varepsilon>0$, $\delta_A\{n\in\mathbb{N} : |x_n - L| \geq \varepsilon\} = 0$; or equivalently $\lim_j \sum_{n:\,|x_n-L|\geq\varepsilon} a_{jn} = 0$. We denote this limit by $st_A - \lim x = L$ (see [16], [19], [23], [26]). The case in

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which $A = C_1$, the Cesáro matrix of order one, reduces to the statistical convergence, and also if $A = I$, the identity matrix, then it coincides with the ordinary convergence. We note that if $A = (a_{jn})$ is a nonnegative regular summability matrix such that $\lim_j \max_n \{a_{jn}\} = 0$, then A-statistical convergence is stronger than convergence [23].

Chlodovsky [9] was the first to notice that the Bernstein polynomials,

$$B_n(f;x) = \sum_{k=0}^{n} \tbinom{n}{k} x^k (1-x)^{n-k} f(\tfrac{k}{n}),$$

converge to middle of jump at the point of simple discontinuity of a function. That is, if x is a point of discontinuity of first kind, then

$$\lim_n B_n(f;x) = \tfrac{f(x^+)+f(x^-)}{2} =: \overline{f(x)}.$$

But this phenomenon does not always take place for general positive linear approximation operators. An example was given by Bojanic and Cheng in [5] where they showed that the Hermit-Fejer interpolation operator,

$$H_n(f;x) = \sum_{k=1}^{n} f(x_{n,k})(1 - x x_{n,k}) \left\{ \tfrac{T_n(x)}{n(x - x_{n,k})} \right\}^2,$$

where the nodes $x_{n,k} = \cos(\frac{(2k-1)\pi}{2n})$ are the zeros of Chebyshev polynomials $T_n(x) = \cos(n \cos^{-1} x)$, does not converge at a point of simple discontinuity. However, Bojanic and Khan [6] showed that the Cesáro averages of the Hermit-Fejer operator, $\frac{1}{n} \sum_{k=1}^{n} H_k(f;x)$, do converge to the mid point of the jump discontinuity. So, it shows that this summability method is stronger than the classical sense in approximation theory. In recent years another form of regular summability transformation has shown to be quite effective in "summing" nonconvergent sequences which may have unbounded subsequences (see [16], [17]). Furthermore, some Korovkin type approximation theorems have been studied via statistical convergence and A-statistical convergence in [11], [12], [13], [20].

The aim of the present paper is to provide an A-statistical approximation theorem for Agratini type operators [1]. Note that Agratini's operators are a Stancu type generalization (see [29]) of the operators in [10]. We also give the rates of A-statistical convergence of these operators by means of the first and second order modulus of continuity and the elements of the Lipschitz class.

2. STATISTICAL APPROXIMATION OF POSITIVE OPERATORS

In this section, we give a generalization of Agratini's operators [1] and obtain an approximation theorem for these operators by using A-statistical convergence.

Let $A = (a_{jn})$ be a nonnegative regular summability matrix and let (α_n), (β_k) and $(\gamma_{n,k})$ be real sequences satisfying the following conditions, respectively:

$$\alpha_n \geq 1 \text{ (for any } n \in \mathbb{N}) \text{ and } st_A - \lim_n \alpha_n = 1, \tag{2.1}$$

$$0 \leq \beta_k \leq \beta_{k+1} \; (k = 0, 1, 2, ...), \tag{2.2}$$

$$0 \leq \gamma_{n,k} \leq \tfrac{c}{n} \; \text{(for some } c > 0 \text{ and any } n \in \mathbb{N},\ k = 0, 1, 2, ...). \tag{2.3}$$

Let $(\rho_{n,k})$ be a sequence of positive integers such that

$$\max\{k, n\} \leq \rho_{n,k} \leq \rho_{n,k+1} \text{ (for any } n \in \mathbb{N} \text{ and } k = 0, 1, 2, ...). \tag{2.4}$$

