REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 35 (2006) no. 2, pp. 183–188
ictp.acad.ro/jnaat

FIXED POINTS AND INTEGRAL INCLUSIONS

ADRIAN PETRUŞEL*

Abstract. The aim of this paper is to present, as applications of some fixed point theorems, existence results for integral equations and inclusions.

MSC 2000. 47H10, 54H25

Keywords. Fixed point, φ-contraction, multivalued operator, integral inclusion.

1. NOTATIONS AND NOTIONS

Throughout this paper, the standard notations and terminologies in nonlinear analysis are used. For the convenience of the reader we recall some of them.

Let (X, d) be a metric space and $f : X \to X$ an operator. By $Fix f := \{x \in X | \ x = f(x)\}$ we will denote the fixed point set of the operator f.

The following concept was introduced by I.A. Rus (see [7]).

Definition 1. *Let $(X, \to)$ be an L-space. An operator $f : X \to X$ is, by definition, a Picard operator if:*

(i) $\operatorname{Fix} f = \{x^*\}$;

(ii) $f^n(x) \to x^*$, *as* $n \to \infty$, *for all* $x \in X$.

We will also use the following symbols:

$\mathcal{P}(X) := \{Y | Y \subset X\}$, $P(X) := \{Y \subset X | \ Y \text{ is nonempty}\}$,

$P_{cl}(X) := \{Y \in P(X) | \ Y \text{ is closed}\}$, $P_{cv}(X) := \{Y \in P(X) | \ Y \text{ is convex }\}$ (for a normed space X).

Let A and B be nonempty subsets of the metric space (X, d). The gap between these sets is

$$D(A, B) = \inf\{d(a, b) | \ a \in A, \ b \in B\}.$$

In particular, $D(x_0, B) = D(\{x_0\}, B)$ (where $x_0 \in X$) is called the distance from the point x_0 to the set B.

The Pompeiu-Hausdorff generalized distance between the nonempty closed subsets A and B of the metric space (X, d) is defined by the following formula:

$$H(A, B) := \max\Big\{ \sup_{a\in A} \inf_{b\in B} d(a, b), \sup_{b\in B} \inf_{a\in A} d(a, b)\Big\}.$$

*"Babeş-Bolyai" University Cluj-Napoca, Department of Applied Mathematics, Kogălniceanu 1, 400084 Cluj-Napoca, Romania, e-mail: petrusel@math.ubbcluj.ro.

It is well-known that if (X, d) is a complete metric space, then the pair $(P_{cl}(X), H)$ is a (generalized) complete metric space.

The symbol $T : X \multimap Y$ means $T : X \to P(Y)$, i. e. T is a multi-valued operator from X to Y. We will denote by $Graf(T) := \{(x, y) \in X \times Y | y \in T(x)\}$ the graph of T. Recall that the multi-valued operator is called closed if $Graf(T)$ is closed in $X \times Y$.

For $T : X \to P(X)$ the symbol $\operatorname{Fix} T := \{x \in X | \ x \in T(x)\}$ denotes the fixed point set of the multi-valued operator T.

A sequence of successive approximations of T starting from $x \in X$ is a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X with $x_0 = x, \ x_{n+1} \in T(x_n)$, for $n \in \mathbb{N}$.

2. THE SINGLE-VALUED CASE

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. Then φ is said to be a comparison function (see [6]) if it is increasing and $\varphi^k(t) \to 0$, as $k \to +\infty$. As consequence, we also have $\varphi(t) < t$, for each $t > 0$, $\varphi(0) = 0$ and φ is continuous in 0.

In particular, $\varphi(t) = at$, (where $a \in [0, 1[$), $\varphi(t) = \frac{t}{1+t}$, and $\varphi(t) = \ln(1 + t)$, $t \in \mathbb{R}_+$ are examples of comparison functions.

Also, a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a strict comparison function (see [6]) if it is strictly increasing and $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$, for each $t > 0$.

The following auxiliary result is known (see Dugundji-Granas [3]).

Lemma 2. *Let (X, d) be a complete metric space and $f : X \to X$. Assume:*

i) *for each $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that if $d(x, f(x)) < \delta$, then $f(B(x; \varepsilon) \subset B(x; \varepsilon)$;*

ii) $\lim\limits_{n\to+\infty} d(f^{n-1}(x_0), f^n(x_0)) = 0$, *for some $x_0 \in X$.*

Then the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges to a fixed point for f.

We start this section with a straightforward modified version of a well-known result (J. Matkowski see [3] pp. 15 and I. A. Rus [6]).

