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THE EQUIVALENCE BETWEEN THE KRASNOSELSKIJ, MANN AND ISHIKAWA ITERATIONS

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Abstract. We shall prove that Krasnoselskij iteration converges if and only if Mann-Ishikawa iteration converges, for certain classes of strongly pseudocontractive mappings.

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1. INTRODUCTION

Let X be a real Banach space $T : X \to X$ a map and $x_0, u_0 \in X$. In [6] is introduced the following iteration

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n T u_n, \tag{1}$$

where $\{\alpha_n\} \subset (0,1)$. The *Krasnoselskij* iteration is defined by, ([5])

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n, \tag{2}$$

where $\lambda \in (0,1)$.

The map $J : X \to 2^{X^*}$ given by $J(x) := \{f \in X^* : \langle x, f\rangle = \|x\|^2, \|f\| = \|x\|\}, \forall x \in X$, is called *the normalized duality mapping.* Note the following inequality

$$\langle y, j(x)\rangle \leq \|x\| \|y\|, \forall x, y \in X, \forall j(x) \in J(x). \tag{3}$$

Definition 1. *Let X be a real Banach space. Let B be a nonempty subset. A map $T : B \to B$ is called strongly pseudocontractive if there exists $k \in (0,1)$ and a $j(x-y) \in J(x-y)$ such that*

$$\langle Tx - Ty, j(x-y)\rangle \leq k\|x-y\|^2, \forall x, y \in B. \tag{4}$$

In [1], the following open question was given: "are Krasnoselskij iteration and Mann iteration equivalent in sense (of [8]) for enough large classes of mappings?" We shall give a positive answer to this question, for the class of strongly pseudocontractive mappings.

We recall the following results from [2], [10] and [7].

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Lemma 2. [2] *If X is a real normed space, then the following relation is true*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \ \forall x, y \in X, \forall j(x+y) \in J(x+y). \tag{5}$$

Lemma 3. [10] *Let $\{a_n\}$ be a nonnegative sequence which satisfies the following inequality*

$$a_{n+1} \leq (1-\lambda_n)a_n + \sigma_n,$$

where $\lambda_n \in (0,1)$, $\forall n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, *and* $\sigma_n = o(\lambda_n)$. *Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 4. [7] *Let X be a smooth Banach space. Suppose that J is uniformly continuous on any bounded subset of X. Then for any $\varepsilon > 0$ and any bounded subset B there is a $\delta > 0$ such that*

$$\|tx + (1-t)\, y\|^2 \leq 2\langle J(y), x\rangle t + 2\varepsilon t + (1-2t)\|y\|^2, \tag{6}$$

for any $x, y \in B$ and $t \in [0, \delta)$.

The following result is a slight generalization of Lemma 1 from [7].

Lemma 5. *Let $\{a_n\}$ be a nonnegative sequence which satisfies*

$$a_{n+1} \leq (1-\lambda_n)a_n + \lambda_n\varepsilon + \lambda_n\delta_n, \forall n \geq n_0,$$

for some fixed n_0 and $\varepsilon > 0$ where $\lambda_n \in (0,1)$, $\forall n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\lim_{n\to\infty} \delta_n = 0$. *Then*

$$0 \leq \lim_{n\to\infty} \sup a_n \leq 2\varepsilon.$$

Proof. Since $\lim_{n\to\infty} \delta_n = 0$, there exists an n_0 such that $\delta_n \leq \varepsilon$, $\forall n \geq n_0$. Thus

$$a_{n+1} \leq (1-\lambda_n)\, a_n + 2\lambda_n\varepsilon, \forall n \geq n_0. \tag{7}$$

Using (7) it can be shown that

$$a_{n+1} \leq (1-\lambda_n)(1-\lambda_{n-1}) \ldots (1-\lambda_1)\, a_1 + 2\varepsilon.$$

Since $\sum \lambda_n = \infty$, the conclusion follows. □

The following Remark is from [3].