Assume now that a sequence of functions (φ_n) has the following properties:

(1°) Let $a \in (0, 1)$. Every function φ_n is analytic on a domain containing the disk $\{z \in \mathbb{C} : |z| \leq a\}$,

(2°) $\varphi_n^{(0)}(0) = \varphi_n(0) > 0$ (for every $n \in \mathbb{N}$),

(3°) $\varphi_n^{(k)}(0) = \alpha_n(\rho_{n,k} + \beta_k)(1 + \gamma_{n,k})\varphi_n^{(k-1)}(0)$ (for every $k, n \in \mathbb{N}$),

where $\varphi_n^{(k)}(0)$ denotes $\dfrac{d^k}{dx^k}\varphi_n(0)$ (for every $k, n \in \mathbb{N}$), and also the sequences (α_n), (β_k), $(\gamma_{n,k})$ and $(\rho_{n,k})$ satisfy the conditions (2.1), (2.2), (2.3) and (2.4), respectively.

We now introduce the sequence of operators Ω_n on $C[0, a]$, the space of all continuous functions on $[0, a]$, by

$$\Omega_n(f; x) = \frac{1}{\varphi_n(x)} \sum_{k=0}^{\infty} f\left(\frac{k}{\rho_{n,k}+\beta_k}\right) \varphi_n^{(k)}(0) \frac{x^k}{k!}, \; (f \in C[0, a],\ n \in \mathbb{N}). \tag{2.5}$$

Now we analyze our operators Ω_n and give their applications in approximation theory settings. To obtain that we first assume

$$m_{n,k}(x) = \binom{n+k}{k} x^k (1 - x)^{n+1}.$$

If we choose $\beta_k = 1$, $\rho_{n,k} = n + k$, $\varphi_n(x) = (1 - x)^{-n-1}$, then our operators Ω_n given by (2.5) turn out to be unmodified Meyer-König and Zeller Operators [25]

$$M_n(f; x) = \sum_{k=0}^{\infty} m_{n,k}(x) f\left(\frac{k}{k+n+1}\right).$$

Choosing $\beta_k = 0$, $\rho_{n,k} = n + k$, $\varphi_n(x) = (1 - x)^{-n-1}$, then the operators Ω_n reduce to Cheney and Sharma's operators [8]

$$S_n(f; x) = \sum_{k=0}^{\infty} m_{n,k}(x) f\left(\frac{k}{k+n}\right).$$

Taking $\beta_k = 1$, $\rho_{n,k} = n + k$, $\varphi_n(x) = (1-x)^{-n-1} \exp\left(\frac{tx}{x-1}\right)$ for $t \leq 0$, then the operators Ω_n coincide with the operators of Khan [21]

$$K_n(f;x) = (1-x)^{n+1} \exp\left(\tfrac{tx}{1-x}\right) \sum_{k=0}^{\infty} L_k^n(t) x^k f\left(\tfrac{k}{k+n+1}\right),$$

where $L_k^n(t)$ denotes the Laguerre polynomial of degree k defined by

$$L_k^n(t) = \sum_{r=0}^{k} (-1)^r \tfrac{\Gamma(k+n+1)}{(k+r)!\Gamma(n+r+1)r!} t^r.$$

We should note that choosing $\rho_{n,k} = k+n$ in (2.4) and replacing the matrix A by the identity matrix I conditions (2.1)-(2.4) reduce to all those in [1].

It is easy to see that each Ω_n is positive and linear, and also $\Omega_n(1;x) = 1$ (for every $n \in \mathbb{N}$) holds.

Throughout the paper we denote the usual norm of the space $C[0,a]$ by $\|\cdot\|$, i.e.,

$$\|f\| = \sup_{x\in[0,a]} |f(x)|, \quad (f \in C[0,a]).$$

To construct our A-statistical approximation theorem for the sequence $\{\Omega_n\}$ we need the following two lemmas whose proofs can immediately be obtained with the similar methods used in [11] and [14].