Theorem 3. *Let (X, d) be a complete metric space and $f : X \to X$ such that $d(f(x), f(y)) \leq \varphi(d(x, y))$, for all $x, y \in X$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following assumptions:*

i) *φ is increasing and upper semi-continuous;*

ii) *the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) := t - \varphi(t)$, is strictly increasing and $\lim\limits_{t\to\infty} \psi(t) = \infty$.*

Then:

a) *the operator f is Picard (denote by x_f^* the unique fixed point);*

b) *if $g : X \to X$ is an operator having at least a fixed point $x_g^* \in X$ and there exists $\eta > 0$ such that $d(f(x), g(x)) \leq \eta$, for each $x \in X$, then $d(x_f^*, x_g^*) \leq \psi^{-1}(\eta)$.*

Proof. Let $x_0 \in X$ be arbitrary. Denote $d_n := d(f^{n-1}(x_0), f^n(x_0)), n \in \mathbb{N}^n$. Then $d_n \leq \varphi(d_{n-1}) \leq d_{n-1}, n \in \mathbb{N}^n$. Hence the sequence $(d_n)_{n\in\mathbb{N}^n}$ converges to a certain element $d \in \mathbb{R}_+$.

Suppose $d > 0$. Then $\psi(d) > \psi(0) = 0$. So $d > \varphi(d)$. On the other hand, since $d_n \leq \varphi(d_{n-1})$ and using the upper semi-continuity of φ, we get that $d \leq \limsup_{n\to+\infty} \varphi(d_{n-1}) \leq \varphi(d)$. The contradiction proves that $d = 0$. Hence $\lim_{n\to+\infty} d(f^{n-1}(x_0), f^n(x_0)) = 0$, for each $x_0 \in X$ (This property is usually called the asymptotic regularity of f).

Let $\varepsilon > 0$ and define $\delta(\varepsilon) := \varepsilon - \varphi(\varepsilon)$. Suppose $d(x, f(x)) < \delta$ and let $z \in B(x; \varepsilon)$. We have:
$d(f(z), x) \leq d(f(z), f(x)) + d(f(x), x) \leq \varphi(d(x, z)) + \delta \leq \varepsilon$. The conclusion follows from Lemma 2.

For the second conclusion observe that
$d(x_f^*, x_g^*) \leq d(f(x_f^*), f(x_g^*)) + d(f(x_g^*), g(x_g^*)) \leq \varphi(d(x_f^*, x_g^*)) + \eta$. Hence $d(x_f^*, x_g^*) \leq \psi^{-1}(\eta)$. □

As an application of the above result we can prove the following

Theorem 4. *Consider the integral equation:*

$$(1) \quad x(t) = \int_a^b K(t, s, x(s))\mathrm{d}s + g(t), \ t \in [a, b].$$

Suppose:

i) *$K : [a, b] \times [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : [a, b] \to \mathbb{R}^n$ are continuous;*

ii) *there exist a continuous function $p : [a, b]\times[a, b] \to \mathbb{R}_+$ and an increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) := t - \varphi(t)$ is strictly increasing and $\lim_{t\to\infty} \psi(t) = \infty$, such that*

$$\|K(t, s, u) - K(t, s, v)\| \leq p(t, s)\varphi(\|u - v\|), \text{ for each } t, s \in [a, b], \ u, v \in \mathbb{R}^n.$$

iii) $\sup_{t\in[a,b]} \int_a^b p(t, s)\mathrm{d}s \leq 1.$

Then:

a) *the integral equation* (1) *has an unique solution x^* in* $C([a, b], \mathbb{R}^n)$;

b) *if*

$$(2) \quad y(t) = \int_a^b L(t, s, y(s))\mathrm{d}s + h(t), \ t \in [a, b]$$

is another integral equation having at least one solution $y^* \in C([a, b], \mathbb{R}^n)$ and there are $\eta_1, \eta_2 > 0$ such that $\|K(t, s, u) - L(t, s, u)\| \leq \eta_1$ and $\|g(t) - h(t)\| \leq \eta_2$, for each $t, s \in [a, b]$ and $u \in C([a, b], \mathbb{R}^n)$, then $\|x^* - y^*\| \leq \psi^{-1}(\eta_1 \cdot (b - a) + \eta_2)$.

Proof. Define $A : C([a, b], \mathbb{R}^n) \to C([a, b], \mathbb{R}^n)$, by the formula

$$Ax(t) := \int_a^b K(t, s, x(s))\mathrm{d}s + g(t), \ t \in [a, b].$$

Observe that $\|Ax(t)-Ay(t)\| \leq \int_a^b \|[K(t,s,x(s))-K(t,s,y(s))]\|\mathrm{d}s \leq \int_a^b p(t,s)\|\varphi(\|x(s)-y(s)\|)\mathrm{d}s \leq \varphi(\|x-y\|_C)\int_a^b p(t,s)\mathrm{d}s$. Thus

$$\|Ax-Ay\|_C \leq \varphi(\|x-y\|_C), \text{ for each } x,y \in C([a,b],\mathbb{R}^n)$$

(here $\|\cdot\|_C$ stands for the sup-norm in $C([a,b],\mathbb{R}^n)$. The first conclusion follows from Theorem 3.

