BILATERAL APPROXIMATIONS OF THE ROOTS OF SCALAR EQUATIONS BY LAGRANGE-AITKEN-STEFFENSEN METHOD OF ORDER THREE

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Abstract. We study the monotone convergence of two general methods of Aitken-Steffenssen type. These methods are obtained from the Lagrange inverse interpolation polynomial of degree two, having controlled nodes. The obtained results provide information on controlling the errors at each iteration step.

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1. INTRODUCTION

It is well known that the Steffensen, Aitken, and Aitken-Steffensen methods are obtained from the inverse Lagrange interpolation polynomial of degree one, with controlled nodes [8]–[12], [6]. Consider the equation

\[ f(x) = 0 \]

where \( f : [a, b] \to \mathbb{R}, a, b \in \mathbb{R}, a < b \).

We also consider the following three equations, each of them equivalent with equation \( f(x) = 0 \):

\[ x = g(x), \quad g : [a, b] \to [a, b] \]

and

\[ x = g_1(x), \quad g_1 : [a, b] \to [a, b], \]

\[ x = g_2(x), \quad g_2 : [a, b] \to [a, b]. \]

The Steffensen method in given by relations

\[ x_{n+1} = x_n - \frac{f(x_n)}{f(x_n) + f(g(x_n); z)} \quad x_0 \in [a, b], \quad n = 0, 1, 2, \ldots, \]

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and, analogously, the Aitken method is of the following form:

\[(1.5) \quad x_{n+1} = g_1(x_n) - \frac{f(g_1(x_n))}{g_1(x_n) - g_2(x_n)} \quad x_0 \in [a, b], \quad n = 0, 1, 2, \ldots \]

Finally, the Aitken-Steffensen method is given by the relations:

\[(1.6) \quad x_{n+1} = g_1(x_n) - \frac{f(g_1(x_n))}{g_1(x_n) - g_2(g_1(x_n))} \quad x_0 \in [a, b], \quad n = 0, 1, 2, \ldots \]

The order of convergence of all three methods (1.4)–(1.6) is at least two, and this order depends on the functions \(g\) and \(g_1, g_2\) respectively. Essentially, the methods (1.4)–(1.6) are obtained from the method of chord where the interpolation nodes depend on the functions \(g\) and respectively \(g_1, g_2\). In papers [1], [8], [9], [12] some conditions had to be considered in order that all the three methods (1.4)–(1.6) generate two sequences \((u_n)_{n \geq 0}\) and \((v_n)_{n \geq 0}\) with the properties:

\(\alpha\) The sequence \((u_n)_{n \geq 0}\) is increasing and the sequence \((v_n)_{n \geq 1}\) is decreasing:

\(\beta\) \(\lim u_n = \lim v_n = \bar{x}, \) where \(\bar{x}\) is the root of equation (1.1), \(\bar{x} \in [a, b]\).

Practically, such sequences are very interesting, because by inequalities

\[\max \{\bar{x} - u_n, v_n - \bar{x}\} \leq v_n - u_n, \quad n = 0, 1, 2, \ldots,\]

the errors of approximation at every step of iteration may be controlled.

Let \(a_1, a_2, a_3 \in [a, b]\) be three nodes of interpolation, and let \(b_1, b_2, b_3\) the values of the function \(f\), i.e.

\[b_1 = f(a_1), \quad b_2 = f(a_2), \quad b_3 = f(a_3).\]

Suppose that the function \(f : [a, b] \to F\), is bijective where \(F = f([a, b])\).

Then there exists \(f^{-1} : F \to [a, b]\) and the following equality holds [12], [14]:

\[(1.7) \quad f^{-1}(y) = a_1 + [b_1, b_2; f^{-1}](y - b_1) + [b_1, b_2, b_3; f^{-1}](y - b_1)(y - b_2) + [y, b_1, b_2, b_3; f^{-1}](y - b_1)(y - b_2)(y - b_3),\]

for every \(y \in F\).