Lemma 2.1. *Let $A = (a_{jn})$ be a nonnegative regular summability matrix. Then we have*

$$st_A - \lim_n \|\Omega_n(t;x) - x\| = 0,$$

where $\|\,.\,\|_{C[0,a]}$ denotes the ordinary sup norm on the space $C[0,a]$.

Lemma 2.2. *Let $A = (a_{jn})$ be a nonnegative regular summability matrix. Then we have*

$$st_A - \lim_n \left\|\Omega_n(t^2;x) - x^2\right\| = 0.$$

Combining Lemmas 2.1 and 2.2 we have the following main result.

Theorem 2.3. *Let $A = (a_{jn})$ be a nonnegative regular summability matrix. Then, for all $f \in C[0,a]$, we have*

$$st_A - \lim_n \|\Omega_n(f;x) - f(x)\| = 0.$$

Proof. By Lemmas 2.1 and 2.2, we immediately get

$$st_A - \lim_n \left\|\Omega_n(t^i;x) - x^i\right\| = 0, \;\; i = 0,1,2.$$

So, the result follows from Theorem 1 in [20], (see also [12]). We note that Theorem 1 in [20] is given for statistical convergence, but the proof also works for A-statistical convergence. □

When the matrix A is replaced by the identity matrix I in Theorem 2.3, then the following result holds at once.

Corollary 2.4. *For all $f \in C[0,a]$, the sequence $\{\Omega_n(f)\}$ converges uniformly to f on $[0,a]$.*

The following example shows that A-statistical approximation in Theorem 2.3 is stronger than ordinary norm-wise convergence in Corollary 2.4.

Let $A = C_1 = (c_{jn})$, the Cesáro matrix of order one, defined by

$$c_{jn} = \begin{cases} \frac{1}{n}; & \text{if } n \le j \\ 0; & \text{otherwise.} \end{cases}$$

Then C_1-statistical convergence is known as statistical convergence [15]. Assume that (u_n) is defined by

$$u_n = \begin{cases} 0; & \text{if } n \text{ is a square} \\ 1; & \text{if } n \text{ is a nonsquare.} \end{cases}$$

Observe that (u_n) is non-convergent, but it is statistically convergent to 1, i.e., $st - \lim_n u_n = 1$. Let $\{\Omega_n\}$ be the sequence of positive linear operators given (2.5). Now define the operators Ω_n^* by

$$\Omega_n^*(f;x) := u_n \Omega_n(f;x), \ f \in C[0,a], \ 0 < a < 1.$$

Then we may write, for all $f \in C[0,a]$, that

$$\begin{aligned} \Omega_n^*(f;x) - f(x) &= u_n \Omega_n(f;x) - f(x) + u_n f(x) - u_n f(x) \\ &= u_n \left(\Omega_n(f;x) - f(x)\right) + f(x)\left(u_n - 1\right) \end{aligned}$$

and hence

$$\|\Omega_n^*(f;x) - f(x)\| \le \|\Omega_n(f;x) - f(x)\| + C\left|u_n - 1\right|. \tag{2.6}$$

Since, for all $f \in C[0,a]$, the sequence $\{\Omega_n(f)\}$ converges uniformly to f and also (u_n) converges statistically to 1, it follows from (2.6) that

$$st - \lim_n \|\Omega_n^*(f;x) - f(x)\| = 0.$$

Hence Theorem 2.3 holds for the operators Ω_n^*. However, since (u_n) is non-convergent, the sequence $\{\Omega_n^*(f)\}$ is not uniformly convergent to f, which does not satisfy Corollary 2.4.

Before closing this section, we should remark that choosing $\rho_{n,k} = n + k$ and replacing (2.2) by the condition $0 \le \beta_k \le 1 + \beta_{k+1}$ Theorem 2.3 reduces to Theorem 3 of [11].