For the second conclusion, consider $B: C([a,b],\mathbb{R}^n) \to C([a,b],\mathbb{R}^n)$, given by the formula

$$Bx(t) := \int_a^b L(t,s,x(s))\mathrm{d}s + h(t),\ t \in [a,b].$$

Then $\|Ax(t)-Bx(t)\| \leq \int_a^b \|K(t,s,x(s))-L(t,s,x(s))\|\mathrm{d}s + \|g(t)-h(t)\| \leq \eta_1\cdot(b-a)+\eta_2$, for each $t\in[a,b]$. Hence $\|Ax-Bx\|_C \leq \eta_1\cdot(b-a)+\eta_2$.

Then from the second conclusion of Theorem 3 we conclude $\|x^*-y^*\|_C \leq \psi^{-1}(\eta_1\cdot(b-a)+\eta_2)$. □

Example 1. If $g \in C[a,b]$, then the integral equation

$$x(t) = \int_a^b s\cdot\frac{x(s)}{1+x(s)}\mathrm{d}s + g(t), \text{ for each } t\in[a,b]$$

has a unique solution in $C[a,b]$. The conclusion follows by an application of Theorem 4 with $\varphi(t) = \frac{t}{1+t}$.

Example 2. If $g \in C[a,b]$, then the integral equation

$$x(t) = \int_a^b \ln(1+x(s))\mathrm{d}s + g(t), \text{ for each } t\in[a,b]$$

has a unique solution in $C[a,b]$. The conclusion follows by an application of Theorem 4 with $\varphi(t) = \ln(1+t)$.

3. THE MULTI-VALUED CASE

The following result is known in the literature as Wegrzyk's theorem (see [10].

Theorem 5. *Let (X,d) be a complete metric space and $T: X \to P_{cl}(X)$ be such that $H(T(x),T(y)) \leq \varphi(d(x,y))$, for each $x,y\in X$. Assume that $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ is a strict comparison function. Then $\operatorname{Fix} T$ is nonempty and for any $x_0 \in X$ there exists a sequence of successive approximations of T starting from x_0 which converges to a fixed point of T.*

As an application, let us consider the following integral inclusion:

$$(3)\quad x(t) \in \int_a^b K(t,s,x(s))\mathrm{d}s + g(t),\ t\in[a,b].$$

Theorem 6. *Let $K: [a,b]\times[a,b]\times\mathbb{R}^n \to P_{cl,cv}(\mathbb{R}^n)$ and $g: [a,b]\to\mathbb{R}^n$ such that:*

(a) *there exists an integrable function* $M : [a,b] \to \mathbb{R}_+$ *such that for each* $t \in [a,b]$ *and* $u \in \mathbb{R}^n$ *we have* $K(t,s,u) \subset M(s)B(0;1)$*, a.e.* $s \in [a,b]$;

(b) *for each* $u \in \mathbb{R}^n$ $K(\cdot,\cdot,u) : [a,b] \times [a,b] \to P_{cl,cv}(\mathbb{R}^n)$ *is jointly measurable;*

(c) *for each* $(s,u) \in [a,b] \times \mathbb{R}^n$ $K(\cdot,s,u) : [a,b] \to P_{cl,cv}(\mathbb{R}^n)$ *is lower semi-continuous;*

(d) *there is a strict comparison function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that for each* $(t,s) \in [a,b] \times [a,b]$ *and each* $u, v \in \mathbb{R}^n$ *we have that*

$$H(K(t,s,u), K(t,s,v)) \le p(t,s) \cdot \varphi(\|u - v\|)$$

where $p : [a,b] \times [a,b] \to \mathbb{R}_+$ *is a continuous function and*

$$\sup_{t\in[a,b]} \int_a^b p(t,s)\mathrm{d}s \le 1;$$

(e) g *is continuous.*

Then there exists at least one solution for the integral inclusion (3).

Proof. Define the multi-valued operator $T : C([a,b],\mathbb{R}^n) \to \mathcal{P}(C([a,b],\mathbb{R}^n))$ by

$$T(x) := \left\{ v \in C([a,b],\mathbb{R}^n) |\ v(t) \in \int_a^b K(t,s,x(s))\mathrm{d}s + g(t),\ t \in [a,b] \right\}.$$

The proof follows the following steps.

1. $T(x) \in P_{cl}(C([a,b],\mathbb{R}^n))$.

From (e) and Theorem 2 in Rybiński [9] we have that for each $x \in C([a,b],\mathbb{R}^n)$ there exists $k(t,s) \in K(t,s,x(s))$, for all $(t,s) \in [a,b]$, such that $k(t,s)$ is integrable with respect to s and continuous with respect to t. Then

$$v(t) := \int_a^b k(t,s)\mathrm{d}s + g(t),$$

has the property $v \in T(x)$. Moreover, from (a) and (b), via Theorem 8.6.4. in Aubin and Frankowska [1], we get that $T(x)$ is a closed set, for each $x \in C([a,b],\mathbb{R}^n)$.

