REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 35 (2006) no. 2, pp. 189–197
ictp.acad.ro/jnaat

APPROXIMATION PROPERTIES OF MODIFIED STANCU BETA OPERATORS

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Abstract. In this paper we give approximation theorems for modified Stancu beta operators of differentiable functions. The Stancu beta operators were examined in [8, 1, 2, 5] and other papers.

MSC 2000. 41A36, 41A25.

Keywords. Beta operator, degree of approximation, Voronovskaya theorem.

1. INTRODUCTION

1.1. In 1995 D. D. Stancu ([8]) introduced and examined approximation properties of the following beta operators

$$L_n(f;x) := \tfrac{1}{B(nx,n+1)} \int_0^\infty \tfrac{t^{nx-1}}{(1+t)^{nx+n+1}} f(t)\mathrm{d}t, \tag{1.1}$$

$x \in I = (0,\infty)$, $n \in N = \{1,2,...\}$, of real-valued functions f bounded on I, where B is the Euler beta function ([4]) defined by the formula

$$B(a,b) := \int_0^1 t^{a-1}(1-t)^{b-1}\mathrm{d}t \equiv \int_0^\infty \tfrac{t^{a-1}}{(1+t)^{a+b}}\mathrm{d}t, \quad a,b>0. \tag{1.2}$$

In [8] it was proved that if f is continuous and bounded on I, then

$$|L_n(f;x) - f(x)| \le \left(1+\sqrt{x(x+1)}\right)\ \omega_1\left(f;\tfrac{1}{\sqrt{n-1}}\right) \tag{1.3}$$

and

$$|L_n(f;x) - f(x)| \le (3 + x(x+1))\ \omega_2\left(f;\tfrac{1}{\sqrt{n-1}}\right), \tag{1.4}$$

for all $x \in I$ and $n \ge 2$, where $\omega_k(f;\cdot)$, $k = 1,2$, is the modulus of continuity of the order k of f.

It is known ([1], [8]) that if f is continuous and bounded on I with derivatives f' and f'', then

$$\lim_{n\to\infty} n\ (L_n(f;x) - f(x)) = \tfrac{x(x+1)}{2}\ f''(x) \quad \text{at every} \quad x \in I \tag{1.5}$$

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and the order $O\left(\frac{1}{n}\right)$ of approximation of r times differentiable functions f, $r \geq 3$, by L_n cannot be improved.

1.2. In this paper we shall show that the approximation order of differentiable functions by beta operators can be improved by some modification of formula (1.1). We use the Kirov type method given for Bernstein operators in [6] (see also [7]).

Similarly to [3] let $p \in N_0 = N \cup \{0\}$,

$$w_0(x) := 1, \quad w_p(x) := (1 + x^p)^{-1} \quad \text{if} \quad p \geq 1, \;\; x \in I, \tag{1.6}$$

and let C_p be the set of all real–valued functions f defined on I, for which $w_p f$ is uniformly continuous and bounded on I and the norm

$$\|f\|_p \; := \; \sup_{x \in I} \, w_p(x) \, |f(x)| \, . \tag{1.7}$$

We have $C_p \subset C_q$ if $p < q$ and $\|f\|_q \leq \|f\|_p$ for $f \in C_p$.

In this paper we shall consider the functions class C^r, $r \in N_0$, of all $f \in C_r$ having derivatives $f^{(k)} \in C_{r-k}$, $0 \leq k \leq r$. Clearly $C^0 \equiv C_0$.

1.3. Analogously to [8] we denote by

$$b_{u,v}(t) := \tfrac{t^{u-1}}{B(u,v)(1+t)^{u+v}}, \qquad t > 0, \tag{1.8}$$

for positive parameters u and v, where B is the beta function defined by (1.2).

Let $r \in N_0$. For $f \in C^r$ we define the following modified Stancu beta operators:

$$L_{n;r}(f; x) := \int_0^\infty b_{nx,n+1}(t) \, F_r(x,t) \, \mathrm{d}t, \tag{1.9}$$

$x \in I$ and $r \leq n \in N$, where

$$F_r(x,t) := \sum_{j=0}^{r} \tfrac{f^{(j)}(t)}{j!} (x-t)^j, \qquad x, t \in I. \tag{1.10}$$

If $r = 0$ and $f \in C^0$, then by (1.9), (1.10) and (1.1) we have

$$L_{n;0}(f; x) \equiv L_n(f; x) = \int_0^\infty b_{nx,n+1}(t) \, f(t) \, \mathrm{d}t \quad \text{for} \quad x \in I, \;\; n \in N. \tag{1.11}$$

In Section 2 we shall prove that $L_{n;r}$, $r \in N$, $n \geq 2r$, is a linear operator acting from C^r into C_r. Moreover we shall prove that L_n is a linear positive operator acting from C_r into C_r if $n \geq 2r$ and $r \in N$.