3. RATES OF A-STATISTICAL CONVERGENCE

In this section, we compute the rates of A-statistical convergence in Theorem 2.3 by means of the first and second order modulus of continuity and the elements of the Lipschitz class.

Let $f \in C[0,a]$. The first order modulus of continuity of f, denoted by $w(f,\delta)$, is defined to be

$$w(f,\delta)=\sup_{|t-x|<\delta,\ t,x\in[0,a]}|f(t)-f(x)|$$

(see [2], [24] for details).
It is well known that for any $\delta>0$ and each $t,x\in[0,a]$

$$|f(t)-f(x)|\le w(f,\delta)\left(\tfrac{|t-x|}{\delta}+1\right). \tag{3.1}$$

The next result gives the rate of A-statistical convergence of the sequence $\{\Omega_n(f)\}$ (for all $f\in C[0,a]$) in Theorem 2.3 by means of first order modulus of continuity.

THEOREM 3.1. *For all* $f\in C[0,a]$, *we have*

$$\|\Omega_n(f;x)-f(x)\|\le(1+B_3^{1/2})w(f,\delta_n),$$

where

$$\begin{aligned}B_3&=\max\{a,ac,2a^2,2a^2c,a^2c^2\},\\ \delta_n&=\left\{(\alpha_n-1)(\alpha_n+2)+\tfrac{\alpha_n(\alpha_n+1)}{n^2}+\tfrac{\alpha_n(\alpha_n+2)}{n}\right\}^{1/2}.\end{aligned} \tag{3.2}$$

Proof. We will use Popoviciu's technique given in [28]. Let $f\in C[0,a]$. By linearity and monotonicity of Ω_n we obtain

$$\begin{aligned}|\Omega_n(f;x)-f(x)|&=|\Omega_n(f(t)-f(x);x)|\\ &\le\Omega_n(|f(t)-f(x)|\,;x)\\ &=\tfrac{1}{\varphi_n(x)}\sum_{k=0}^{\infty}\left|f\left(\tfrac{k}{\rho_{n,k}+\beta_k}\right)-f(x)\right|\varphi_n^{(k)}(0)\tfrac{x^k}{k!}.\end{aligned}$$

By (3.1) and the Cauchy-Bunyakowsky-Schwarz inequality we have

$$\begin{aligned}|\Omega_n(f;x)-f(x)|&\le\tfrac{w(f,\delta_n)}{\varphi_n(x)}\sum_{k=0}^{\infty}\left(\tfrac{1}{\delta_n}\left|\tfrac{k}{\rho_{n,k}+\beta_k}-x\right|+1\right)\varphi_n^{(k)}(0)\tfrac{x^k}{k!}\\ &=w(f,\delta_n)\left\{\tfrac{1}{\delta_n\varphi_n(x)}\sum_{k=0}^{\infty}\left|\tfrac{k}{\rho_{n,k}+\beta_k}-x\right|\varphi_n^{(k)}(0)\tfrac{x^k}{k!}+1\right\}\\ &\le w(f,\delta_n)\\ &\quad\times\left\{\tfrac{1}{\delta_n}\left[\tfrac{1}{\varphi_n(x)}\sum_{k=0}^{\infty}\left(\tfrac{k}{\rho_{n,k}+\beta_k}-x\right)^2\varphi_n^{(k)}(0)\tfrac{x^k}{k!}\right]^{1/2}+1\right\}\\ &=w(f,\delta_n)\left[\tfrac{1}{\delta_n}(A_n(x))^{1/2}+1\right],\end{aligned}$$

where

$$A_n(x)=\tfrac{1}{\varphi_n(x)}\sum_{k=0}^{\infty}\left(\tfrac{k}{\rho_{n,k}+\beta_k}-x\right)^2\varphi_n^{(k)}(0)\tfrac{x^k}{k!}. \tag{3.3}$$