If \(\bar{x} \in [a, b]\) is the root of equation (1.1), then \(\bar{x} = f^{-1}(0)\), and by (1.7) one obtains the following representation of \(\bar{x}\):

\[(1.8) \quad \bar{x} = a_1 - [b_1, b_2; f^{-1}]b_1 + [b_1, b_2, b_3; f^{-1}]b_1b_2 - [0, b_1, b_2, b_3; f^{-1}]b_1b_2b_3.\]

By relations (see [12])

\[\frac{1}{[a_1, a_2; f]}\]
and
\[ [b_1, b_2, b_3; f^{-1}] = -\frac{[a_1, a_2, a_3; f]}{[a_1, a_2; f][a_1, a_3; f][a_2, a_3; f]}, \]
using (1.8), one obtains the following approximation for \( \bar{x} \):
\[ a_4 = a_1 - \frac{f(a_1)}{[a_1, a_2; f]} - \frac{[a_1, a_2, a_3; f]f(a_1)f(a_2)}{[a_1, a_2; f][a_1, a_3; f][a_2, a_3; f]}, \]
and the error:
\[ \bar{x} - a_4 = -[0, b_1, b_2, b_3; f^{-1}]f(a_1)f(a_2)f(a_3). \]
If \( f \in C^3([a, b]) \) and \( f'(x) \neq 0, \forall x \in [a, b] \), then \( f^{-1} \in C^3(F) \) and the following equality holds (see [12, 14]):
\[ [f^{-1}(y)]''' = \frac{3f''(x)^2 - f'(x)f'''(x)}{f'(x)^3}, \]
where \( y = f(x) \).

Using the mean value formula for divided differences (see [12]) it follows that there exists \( \eta \in F \) such that
\[ [0, b_1, b_2, b_3; f^{-1}] = \frac{(f^{-1})'''(\eta)}{6}. \]
Because \( f \) is bijective, it follows that there exists \( \xi \in [a, b] \) such that \( \eta = f(\xi) \), and by (1.11) and (1.12) one obtains:
\[ [0, b_1, b_2, b_3; f^{-1}] = \frac{3f''(\xi)^2 - f'(\xi)f'''(\xi)}{6[f'(\xi)]^3}. \]
By (1.9), if one considers particular nodes \( a_1, a_2, a_3 \) it is possible to obtain different methods of Steffensen type, of Aitken type or of Aitken-Steifensenn type.

Let \( x_n \in [a, b] \) be an approximation of the root \( \bar{x} \) of equation (1.1).
If one considers \( a_1 = x_n, a_2 = g(x_n), a_3 = g(g(x_n)) \), then it follows the following method of Steffensen type [8, 9, 12]:
\[ x_{n+1} = x_n - \frac{f(x_n)}{[x_n, g(x_n); f]} - \frac{[x_n, g(x_n), g(g(x_n)); f]f(x_n)f(g(x_n))}{[x_n, g(x_n); f][x_n, g(g(x_n)); f][g(x_n), g(g(x_n)); f]}, \]
\( x_0 \in [a, b], n = 0, 1, 2, \ldots \).
If \( a_1 = x_n, a_2 = g_1(x_n), a_3 = g_2(g_1(x_n)) \), then one obtains the following method of Aitken-Steffensen type:
\[ x_{n+1} = x_n - \frac{f(x_n)}{(x_n, g_1(x_n); f)} - \frac{[x_n, g_1(x_n), g_2(g_1(x_n)); f]f(x_n)f(g_1(x_n))}{[x_n, g_1(x_n); f][x_n, g_2(g_1(x_n)); f][g_1(x_n), g_2(g_1(x_n)); f]}, \]
Using the symmetry of Lagrange polynomial with respect to nodes, by permutations of $a_1, a_2, a_3$ in methods (1.14)–(1.16), one obtains the same results for $x_{n+1}$.