1.4. It is known ([1], [8]) that operators $L_n(f)$ given by (1.1) are well defined for functions $f_s(x) = x^s$, $x \in I$, $s \in N$, $n \geq s$, and $L_n(f_s)$ are algebraic polynomials of the order s. In [1] and [8] it was proved that

$$L_n(1; x) \; = \; 1, \qquad L_n(t; x) \; = \; x \quad \text{for} \quad x > 0, \;\; n \geq 1, \tag{1.12}$$

$$L_n\left((t-x)^2; x\right) \; = \; \tfrac{x(x+1)}{n-1}, \quad x > 0, \;\; n \geq 2, \tag{1.13}$$

and for every $s \in N$ there exists a positive constant $M_1(s)$ depending only on s such that

$$w_{2s}(x)\, L_n\left((t-x)^{2s}; x\right) \;\leq\; M_1(s)\, n^{-s} \quad \text{for} \quad x>0, \;\; n \geq 2s. \tag{1.14}$$

2. LEMMAS

In this paper we shall denote by $M_i(\alpha, \beta)$, $i \in N$, suitable positive constants depending only on indicated parameters α and β.

We shall apply the following inequalities

$$(w_p(x))^2 \;\leq\; w_{2p}(x), \quad \tfrac{1}{(w_p(x))^2} \;\leq\; \tfrac{4}{w_{2p}(x)}, \quad \tfrac{w_{p+s}(x)}{w_p(x) w_s(x)} \leq 3, \tag{2.1}$$

for $x > 0$ and $p, s \in N_0$, which can be easily obtained from (1.6).

Applying the Hölder inequality and (1.12), (2.1) and (1.14), we immediately obtain

Lemma 2.1. *For every $s \in N$ there exists $M_2(s) = \text{const.} > 0$ such that*

$$w_s(x)\, L_n(|t-x|^s; x) \leq M_2(s)\, n^{-\frac{s}{2}} \quad \text{for} \quad x > 0, \;\; n \geq 2s.$$

Lemma 2.2. *Let $p, s \in N$. Then there exist positive constants $M_3(p)$ and $M_4(p,s)$ such that*

$$w_p(x)\, L_n\left(\tfrac{1}{w_p(t)}; x\right) \leq M_3(p) \quad \text{for} \quad x>0, \;\; n \geq 2p, \tag{2.2}$$

and

$$w_{p+s}(x)\, L_n\left(\tfrac{|t-x|^s}{w_p(t)}; x\right) \leq M_4(p,s)\, n^{-s/2}, \tag{2.3}$$

for $x > 0$ and $n \geq 2(p+s)$.

Proof. By (1.6) we have

$$\tfrac{1}{w_p(t)} = 1 + t^p \leq 2^p\left(1 + x^p + |t-x|^p\right), \quad t, x \in I, \;\; p \in N,$$

which by (1.1), (1.2) and (1.12) implies that

$$L_n\left(\tfrac{1}{w_p(t)}; x\right) \leq 2^p\left(1 + x^p + L_n\left(|t-x|^p; x\right)\right).$$

Now by (1.6) and Lemma 2.1 follows (2.2).

Applying the Hölder inequality and (1.14), (2.1), (2.2) and (1.6), we get

$$\begin{aligned} w_{p+s}(x)\, L_n\left(\tfrac{|t-x|^s}{w_p(t)}; x\right) &\leq 3\, w_p(x)\, w_s(x)\, L_n\left(\tfrac{|t-x|^s}{w_p(t)}; x\right) \leq \\ &\leq 6\left(w_{2s}(x)\, L_n\left((t-x)^{2s}; x\right)\right)^{1/2}\left(w_{2p}(x)\, L_n\left(\tfrac{1}{w_{2p}(t)}; x\right)\right)^{1/2} \\ &\leq M_4(p,s)\, n^{-s/2} \quad \text{for} \quad x>0, n \geq 2(p+s), \end{aligned}$$

and we complete the proof. □

Lemma 2.3. *Let* $p \in N$. *Then there exists* $M_5(p) = \text{const.} > 0$ *such that for every* $f \in C_p$ *we have*

$$\|L_n(f)\|_p \le M_5(p)\,\|f\|_p \quad \text{for} \quad n \ge 2p. \tag{2.4}$$

If $f \in C_0$, *then* $\|L_n(f)\|_0 \le \|f\|_0$ *for* $n \ge 1$.
The formula (1.1) *and* (2.4) *show that* L_n, $n \ge 2p$, *is a positive linear operator from the space* C_p *into* C_p, $p \in N_0$.