This implies that

$$\|\Omega_n(f;x) - f(x)\| \le w(f,\delta_n)\left[\tfrac{1}{\delta_n}\left(\sup_{x\in[0,a]} A_n(x)\right)^{1/2} + 1\right]. \tag{3.4}$$

For each $x \in [0,a]$, one can write

$$\begin{aligned} A_n(x) &= \Omega_n(t^2;x) - 2x\Omega_n(t;x) + x^2 \\ &\le \left|\Omega_n(t^2;x) - x^2\right| + 2x\left|\Omega_n(t;x) - x\right|. \end{aligned}$$

So, by (2.9) and (2.12) we get

$$\begin{aligned} \sup_{x\in[0,a]} A_n(x) &\le \left\|\Omega_n(t^2;x) - x^2\right\|_{C[0,a]} + 2a\left\|\Omega_n(t;x) - x\right\|_{C[0,a]} \\ &\le B_2\left\{\alpha_n^2 - 1 + \tfrac{\alpha_n(\alpha_n+1)}{n^2} + \tfrac{\alpha_n(\alpha_n+1)}{n}\right\} + 2aB_1\left\{\alpha_n - 1 + \tfrac{\alpha_n}{n}\right\} \\ &\le B_3\left\{(\alpha_n - 1)(\alpha_n + 2) + \tfrac{\alpha_n(\alpha_n+1)}{n^2} + \tfrac{\alpha_n(\alpha_n+2)}{n}\right\} \\ &= B_3\delta_n^2, \end{aligned} \tag{3.5}$$

where $B_3 = \max\{2aB_1, B_2\} = \max\{a, ac, 2a^2, 2a^2c, a^2c^2\}$. Combining (3.5) with (3.4) we can write

$$\|\Omega_n(f;x) - f(x)\| \le (1 + {B_3}^{1/2})w(f,\delta_n),$$

whence the result. □

Let $f \in C[0,a]$. Then the second order modulus of continuity of f denoted by $w_2(f,\delta)$ is defined as

$$w_2(f,\delta) = \sup\left\{|f(x+h) - 2f(x) + f(x-h)| : (x \mp h) \in [0,1],\ \ |h| \le \delta\right\}$$

This modulus is also known as Zygmund's modulus for the function f.

In order to estimate this order of approximation via second modulus of continuity we will benefit the Peetre's K-functional.

Now we denote the space of the functions f such that $f, f', f'' \in C[0,a]$ by $C^2[0,a]$ and define the following norm in the space $C^2[0,a]$ by

$$\|f\|_{C^2[0,a]} := \|f\| + \|f'\| + \|f''\|.$$

Then the following Peetre's K-functional [3] (see also [7]) is given by

$$K(f,\delta) = \inf_{g\in C^2[0,a]}\{\|f - g\| + \delta\|g\|_{C^2[0,a]}\}. \tag{3.6}$$

Theorem 3.2. *If* $f \in C[0,a]$ *then we have*

$$\|\ \Omega_n(f;x) - f(x)\ \| \le 2K\left(f, (\sqrt{B_3}\delta_n)^2\right) \tag{3.7}$$

where B_3 *and* δ_n *are the same as in* (3.2).

Proof. If $g \in C^2[0,a]$ then we have

$$g(s)-g(x)=g'(x)(s-x)+\int_x^s g''(u)(s-u)\mathrm{d}u. \tag{3.8}$$

Applying the operator Ω_n to the (3.8) we get

$$|\Omega_n(g;x)-g(x)| \le \left\{|\Omega_n((t-x);x)|+\tfrac{1}{2}\Omega_n((t-x)^2;x)\right\}\|g\|_{C^2[0,a]}. \tag{3.9}$$