In [14] conditions are given in order that method (1.14) generates sequences approximating the root of equation (1.1) bilaterally. In this paper we study methods (1.15) and (1.16) and we obtain same conditions in order that the sequences generated by there methods are bilateral approximations of the root $\bar{x}$ of equations (1.1).

2. THE CONVERGENCE OF AITKEN-STEFFENSEN METHOD

In the following we study the method (1.15) and we search the conditions on the sequences $(x_n)_{n \geq 0}$, $(g_1(x_n))_{n \geq 0}$ and $(g_2(g_1(x_n)))_{n \geq 0}$ generated from this method in order that they are monotonic sequences, bilaterally approximating the root $\bar{x}$ of equation (1.1).

We need the following hypothesis:

a) $g_1$ is increasing on $[a, b]$;

b) $g_2$ is continuous and decreasing on $[a, b]$;

c) equation (1.1) has a solution $\bar{x} \in [a, b]$ and $g_1(\bar{x}) = g_2(\bar{x}) = \bar{x}$,

d) function $f$ is in $C^3([a, b])$, and for every $x \in [a, b]$ the following relation is fulfilled:

\[
3[f''(x)]^2 - f'(x)f'''(x) < 0;
\]
e) function $g_1$ satisfies inequality

\[|g_1(x) - g_1(\bar{x})| \leq L |x - \bar{x}|, \forall x \in [a, b],\]

where $0 < L < 1$.

Concerning the convergence of sequence $(x_n)_{n \geq 0}$ generated by (1.15), the following Theorem holds:

**Theorem 2.1.** Let $x_0 \in [a, b]$ and $f, g_1, g_2$ verify the following conditions:

i) $f$ is increasing on $[a, b]$;

ii) $f$ is convex on $[a, b]$;

iii) functions $g_1, g_2$ and $f$ verify hypotheses a)–e);

iv) $x_0 > \bar{x}$ and $g_2(g_1(x_0)) > a$.
Then the sequences \((x_n)_{n \geq 0}\), \((g_1(x_n))_{n \geq 0}\) and 
\((g_2(g_1(x_n)))_{n \geq 0}\) generated by (1.15) verify the properties:

1. \((x_n)_{n \geq 0}\) and 
\((g_1(x_n))_{n \geq 0}\) are decreasing and bounded from below by \(x\):

2. \((g_2(g_1(x_n)))_{n \geq 0}\) is increasing and bounded from above by \(\bar{x}\);

3. At every iteration step the following inequalities hold:

\[
(2.2) \quad x_n - \bar{x} \leq x_n - g_2(g_1(x_n)), n = 0, 1, \ldots;
\]

\[
(2.3) \quad f(x_0) > 0, \quad f(g_1(x_0)) < 0,
\]

and

\[
f(g_2(g_1(x_0))) < 0,
\]

and by considering \(i_1\) and \(ii_1\) for \(n = 0\) in (1.15) it follows that \(x_1 < x_0\). Using (1.13), by hypotheses \(d)\) and (1.10) for \(f_1 = x_1 = b_1 = f(x_0), b_2 = f(g_1(x_0)), b_3 = f(g_2(g_1(x_0)))\) it follows \(x - x_1 < 0\), i.e. \(x_1 > \bar{x}\). By hypotheses \(a)\) and \(x_1 < x_0\) it follows \(g_1(x_1) < g_1(x_0)\) and then, by \(b)\) one obtains \(g_2(g_1(x_1)) \geq g_2(g_1(x_0))\), and \(g_2(g_1(x_1)) < \bar{x}\). Let \(x_m, m \in \mathbb{N}\) be an element of sequence \((x_n)_{n \geq 0}\) generated by (1.15) and suppose that \(x_m > \bar{x}\). Then one obtains the relations:

\[
(2.3) \quad g_2(g_1(x_m)) < g_2(g_1(x_{m+1})) < \bar{x} < g_1(x_{m+1}) < x_{m+1} < g_1(x_m) < x_m.
\]