Proof. By (1.1), (1.2), (1.8) and (1.6) we have for $f \in C_p$, $p \in N_0$,

$$w_p(x)\,|L_n(f;x)| \le \|f\|_p\, w_p(x)\, L_n\left(\tfrac{1}{w_p(t)};x\right), \quad x > 0, \;\; n \ge 2p,$$

which by (2.2) and (1.7) yields (2.4). □

Lemma 2.4. *Let* $r \in N$. *Then there exists* $M_6(r) = \text{const.} > 0$ *such that*

$$\|L_{n;r}(f)\|_r \le M_6(r) \sum_{j=0}^{r} \left\|f^{(j)}\right\|_{r-j}, \tag{2.5}$$

for every $f \in C^r$ *and* $n \ge 2r$.

The formulas (1.9) *and* (1.10) *and* (2.5) *show that* $L_{n;r}$, $n \ge 2r$, *is a linear operator from the space* C^r *into* C_r, $r \in N$.

Proof. It is obvious that if $f \in C^r$, $r \in N$, then there exist $\|f^{(j)}\|_{r-j}$ for $0 \le j \le r$. This fact and (1.8)–(1.11) and (1.7) imply that

$$\begin{aligned} |L_{n;r}(f;x)| &\le \sum_{j=0}^{r} \tfrac{1}{j!}\, L_n\left(\left|f^{(j)}(t)\right| |t-x|^j; x\right) \le \\ &\le \sum_{j=0}^{r} \left\|f^{(j)}\right\|_{r-j} L_n\left(\tfrac{|t-x|^j}{w_{r-j}(t)}; x\right) \end{aligned}$$

for $f \in C^r$. Now applying Lemma 2.1 and Lemma 2.2 we obtain

$$w_r(x)\,|L_{n;r}(f;x)| \le M_6(r) \sum_{j=0}^{r} \left\|f^{(j)}\right\|_{r-j} n^{-j/2},$$

for $x > 0$ and $n \ge 2r$, which by (1.7) yields (2.5). □

3. THEOREMS

3.1. First we shall give theorems on the order of approximation of function f by operators L_n and $L_{n;r}$. We shall use the modulus of continuity $\omega_k(f)$ of the order $k = 1, 2$ of $f \in C_p$, i.e.

$$\omega_k(f;C_p;t) := \sup_{0 \le h \le t} \left\|\Delta_h^k f(\cdot)\right\|_p, \qquad t \ge 0, \tag{3.1}$$

where $\Delta_h^1 f(x) \equiv \Delta_h f(x) = f(x+h) - f(x)$ and $\Delta_h^2 f(x) \equiv \Delta_h(\Delta_h f(x)) = f(x) - 2f(x+h) + f(x+2h)$ for $x > 0$ and $h \ge 0$.

THEOREM 3.1. *Suppose that $f \in C_p$, $p \in N_0$, is twice differentiable function on I and f' and f'' belong to C_p also. Then there exists $M_7(p) = \text{const.} > 0$ such that*

$$\|L_n(f) - f\|_{p+2} \;\leq\; M_7(p)\, \|f''\|_p \, n^{-1}, \tag{3.2}$$

for $n \geq 2p + 4$.

Proof. For f satisfying our assumptions we have

$$f(t) = f(x) + f'(x)(t-x) + \int_x^t \int_x^s f''(u)\,\mathrm{d}u\,\mathrm{d}s, \qquad t, x > 0.$$

By elementary calculations we get

$$f(t) = f(x) + f'(x)(t-x) + \int_x^t (t-u) f''(u)\,\mathrm{d}u$$

and next by (1.1) and (1.12) we can write

$$L_n(f(t);x) \;=\; f(x) + L_n\left(\int_x^t (t-u) f''(u)\,\mathrm{d}u; x\right)$$

and

$$|L_n(f(t);x) - f(x)| \;\leq\; L_n\left(\left|\int_x^t (t-u) f''(u)\,\mathrm{d}u\right|; x\right).$$