On the other hand, since Ω_n is a linear operator, we have

$$|\Omega_n(f;x)-f(x)| \le |\Omega_n(f-g;x)|+|f(x)-g(x)|+|\Omega_n(g;x)-g(x)|.$$

Thus, by using $\Omega_n(1;x)\equiv 1$ and (3.9), we can write

$$\begin{aligned}\|\Omega_n(f;x)-f(x)\| &\le 2\|f-g\|+\{\|\Omega_n((t-x);x)\| \\ &\quad +\tfrac{1}{2}\left\|\Omega_n((t-x)^2;x)\right\|\Big\}\|g\|_{C^2[0,a]}.\end{aligned} \tag{3.10}$$

After some simple calculations, using (3.4) in (3.10) we can write

$$\begin{aligned}\|\Omega_n(f;x)-f(x)\| &\le 2\|f-g\|+2B_3\{(\alpha_n-1)(\alpha_n+2) \\ &\quad +\tfrac{\alpha_n(\alpha_n+1)}{n^2}+\tfrac{\alpha_n(\alpha_n+2)}{n}\Big\}\|g\|.\end{aligned} \tag{3.11}$$

By taking infimum over $g \in C^2\ [0,a]$ on the both sides of (3.11), we obtain (3.7). □

The following theorem estimates the rate of convergence of the sequence $\{\Omega_n\}$ to the function f via Zygmund modulus.

THEOREM 3.3. *If* $f \in C[0,a]$ *then for each* $0 \le \left(\sqrt{B_3}\delta_n\right)^2 \le 1,$ *we have*

$$\|\Omega_n(f;x)-f(x)\| \le C_f \max\left\{w_2(f,\sqrt{B_3}\delta_n),\ \left(\sqrt{B_3}\delta_n\right)^2\right\} \tag{3.12}$$

where B_3 *and* δ_n *are the same as in* (3.2).

Proof. By using the inequality (see Proposition 3.4.1 of [7])

$$K(f,\delta) \le C_1\left(w_2(f,\sqrt{\delta})+\min\{1,\delta\}\|f\|\right)$$

in (3.7) for $\delta=\left(\sqrt{B_3}\delta_n\right)^2$, we have

$$\|\Omega_n(f;x)-f(x)\| \le 2\,C_1\left(w_2(f,\sqrt{B_3}\delta_n)+\left(\sqrt{B_3}\delta_n\right)^2\|f\|\right). \tag{3.13}$$

If $\left(\sqrt{B_3}\delta_n\right)^2 \le w_2(f,\sqrt{B_3}\delta_n)$ then from (3.13), we have

$$\|\Omega_n(f;x)-f(x)\| \le 2C_1(1+\|f\|)\,w_2(f,\sqrt{B_3}\delta_n) \tag{3.14}$$

else if $w_2(f,\sqrt{B_3}\delta_n) < \left(\sqrt{B_3}\delta_n\right)^2$ then from (3.13), we have

$$\|\Omega_n(f;x)-f(x)\| \le 2C_1(1+\|f\|)\left(\sqrt{B_3}\delta_n\right)^2 \tag{3.15}$$

by choosing $C_f := 2C_1(1+\|f\|)$ in (3.14) and (3.15), we get (3.12). □

We will now study the rate of A-statistical convergence of the positive linear operators Ω_n by means of the elements of the Lipschitz class $Lip_M(\alpha)$, where $M > 0$ and $0 < \alpha \leq 1$.

We recall that a function $f \in C[0,a]$ belongs to $Lip_M(\alpha)$ if the inequality

$$|f(t) - f(x)| \leq M\,|t-x|^{\alpha}\,,\ (t, x \in [0,a],\ 0 < \alpha \leq 1)$$

holds.

THEOREM 3.4. *For all $f \in Lip_M(\alpha)$, we have*

$$\|\Omega_n(f;x) - f(x)\| \leq M B_3^{\alpha/2} \delta_n^{\alpha},$$

where B_3 and δ_n are the same as in (3.2).