2. $H(T(x_1), T(x_2)) \le \varphi(\|x_1 - x_2\|)$, for each $x_1, x_2 \in C([a,b],\mathbb{R}^n)$.

Let $x_1, x_2 \in C([a,b],\mathbb{R}^n)$ and $v_1 \in T(x_1)$. Then

$$v_1(t) \in \int_a^t K(t,s,x_1(s))\mathrm{d}s + g(t), \quad t \in [a,b].$$

It follows that

$$v_1(t) = \int_a^b k_1(t,s)\mathrm{d}s + g(t),$$

$t \in [a,b]$, where $k_1(t,s) \in K(t,s,x_1(s))$, $(t,s) \in [a,b] \times [a,b]$.

From (d) we have $H(K(t,s,x_1(s)), K(t,s,x_2(s)) \le p(t,s)\varphi(\|x_1(s) - x_2(s)\|)$ $\le p(t,s)\varphi(\|x_1 - x_2\|)$, so there exists $w \in K(t,s,x_2(s))$ such that $\|k_1(t,s) - w\| \le p(t,s)\varphi(\|x_1 - x_2\|)$, $t, s \in [a,b]$.

Let us define $U : [a,b] \times [a,b] \to \mathcal{P}(\mathbb{R}^n)$, by $U(t,s) = \{w|\ \|k_1(t,s) - w\| \leq p(t,s)\varphi(\|x_1 - x_2\|)\}$. Since the multi-valued operator $V(t,s) := U(t,s) \cap K(t,s,x_2(s))$ is jointly measurable and lower semi-continuous in t there exists $k_2(t,s)$ a selection for V, jointly measurable (and hence integrable in s) and continuous in t. So, $k_2(t,s) \in K(t,s,x_2(s))$ and $\|k_1(t,s) - k_2(t,s)\| \leq p(t,s)\varphi(\|x_1 - x_2\|)$, for each $t,s \in [a,b]$.

Consider $v_2(t) = \int_a^b k_2(t,s)\mathrm{d}s + g(t)$, $t \in [a,b]$. We have:

$$\begin{aligned}\|v_1(t) - v_2(t)\| &\leq \int_a^b \|k_1(t,s) - k_2(t,s)\|\mathrm{d}s \leq \int_a^b p(t,s)\varphi(\|x_1 - x_2\|)\mathrm{d}s \\ &\leq \varphi(\|x_1 - x_2\|).\end{aligned}$$

A similar relation can be obtained by interchanging the roles of x_1 and x_2. So the second step follows.

The conclusion follows from Theorem 5. □

Example 3. If $g \in C[a,b]$, and $K(s,u) := [\frac{u}{2(1+u)}, \frac{u}{1+u}]$ then the integral inclusion

$$x(t) \in \int_a^b K(s,x(s))\mathrm{d}s + g(t), \text{ for each } t \in [a,b]$$

has at least one unique solution in $C[a,b]$. The conclusion follows by an application of Theorem 6 with $\varphi(t) = \frac{t}{1+t}$.

REFERENCES

[1] Aubin, J.-P. and Frankowska, H., *Set-valued analysis*, Birkhauser, Basel, 1990.

[2] Biles, D.C., Robinson, M.P. and Spraker, J.S., *Fixed point approaches to the solution of integral inclusions*, Topol. Meth. Nonlinear Anal., **25**, pp. 297–311, 2005.

[3] Dugundji J. and Granas, A., *Fixed point theory*, Springer-Verlag, Berlin, 2003.

[4] Hu, S. and Papageorgiou, N.S., *Handbook of multivalued analysis, Vol. I and II*, Kluwer Acad. Publ., Dordrecht, 1997 and 1999.

[5] Petruşel, A., *Operatorial inclusions*, House of the Book of Science, 2002.

[6] Rus, I.A., *Generalized contractions*, Cluj University Press, Cluj-Napoca, 2001.

[7] Rus, I.A., *Picard operators and applications*, Scientiae Mathematicae Japonicae, **58**, pp. 191–219, 2003.

[8] Rus, I.A., Petruşel, A. and Sîntămărian, A., *Data dependence of the fixed point set of some multivalued weakly Picard operators*, Nonlinear Anal., **52**, pp. 1947–1959, 2003.

[9] Rybinski, L., *On Carathédory type selections*, Fund. Math., **125**, pp. 187–193, 1985.

[10] Wegrzyk, R., *Fixed point theorems for multifunctions and their applications to functional equations*, Disscus. Math., **201**, 1982.

Received by the editors: April 12, 2006.