Using the inequality

$$\begin{aligned}\left|\int_x^t (t-u) f''(u)\mathrm{d}u\right| \;&\leq\; \|f''\|_p \left|\int_x^t \tfrac{|t-u|}{w_p(u)}\,\mathrm{d}u\right| \\ &\leq\; \|f''\|_p \left(\tfrac{1}{w_p(t)} + \tfrac{1}{w_p(x)}\right)(t-x)^2 \quad \text{for} \quad t, x > 0,\end{aligned}$$

and (2.1) we get

$$\begin{aligned} w_{p+2}(x)\,|L_n(f;x) - f(x)| \leq \|f''\|_p \Big(& w_{p+2}(x) L_n\left(\tfrac{(t-x)^2}{w_p(t)}; x\right) \\ &+ 3w_2(x)\, L_n((t-x)^2; x)\Big)\end{aligned}$$

and next by (2.3), (1.6), (1.13) and (1.7) we obtain (3.2) for $n \geq 2p + 4$. □

Now we shall prove analogue of (1.4) for $f \in C_p$.

THEOREM 3.2. *Let $p \in N_0$. Then there exists $M_8(p) = \text{const.} > 0$ such that for every $f \in C_p$ and $n \geq 2p + 4$ we have*

$$\|L_n(f) - f\|_{p+2} \;\leq\; M_8(p)\, \omega_2\left(f; C_p; \tfrac{1}{\sqrt{n}}\right). \tag{3.3}$$

Proof. Similarly to [3] we apply the following Steklov function f_h of $f \in C_p$:

$$f_h(x) \;:=\; \tfrac{4}{h^2} \int_0^{h/2} \int_0^{h/2} (2f(x+s+t) - 2f(x + 2(s+t)))\,\mathrm{d}s\,\mathrm{d}t, \quad x, h > 0.$$

It is known ([3]) that f_h and derivatives f_h', f_h'' belong to C_p if $f \in C_p$. Moreover for $h > 0$ we have

$$\|f - f_h\|_p \;\leq\; \omega_2(f; C_p; h), \tag{3.4}$$

$$\|f_h''\|_p \;\leq\; 9\,h^{-2}\,\omega_2(f;C_p;h). \tag{3.5}$$

From the above properties of f_h and linearity of L_n we deduce that

$$\begin{aligned}|L_n(f(t);x)-f(x)| \;&\leq\; |L_n(f(t)-f_h(t);x)| \\ &\quad+\; |L_n(f_h(t);x)-f_h(x)|+|f_h(x)-f(x)|,\end{aligned}$$

for $x>0$, $h>0$ and $n\geq 2p+2$. Applying (2.4) and (3.4) we get

$$w_p(x)\,|L_n(f(t)-f_h(t);x)| \;\leq\; M_5(p)\|f-f_h\|_p \;\leq\; M_5(p)\,\omega_2(f;C_p;h). \tag{3.6}$$

Theorem 3.1 and (3.5) imply that

$$\begin{aligned}w_{p+2}(x)\,|L_n(f_h(t);x)-f_h(x)| \;&\leq\; M_7(p)\,\|f_h''\|_p\,n^{-1} \\ &\leq\; M_7(p)\,h^{-2}\,n^{-1}\,\omega_2(f;C_p;h).\end{aligned} \tag{3.7}$$

Using (1.6), (2.1), (3.6), (3.7) and (3.4), we get

$$w_{p+2}(x)\,|L_n(f;x)-f(x)| \;\leq\; M_9(p)\,\omega_2(f;C_p;h)\,\left(1+h^{-2}n^{-1}\right) \tag{3.8}$$

for $x,h>0$ and $n\geq 2p+4$. Putting $h=\frac{1}{\sqrt{n}}$ in (3.8), we obtain the desired inequality (3.3) by (1.7). □

Now we shall give an analogue of (1.3) for operators $L_{n;r}$.

THEOREM 3.3. *Let $r\in N$. Then there exists $M_{10}(r)=\text{const.}>0$ such that for every $f\in C^r$ we have*

$$\|L_{n;r}(f)-f\|_{r+1} \;\leq\; M_{10}(r)\,n^{-r/2}\,\omega_1\left(f^{(r)};C_0;\tfrac{1}{\sqrt{n}}\right),\qquad n\geq 2r+2. \tag{3.9}$$

Proof. The formulas (1.8)–(1.12) imply that

$$L_{n;r}(f;x)-f(x) \;=\; L_n\left(F_r(x,t)-f(x);x\right), \tag{3.10}$$

for every $f\in C^r$, $x\in I$ and $n\geq 2r+2$.