Remark 1. *If X is real Banach space with a uniformly convex dual, then $J(\cdot)$ is single valued and uniformly continuous on every bounded set of X.*

2. MAIN RESULTS

Theorem 6. *Let X be a real Banach space with a uniformly convex dual and B a nonempty, closed, convex, bounded subset of X. Let $T : B \to B$ be a continuous and strongly pseudocontractive operator. Suppose that*

$$\lim_{n\to\infty} \alpha_n = 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = \infty. \tag{8}$$

If the Krasnoselskij iteration (2) *converges to the fixed point of* T *and* $\|x_n - Tx_n\| = o(\alpha_n)$*, as* $n \to \infty$*, then the Mann iteration* (1) *converges to the fixed point of* T*.*

Proof. Since T is strongly pseudocontractive, the fixed point is unique. Denote the fixed point by x^*. If the iteration (2) converges to the fixed point of T, then $\|x_{n+1} - x_n\| = \lambda \|x_n - Tx_n\| \to 0$ as $n \to \infty$. Using (3), (4) and (5) we obtain the following relations:

$$\begin{aligned}
&\|x_{n+1} - u_{n+1}\|^2 \\
&= \|(1-\alpha_n)(x_n - u_n) + \alpha_n(Tx_n - Tu_n) - \lambda(x_n - Tx_n) + \alpha_n(x_n - Tx_n)\|^2 \\
&\le (1-\alpha_n)^2 \|x_n - u_n\|^2 + \\
&+ 2\langle(\alpha_n(Tx_n - Tu_n) - \lambda(x_n - Tx_n) + \alpha_n(x_n - Tx_n)), J(x_{n+1} - u_{n+1})\rangle \\
&\le (1-\alpha_n)^2 \|x_n - u_n\|^2 + 2\alpha_n \langle(Tx_n - Tu_n, J(x_{n+1} - u_{n+1})\rangle \\
&+ 2(\alpha_n - \lambda)\langle x_n - Tx_n, J(x_{n+1} - u_{n+1})\rangle \\
&\le (1-\alpha_n)^2 \|x_n - u_n\|^2 + 2\alpha_n \langle(Tx_n - Tu_n, J(x_n - u_n)\rangle \\
&+ 2\alpha_n \langle(Tx_n - Tu_n, J(x_{n+1} - u_{n+1}) - J(x_n - u_n)\rangle \\
&+ 2(\alpha_n - \lambda)\langle x_n - Tx_n, J(x_{n+1} - u_{n+1})\rangle,
\end{aligned}$$

that is,

$$\begin{aligned}
(9) \qquad &\|x_{n+1} - u_{n+1}\|^2 \le \\
&\le (1-\alpha_n)^2 \|x_n - u_n\|^2 + 2\alpha_n k \|x_n - u_n\|^2 \\
&+ 2\alpha_n \langle(Tx_n - Tu_n, J(x_{n+1} - u_{n+1}) - J(x_n - u_n)\rangle \\
&+ 2(\alpha_n - \lambda)\langle x_n - Tx_n, J(x_{n+1} - u_{n+1})\rangle \\
&\le (1-\alpha_n)^2 \|x_n - u_n\|^2 + 2\alpha_n k \|x_n - u_n\|^2 \\
&+ 2\alpha_n \langle(Tx_n - Tu_n, J(x_{n+1} - u_{n+1}) - J(x_n - u_n)\rangle \\
&+ 2|\alpha_n - \lambda| \|x_n - Tx_n\| \|x_{n+1} - u_{n+1}\| \\
&\le (1-\alpha_n)^2 \|x_n - u_n\|^2 + 2\alpha_n k \|x_n - u_n\|^2 \\
&+ 2\alpha_n \langle(Tx_n - Tu_n, J(x_{n+1} - u_{n+1}) - J(x_n - u_n)\rangle \\
&+ 2|\alpha_n - \lambda| M_1 \|x_n - Tx_n\|,
\end{aligned}$$