Inequality \(x_{m+1} < g_1(x_m)\) follows from equality:

\[
x_{m+1} = x_m - \frac{f(x_m)}{\left[ x_m, g_1(x_m); f \right]}
\]

\[
- \frac{f(x_m)}{\left[ x_m, g_1(x_m); f \right]} f(x_m) f(g_1(x_m))
\]

\[
\frac{f(x_m)}{\left[ x_m, g_1(x_m); f \right]} \left[ x_m, g_1(x_m); f \right] f(g_1(x_m)) f(g_2(g_1(x_m))) f(g_1(x_0))
\]

\[
\frac{f(x_m)}{\left[ x_m, g_1(x_m); f \right]} \left[ x_m, g_1(x_m); f \right] f(g_1(x_m)) f(g_2(g_1(x_m))) f(g_1(x_0))
\]

and from the hypothesis of the theorem.

From relations (2.3) it follows (2.2) and conclusions \(j_1\) and \(ii_1\). Conclusion \(jv_1\) is obvious. The theorem is proved.

**Remark 2.2.** If function \(f\) is concave and decreasing on \([a, b]\), then function \(h: [a, b] \rightarrow \mathbb{R}\) defined by \(h(x) = -f(x)\) in convex and increasing. The relation (2.1) is also verified for \(h\). Consequently the sequence generated by (1.15) for function \(h\), verifies all the conditions of Theorem 2.1 and then the conclusions of this theorem hold.
An analogous proof with that from Theorem 2.1 is valid for the following:

**Theorem 2.3.** Let \( x_0 \in [a, b] \) and \( f, g_1, g_2 \) verify the following conditions:

i) \( f \) is increasing on \([a, b]\);

ii) \( f \) is concave on \([a, b]\);

iii) \( g_1, g_2 \) and \( f \) verify the hypotheses \( a) - e) \);

iv) \( x_0 < \bar{x} \) and \( g_2(g_1(x_0)) \leq b \).

Then sequences \((x_n)_{n \geq 0}, (g_1(x_n))_{n \geq 0}\) and \((g_2(g_1(x_n)))_{n \geq 0}\) generated by (1.15) have the following properties:

j) \( (x_n)_{n \geq 0} \) and \((g_1(x_n))_{n \geq 0}\) are decreasing and bounded from above by \( \bar{x} \);

jj) \( (g_2(g_1(x_n)))_{n \geq 0} \) is increasing and bounded from below by \( \bar{x} \);

jjj) the following relations hold:

\[
2.4 \quad \bar{x} - x_n \leq g_2(g_1(x_n)) - x_n, \quad n = 0, 1, 2, \ldots,
\]

jv) \( \lim_{n \to \infty} x_n = \lim_{n \to \infty} g_1(x_n) = \lim_{n \to \infty} g_2(g_1(x_n)) = \bar{x} \).

**Remark 2.4.** If function \( f \) is decreasing and convex, then \( h_1 : [a, b] \to \mathbb{R} \) defined by \( h_1(x) = -f(x) \) is increasing and concave, \( h_1 \) verifies all hypotheses of Theorem 2.3 and consequently all the conclusions j2 - jv2 hold.

### 3. Convergence of Aitken Type Method

In order to study the convergence of the sequences generated by (1.16) we must suppose that functions \( f, g_1 \) and \( g_2 \) verify hypotheses \( a) - e) \) of section 2.

The following theorem holds:

**Theorem 3.1.** Let \( x_0 \in [a, b] \) and the functions \( f, g_1 \) and \( g_2 \) verify the conditions:

i) \( f \) is increasing an \([a, b]\);

ii) \( f \) is convex an \([a, b]\);

iii) \( g_1 \) and \( g_2 \) verifies the hypotheses \( a) - e) \);

iv) \( x_0 > \bar{x} \) and \( g_2(x_0) > a \).