Similarly to [6] and [7] we use the following modified Taylor formula of $f\in C^r$ at a fixed point $t>0$:

$$f(x) \;=\; \sum_{j=0}^{r}\tfrac{f^{(j)}(t)}{j!}(x-t)^j+\tfrac{(x-t)^r}{(r-1)!}\,I_r(x,t),\qquad x>0,$$

where

$$I_r(x,t) \;:=\; \int_0^1(1-u)^{r-1}\left(f^{(r)}(t+u(x-t))-f^{(r)}(t)\right)\mathrm{d}u. \tag{3.11}$$

Hence we have

$$f(x)-F_r(x,t) \;=\; \tfrac{(x-t)^r}{(r-1)!}\,I_r(x,t),\qquad t,x>0,$$

which used to (3.10) implies that

$$\begin{aligned}|L_{n;r}(f;x)-f(x)| \;&\leq\; L_n\left(|F_r(x,t)-f(x)|;x\right) \\ &\leq\; \tfrac{1}{(r-1)!}\,L_n(|t-x|^r\,|I_r(x,t)|;x)\end{aligned}$$

for $x>0$ and $n\geq 2r+2$. Applying elementary properties of ω_1 defined by (3.1), we get from (3.11)

$$\begin{aligned}|I_r(x,t)| &\leq \int_0^1 (1-u)^{r-1}\omega_1\left(f^{(r)};C_0;u|x-t|\right)\mathrm{d}u\\ &\leq \omega_1\left(f^{(r)};C_0;|t-x|\right)\int_0^1(1-u)^{r-1}\mathrm{d}u\\ &\leq \tfrac{1}{r}\,\omega_1\left(f^{(r)};C_0;\tfrac{1}{\sqrt{n}}\right)\left(\sqrt{n}|t-x|+1\right).\end{aligned}$$

From the above and by (1.6) and Lemma 2.1, we obtain

$$\begin{aligned}&w_{r+1}(x)\ |L_{n;r}(f;x)-f(x)|\leq\\ &\leq \tfrac{1}{r!}\,\omega_1\left(f^{(r)};C_0;\tfrac{1}{\sqrt{n}}\right)\left(\sqrt{n}\,w_{r+1}(x)\,L_n\left(|t-x|^{r+1};x\right)+w_r(x)\,L_n\left(|t-x|^r;x|\right)\right)\\ &\leq\ M_{10}(r)\,n^{-r/2}\,\omega_1\left(f^{(r)};C_0;\tfrac{1}{\sqrt{n}}\right),\end{aligned}$$

for $x>0$ and $n\geq 2r+2$, which by (1.7) gives (3.9). □

From Theorem 3.2 and Theorem 3.3 we can derive the following corollaries.

Corollary 3.4. *If $f\in C_p$, $p\in N_0$, then*

$$\lim_{n\to\infty}\left(L_n(f;x)-f(x)\right)\ =\ 0\quad\text{at every}\quad x>0.$$

The above convergence is uniform on every interval $[a,b]$, $0<a<b<\infty$.

Corollary 3.5. *If $f\in C^r$, $r\in N$, then*

$$\lim_{n\to\infty} n^{r/2}\ \|L_{n;r}(f)-f\|_{r+1}\ =\ 0.$$

Corollary 3.6. *The order of approximation of $f\in C^r$, $r\geq 2$, by operators $L_{n;r}(f)$ is better than by $L_n(f)$.*

3.2. In this section we shall prove the Voronovskaya type theorem for operators $L_{n;r}$, i.e. we shall give an analogue of (1.5).

Theorem 3.7. *Suppose that $r\in N$ and f is a function belonging to C^r and having derivatives $f^{(r+1)}$ and $f^{(r+2)}$ continuous and bounded on I. Then for every $x>0$ we have*

$$\begin{aligned}L_{n;r}(f;x)-f(x)\ &=\ \tfrac{(-1)^r f^{(r+1)}(x)L_n((t-x)^{r+1};x)}{(r+1)!}\\ &+\ \tfrac{(-1)^r(r+1)f^{(r+2)}(x)L_n((t-x)^{r+2};x)}{(r+2)!}\\ &+\ o_x\left(n^{(r+2)/2}\right)\quad\text{as}\quad n\to\infty.\end{aligned}\tag{3.12}$$