Proof. Let $f \in Lip_M(\alpha)$ and $0 < \alpha \leq 1$. By linearity and monotonicity of Ω_n we have

$$\begin{aligned} |\Omega_n(f;x) - f(x)| &\leq \Omega_n(|(f(t) - f(x)|\,;x) \\ &= \tfrac{1}{\varphi_n(x)} \sum_{k=0}^{\infty} \left| f\left(\tfrac{k}{\rho_{n,k}+\beta_k}\right) - f(x)\right| \varphi_n^{(k)}(0)\tfrac{x^k}{k!} \\ &\leq \tfrac{M}{\varphi_n(x)} \sum_{k=0}^{\infty} \left|\tfrac{k}{\rho_{n,k}+\beta_k} - x\right|^{\alpha} \varphi_n^{(k)}(0)\tfrac{x^k}{k!}. \end{aligned}$$

Applying the Hölder inequality with $p = \dfrac{2}{\alpha}$, $q = \dfrac{2}{2-\alpha}$ we get

$$\begin{aligned} |\Omega_n(f;x) - f(x)| &\leq M\left[\tfrac{1}{\varphi_n(x)} \sum_{k=0}^{\infty} \left(\tfrac{k}{\rho_{n,k}+\beta_k} - x\right)^2 \varphi_n^{(k)}(0)\tfrac{x^k}{k!}\right]^{\frac{\alpha}{2}} \\ &\quad\times\left[\tfrac{1}{\varphi_n(x)} \sum_{k=0}^{\infty} \varphi_n^{(k)}(0)\tfrac{x^k}{k!}\right]^{\frac{2-\alpha}{2}} \\ &= M(A_n(x))^{\alpha/2}, \end{aligned}$$

where $A_n(x)$ is given by (3.3). Combining this with (3.5) that

$$\|\Omega_n(f;x) - f(x)\| \leq M B_3^{\alpha/2} \delta_n^{\alpha}$$

whence the result. □

4. A GENERALIZATION OF r-TH ORDER OF THE OPERATORS Ω_N

By $C^{[r]}[0,a]$ $(0 < a < 1,\ r = 0,1,2,...)$ we denote the set of the functions f having the continuous r-th derivative $f^{(r)}$ $(f^{(0)}(x) = f(x))$ on the segment $[0,a]$ (see [22], and also [10]).

We consider the following generalization of the positive linear operators Ω_n defined by (2.5)

$$\Omega_{n,r}(f;x)=\frac{1}{\varphi_n(x)}\sum_{k=0}^{\infty}\sum_{i=0}^{r}f^{(i)}\left(\frac{k}{\rho_{n,k}+\beta_k}\right)\left(x-\frac{k}{\rho_{n,k}+\beta_k}\right)^i\frac{1}{i!}\varphi_n^{(k)}(0)\frac{x^k}{k!}, \tag{4.1}$$

where $f\in C^{[r]}[0,a]$, $r=0,1,2,...$, and $n\in\mathbb{N}$. We call the operators (4.1) the r-th order of the operators Ω_n (see, for instance, [10], [22]). Note that taking $r=0$ we get the sequence $\{\Omega_n\}$ defined by (2.5).

Now we have the following

Theorem 4.1. *For all $f\in C^{[r]}[0,a]$ such that $f^{(r)}\in Lip_M(\alpha)$, we get*

$$\|\Omega_{n,r}(f;x)-f(x)\|\le\frac{M}{(r-1)!}\frac{\alpha}{\alpha+r}B(\alpha,r)\left\|\Omega_n(|t-x|^{r+\alpha};x)\right\|, \tag{4.2}$$

where $B(\alpha,r)$ is the beta function and $r,n\in\mathbb{N}$.