Then sequences \((x_n)_{n \geq 0}, (g_1(x_n))_{n \geq 0}\) and \((g_2(x_n))_{n \geq 0}\) generated by (1.16) have the properties:

j) \( (x_n)_{n \geq 0} \) and \((g_1(x_n))_{n \geq 0}\) are decreasing and bounded from above by \( \bar{x} \);

jj) \( (g_2(x_n))_{n \geq 0} \) is decreasing and bounded from below by \( \bar{x} \);

jjj) the following relations hold:

\[
x_n - \bar{x} \leq x_n - g_2(x_n), \quad n = 0, 1, \ldots;
\]

jv) \( \lim_{n \to \infty} x_n = \lim_{n \to \infty} g_1(x_n) = \lim_{n \to \infty} g_2(x_n) = \bar{x} \).
Proof. Let be \( x_m \in [a, b], x_m > \bar{x} \) and \( g_2(x_m) > a \), where \( m \in \mathbb{N} \). By a) it follows that \( g_1(x_m) > \bar{x} \) and by b) \( g_2(x_m) < \bar{x} \). Using e) one obtains \( g_1(x_m) < x_m \). By hypothesis \( i_3 \) and \( ii_3 \), and by the above relations and (1.16), for \( n = m \) it follows \( x_{m+1} < x_m \).

For \( a_1 = x_m, a_2 = g_1(x_m) \) and \( a_3 = g_2(x_m) \) in (1.10), by \( i_3 \) and hypothesis d) it follows that \( x_{m+1} > \bar{x} \). Observe that \( x_{m+1} \) in (1.16) may be expressed in the following way:

\[
x_{m+1} = g_1(x_m) - \frac{f(g_1(x_m))}{[g_1(x_m), x_m; f]} - \frac{[x_m, g_1(x_m), g_2(x_m); f] f(g_1(x_m)) f(x_m)}{[x_m, g_1(x_m); f][x_m, g_2(x_m); f][g_1(x_m), g_2(x_m); f]}
\]

and then \( x_{m+1} < g_1(x_m) \). By relation \( x_{m+1} < x_m \) it follows that \( g_2(x_{m+1}) > g_2(x_m) \). Consequently it follows that for every \( m \in \mathbb{N} \), the following relations hold:

\[
g_2(x_m) < g_2(x_{m+1}) < \bar{x} < g_1(x_{m+1}) < x_{m+1} < g_1(x_1) < x_m.
\]

By these relations conclusions \( iii_3 \) and \( iv_3 \) also follow. \( \square \)

Remark 3.2. If function \( f \) is decreasing and concave, then function \( h(x) = -f(x), x \in [a, b] \) verifies all the hypothesis of Theorem 3.1 and consequently the sequence \((x_n)_{n \geq 1}\) generated by (1.16) verifies all the conclusions in Theorem 3.1.

Analogously, the following result can be proved

Theorem 3.3. Let \( x_0 \in [a, b] \) and functions \( f, g_1, g_2 \) have the following properties:

- \( i_4 \) \( f \) is increasing on \([a, b]\);
- \( ii_4 \) \( f \) is concave on \([a, b]\);
- \( iii_4 \) \( f, g_1, \) and \( g_2 \) verify hypothesis a)-e);
- \( iv_4 \) \( x_0 < \bar{x} \) and \( g_2(x_0) \leq b \).

Then sequences \((x_n)_{n \geq 0}, (g_1(x_n))_{n \geq 0}\) and \((g_2(x_n))_{n \geq 0}\) generated by (1.16) verify the properties:

- \( i_4 \) \( (x_n)_{n \geq 0} \) and \((g_1(x_n))_{n \geq 0}\) are increasing and bounded from above by \( \bar{x} \);
- \( ii_4 \) sequence \((g_2(x_n))_{n \geq 0}\) is decreasing and bounded from below by \( \bar{x} \);
- \( iii_4 \) the following relations hold:

\[
\bar{x} - x_n \leq g_2(x_n) - x_n, n = 0, 1, 2, \ldots,
\]

- \( iv_4 \) \( \lim x_n = \lim g_1(x_n) = \lim g_2(x_n) = \bar{x} \).