Proof. Fix function f and $x>0$. Then for every derivative $f^{(j)}$, $0\leq j\leq r$, we can write the Taylor formula at x:

$$f^{(j)}(t)\ =\ \sum_{i=0}^{r+2-j}\tfrac{f^{(j+i)}(x)}{i!}\,(t-x)^i+\varphi_j(t,x)(t-x)^{r+2-j},\quad t>0,$$

where $\varphi_j(t) \equiv \varphi_j(t,x)$ is function belonging to C_0 and $\lim_{t\to x} \varphi_j(t) = \varphi_j(x) = 0$. Hence F_r defined by (1.10) can be written in the form

$$F_r(x,t) = \sum_{j=0}^{r} \frac{(-1)^j}{j!} \sum_{i=0}^{r+2-j} \frac{f^{(j+i)}(x)}{i!} (t-x)^{j+i} + \Phi_r(t)(t-x)^{r+2}, \quad t>0,$$

with the function

$$\Phi_r(t) := \sum_{j=0}^{r} \frac{(-1)^j}{j!} \varphi_j(t). \tag{3.13}$$

By elementary calculations we get

$$\begin{aligned} F_r(x,t) &= \sum_{j=0}^{r} (-1)^j \sum_{s=j}^{r+2} \binom{s}{j} \frac{f^{(s)}(x)(t-x)^s}{s!} + \Phi_r(t)(t-x)^{r+2} \\ &= \sum_{s=0}^{r} \frac{f^{(s)}(x)(t-x)^s}{s!} \sum_{j=0}^{s} \binom{s}{j} (-1)^j \\ &\quad + \frac{f^{(r+1)}(x)(t-x)^{r+1}}{(r+1)!} \sum_{j=0}^{r} \binom{r+1}{j} (-1)^j \\ &\quad + \frac{f^{(r+2)}(x)(t-x)^{r+2}}{(r+2)!} \sum_{j=0}^{r} \binom{r+2}{j} (-1)^j + \Phi_r(t)(t-x)^{r+2}. \end{aligned} \tag{3.14}$$

Applying equalities for $m \in N_0$:

$$\sum_{j=0}^{m} \binom{m}{j} (-1)^j = \begin{cases} 1 & \text{if } m=0, \\ 0 & \text{if } m \geq 1, \end{cases}$$

$$\sum_{j=0}^{m} \binom{m+1}{j} (-1)^j = (-1)^m$$

and

$$\sum_{j=0}^{m} \binom{m+2}{j} (-1)^j = (m+1)(-1)^j,$$

we derive from (1.9)–(1.11) and (3.14) the following formula:

$$\begin{aligned} L_{n;r}(f;x) \;=\; & f(x) + \frac{(-1)^r f^{(r+1)}(x)\, L_n\big((t-x)^{r+1};x\big)}{(r+1)!} \\ & + \frac{(-1)^r (r+1) f^{(r+2)}(x)\, L_n\big((t-x)^{r+2};x\big)}{(r+2)!} \\ & + L_n\left(\Phi_r(t)(t-x)^{r+2};x\right), \qquad \text{for} \quad n \geq 2r+2. \end{aligned} \tag{3.15}$$

The properties of φ_j, $0 \leq j \leq r$, imply that Φ_r given by (3.13) is a function belonging to C_0 and $\lim_{t\to x} \Phi_r(t) = \Phi_r(x) = 0$. Hence there exists $L_n(\Phi_r^2(t);x)$

for $x > 0$ and $n \geq 1$ and by Corollary 3.4 we have

$$\lim_{n\to\infty} L_n\left(\Phi_r^2(t);x\right) \;=\; \Phi_r^2(x) \;=\; 0. \tag{3.16}$$

Applying the Hölder inequality, we get

$$\left|L_n\left(\Phi_r(t)(t-x)^{r+2};x\right)\right| \;\leq\; \left(L_n\left(\Phi_r^2(t);x\right)\right)^{1/2}\left(L_n\left((t-x)^{2r+4};x\right)\right)^{1/2}$$

for $n \geq 2r+4$, which by (3.16) and (1.12) gives

$$L_n\left(\Phi_r(t)(t-x)^{r+2};x\right) \;=\; o_x\left(n^{-(r+2)/2}\right) \quad \text{as} \quad n \to \infty. \tag{3.17}$$

Using (3.17) to (3.15), we obtain (3.12). □

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Received by the editors: January 8, 2005.