for some positive constant M_1. Observe that $\{\|Tx_n - Tu_n\|\}$ is bounded. We prove now that

$$(10) \qquad J(x_{n+1} - u_{n+1}) - J(x_n - u_n) \to 0, (n \to \infty).$$

To prove (10) it is sufficient to show that

$$\begin{aligned}
&\|(x_{n+1} - u_{n+1}) - (x_n - u_n)\| \\
&= \|(x_{n+1} - x_n) - (u_{n+1} - u_n)\| \\
&= \|-\lambda x_n + \lambda Tx_n + \alpha_n u_n - \alpha_n Tu_n\| \le
\end{aligned}$$

$$\leq \lambda \|x_n - Tx_n\| + \alpha_n \|u_n - Tu_n\|$$
$$\leq \lambda \|x_n - Tx_n\| + \alpha_n M_3 \to 0, (n \to \infty).$$

Set $M_2 = \sup_n\{\|Tx_n - Tu_n\|\}$, and define

$$\sigma_n := 2\alpha_n M_2 \|J(x_{n+1} - u_{n+1}) - J(x_n - u_n)\| + + 2|\alpha_n - \lambda| M_1 \|x_n - Tx_n\|. \tag{11}$$

Note that the sequences $\{u_n\}$, $\{x_n\}$, $\{Tx_n\}$ and $\{Tu_n\}$ are bounded. Hence, M_1, M_2 and M_3 above are finite. Inserting (11) into (9), we obtain

$$\|x_{n+1} - u_{n+1}\|^2 \leq (1-\alpha_n)^2 \|x_n - u_n\|^2 + 2\alpha_n k \|x_n - u_n\|^2 + \sigma_n. \tag{12}$$

The condition $\lim_{n\to\infty} \alpha_n = 0$ implies the existence of a positive integer n_0 such that, for all $n \geq n_0$,

$$\alpha_n \leq (1-k). \tag{13}$$

Substituting (13) into (12), we obtain

$$1 - 2(1-k)\alpha_n + \alpha_n^2 \leq 1 - 2(1-k)\alpha_n + (1-k)\alpha_n = 1 - (1-k)\alpha_n. \tag{14}$$

Finally,

$$\|x_{n+1} - u_{n+1}\|^2 \leq (1 - (1-k)\alpha_n) \|x_n - u_n\|^2 + \sigma_n. \tag{15}$$

Set $a_n := \|x_n - u_n\|^2$, $\lambda_n := (1-k)\alpha_n \in (0,1)$, and use Lemma 3, to obtain $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \|x_n - u_n\|^2 = 0$; i.e.

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{16}$$

The inequality $0 \leq \|x^* - u_n\| \leq \|x_n - x^*\| + \|x_n - u_n\|$ and (16) imply that $\lim_{n\to\infty} u_n = x^*$. □

THEOREM 7. *Let X be a smooth Banach space with a uniformly convex dual and B a nonempty, closed, convex, bounded subset of X. Let $T : B \to B$ be a continuous and strongly pseudocontractive operator such that*

$$\lambda (1-k) \in (0, 1/2). \tag{17}$$

Suppose condition (8) *is satisfied. If the Mann iteration* (1) *converges to the fixed point of T, then the Krasnoselskij iteration* (2) *converges to the fixed point of T.*

Proof. Since the Mann iteration converges and T is continuous, we have

$$\lim_{n\to\infty} \|u_n - Tu_n\| = 0.$$

For a given $\varepsilon > 0$, since B is bounded, there exists a $\delta > 0$ as in Lemma 4 so that (6) is satisfied. Use (1), (2) and (4) to obtain,