Remark 3.4. If function \( f \) is decreasing and convex, then function \( h(x) = -f(x), x \in [a, b] \) is increasing and concave. It follows that sequence \((x_n)_{n \geq 0}\) generated by (1.16) verifies the hypotheses and all the conclusions of Theorem 3.3.
4. THE DETERMINATION OF THE AUXILIARY FUNCTIONS

In the following, for every situation concerning the monotonicity and convexity of function $f$, functions $g_1$ and $g_2$ can be determined such that conditions a), b), c) and e) should be verified. In the following we thoroughly present the case in which function $f$ is increasing and convex.

Supposing that $f'(x) > 0$ for every $x \in [a, b]$, one considers the functions:

\begin{align}
  g_1(x) &= x - \frac{f(x)}{f'(b)}, \\
  g_2(x) &= x - \frac{f(x)}{f'(a)}.
\end{align}

Then

\[ g_1'(x) = 1 - \frac{f'(x)}{f'(b)} \geq 0 \]

for every $x \in [a, b]$ and, consequently $g_1$ is increasing. Analogously

\[ g_2'(x) = 1 - \frac{f'(x)}{f'(a)} \leq 0 \]

for every $x \in [a, b]$ and then $g_2$ is a decreasing function. It follows that $g_1$ and $g_2$ verify hypotheses a) and b). The hypothesis e) is also verified because $g_1'(x) < 1$, for every $x \in [a, b]$. In $iv_1$) the inequality $g_2(g_1(x_0)) \geq a$ holds. Because $g_1(x_0) < x_0$ and $x_0 > \bar{x}$, the above inequality is verified if $g_2(x_0) \geq a$. This follows, because $g_2(g_1(x_0))) > g_2(x_0)$ ($g_2$ is a decreasing functions). This means that the following relation must hold:

\[ x_0 - \frac{f(x_0)}{f'(a)} \geq a \]

i.e.

\[ x_0 \geq a + \frac{f(x_0)}{f'(a)}. \]

Because $f(x_0) > 0$ and $f'(a) > 0$, for the veridicity of last inequality it is sufficient that

\[ a + \frac{f(x_0)}{f'(a)} \leq \bar{x}, \]

where $\bar{x}$ is the root of equation (1.1). This last inequality may be realized if $x_0$ is sufficiently close to $\bar{x}$.

Consequently, the hypotheses of theorems 2.1 and 3.1 are realized for $g_1$ and $g_2$ considered above. The other cases may be similarly analyzed.

5. ORDER OF CONVERGENCE

In the following we prove that every method (1.15) and (1.16) have the order of convergence three. The order of convergence for method (1.14) was treated in [14], and this order is at least three. Assume the following hypotheses:
α) function $g_2$ verifies relation

$$|g_2(x) - g_2(\bar{x})| \leq p|x - \bar{x}|,$$

for every $x \in [a, b]$, where $p > 0$, $p \in \mathbb{R}$;

β) $m \leq |f'(x)| \leq M$, for every $x \in [a, b]$, where $m > 0$ and $M > 0$ are real numbers.

γ) $|3[f'(x)]^2 - f'(x)f''(x)| \leq q$, for every $x \in [a, b]$, where $q > 0$, $q \in \mathbb{R}$.

In hypotheses α), β), γ) and e), by (1.10) and (1.13), for sequence $(x_n)_{n \geq 0}$ generated by (1.15), one obtains:

$$|\bar{x} - x_{n+1}| \leq \frac{qM^3L^2p}{6m^3} |\bar{x} - x_n|^3, n = 0, 1, 2, \ldots$$

and this means that the order of convergence for method (1.15) is at least three.

With the same hypotheses, for tree sequence $(x_n)_{n \geq 0}$ generated by (1.16) it follows:

$$|\bar{x} - x_{n+1}| \leq \frac{qM^3L^2p}{6m^3} |\bar{x} - x_n|^3, n = 0, 1, 2, \ldots$$

i.e. the order of convergence of (1.16) is also at least three.

REFERENCES


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