Proof. By (4.1) we get

$$\begin{aligned}f(x)-\Omega_{n,r}(f;x)=&\left[f(x)-\sum_{i=0}^{r}f^{(i)}\left(\frac{k}{\rho_{n,k}+\beta_k}\right)\left(x-\frac{k}{\rho_{n,k}+\beta_k}\right)^i\frac{1}{i!}\right]\\&\times\varphi_n^{(k)}(0)\frac{x^k}{k!}.\end{aligned} \tag{4.3}$$

It is known from Taylor's formula that

$$\begin{aligned}&f(x)-\sum_{i=0}^{r}f^{(i)}\left(\frac{k}{\rho_{n,k}+\beta_k}\right)\left(x-\frac{k}{\rho_{n,k}+\beta_k}\right)^i\frac{1}{i!}=\\&=\frac{1}{(r-1)!}\left(x-\frac{k}{\rho_{n,k}+\beta_k}\right)^r\int_0^1(1-t)^{r-1}\left\{f^{(r)}\left(\frac{k}{\rho_{n,k}+\beta_k}\right.\right.\\&\quad\left.\left.+t\left(x-\frac{k}{\rho_{n,k}+\beta_k}\right)\right)-f^{(r)}\left(\frac{k}{\rho_{n,k}+\beta_k}\right)\right\}\mathrm{d}t.\end{aligned} \tag{4.4}$$

Because of $f^{(r)}\in Lip_M(\alpha)$ one can get

$$\begin{aligned}&\left|f^{(r)}\left(\frac{k}{\rho_{n,k}+\beta_k}+t\left(x-\frac{k}{\rho_{n,k}+\beta_k}\right)\right)-f^{(r)}\left(\frac{k}{\rho_{n,k}+\beta_k}\right)\right|\le\\&\le Mt^\alpha\left|x-\frac{k}{\rho_{n,k}+\beta_k}\right|^\alpha.\end{aligned} \tag{4.5}$$

From the well known expression of the beta function we can write

$$\int_0^1(1-t)^{r-1}t^\alpha\mathrm{d}t=B(1+\alpha,r)=\frac{\alpha}{\alpha+r}B(\alpha,r). \tag{4.6}$$

Now by using (4.5) and (4.6) in (4.4) we conclude that

$$\begin{aligned}&\left|f(x)-\sum_{i=0}^{r}f^{(i)}\left(\frac{k}{\rho_{n,k}+\beta_k}\right)\left(x-\frac{k}{\rho_{n,k}+\beta_k}\right)^i\frac{1}{i!}\right|\le\\&\le\frac{M}{(r-1)!}\frac{\alpha}{\alpha+r}B(\alpha,r)\left|x-\frac{k}{\rho_{n,k}+\beta_k}\right|^{r+\alpha}.\end{aligned} \tag{4.7}$$

Taking into consideration (4.3) and (4.7) we have (4.2). □

Now consider the function $g \in C[0,a]$ defined by

$$g(t) = |t - x|^{r+\alpha} \tag{4.8}$$

Since $g(x) = 0$, Theorem 2.3 yields

$$st_A - \lim_n \|\Omega_n(g;x)\| = 0.$$

So, it follows from Theorem 4.1 that, for all $f \in C^{[r]}[0,a]$ such that $f^{(r)} \in Lip_M(\alpha)$, we have

$$st_A - \lim_n \|\Omega_{n,r}(f;x) - f(x)\| = 0.$$

Finally, taking into consideration Theorems 3.1 one can deduce the following result from Theorem 4.1 immediately.

Corollary 4.2. *For all $f \in C^{[r]}[0,a]$ such that $f^{(r)} \in Lip_M(\alpha)$, we have*

$$\|\Omega_{n,r}(f;x) - f(x)\| \leq \tfrac{M}{(r-1)!}\tfrac{\alpha}{\alpha+r} B(\alpha,r)(1 + B_3^{1/2}) w(g,\delta_n),$$

where B_3 and δ_n are the same as in (3.2) *and g is defined by* (4.8)*.*

The last result gives us the rate of A-statistical convergence of the sequence $\{\Omega_{n,r}(f)\}$ by means of the modulus of continuity and the elements of the Lipschitz class $Lip_M(\alpha)$, respectively.

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