$$
\begin{aligned}
&\|x_{n+1} - u_{n+1}\|^2 \\
&= \|x_n - u_n - \lambda x_n + \lambda u_n - \lambda u_n + \alpha_n u_n + \lambda T x_n - \lambda T u_n + \lambda T u_n - \alpha_n T u_n\|^2 \\
&= \left\|(1-\lambda)(x_n - u_n) + \lambda\left(Tx_n - Tu_n - u_n + Tu_n + \frac{\alpha_n}{\lambda}(u_n - Tu_n)\right)\right\|^2 \\
&\le (1-2\lambda)\|x_n - u_n\|^2 + 2\varepsilon\lambda + \\
&+ 2\left\langle \lambda\left(Tx_n - Tu_n - u_n + Tu_n + \frac{\alpha_n}{\lambda}(u_n - Tu_n)\right), J(x_n - u_n)\right\rangle \\
&= (1-2\lambda)\|x_n - u_n\|^2 + 2\varepsilon\lambda + \\
&+ 2\lambda\langle (Tx_n - Tu_n), J(x_n - u_n)\rangle \\
&+ 2\lambda\left\langle \left(-u_n + Tu_n + \frac{\alpha_n}{\lambda}(u_n - Tu_n)\right), J(x_n - u_n)\right\rangle \\
&= (1-2\lambda)\|x_n - u_n\|^2 + 2\varepsilon\lambda + \\
&+ 2\lambda\langle (Tx_n - Tu_n), J(x_n - u_n)\rangle \\
&+ 2(\alpha_n - \lambda)\langle (u_n - Tu_n), J(x_n - u_n)\rangle \\
&\le (1-2\lambda)\|x_n - u_n\|^2 + 2\lambda k\|x_n - u_n\|^2 + 2\varepsilon\lambda + \\
&+ 2|\alpha_n - \lambda|\,\|u_n - Tu_n\|\,\|x_n - u_n\| \\
&= (1 - 2\lambda(1-k))\|x_n - u_n\|^2 + 2\varepsilon\lambda + 2M|\alpha_n - \lambda|\,\|u_n - Tu_n\|,
\end{aligned}
$$

for some positive and finite M which is the supremum of the bounded sequence $\{\|x_n - u_n\|\}$. Denote $a_n = \|x_n - u_n\|^2$, $\lambda_n = 2\lambda(1-k)$, ($\lambda_n \in (0,1)$, by condition 17), and $\sigma_n = 2M|\alpha_n - \lambda|\,\|u_n - Tu_n\|$ and use Lemma 5 to obtain

$$0 \le \limsup \|x_n - u_n\|^2 \le \frac{2\varepsilon}{1-k}.$$

Since $\varepsilon > 0$ is arbitrary, one have $\limsup \|x_n - u_n\|^2 = 0$. Hence $\limsup \|x_n - u_n\| = 0$. The inequality $0 \le \|x^* - x_n\| \le \|u_n - x^*\| + \|x_n - u_n\|$ implies that $\lim_{n\to\infty} x_n = x^*$. □

3. FURTHER RESULTS

The Ishikawa iteration is given by, see [4]:

$$
\begin{aligned}
x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\
y_n &= (1-\beta_n)x_n + \beta_n T x_n,
\end{aligned}
\tag{18}
$$

where the sequences $\{\alpha_n\} \subset (0,1), \{\beta_n\} \subset [0,1)$ satisfy $\lim\limits_{n\to\infty} \alpha_n = \lim\limits_{n\to\infty} \beta_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$. The following result is from [9].

THEOREM 8. [9] *Let X be a real Banach space with a uniformly convex dual and B a nonempty, closed, convex, bounded subset of X. Let $T : B \to B$*

be a continuous and strongly pseudocontractive operator. Then the following assertions are equivalent:

(1) *the Mann iteration* (1) *converges to the fixed point of* T;
(2) *the Ishikawa iteration* (18) *converges to the fixed point of* T.

Theorem 6 and 8 lead to the following corollary.

Corollary 9. *Let* X *be a real smooth Banach space with a uniformly convex dual and* B *a nonempty, closed, convex, bounded subset of* X. *Let* $T : B \to B$ *be a continuous and strongly pseudocontractive operator. Suppose* $\|x_n - Tx_n\| = o(\alpha_n)$, *as* $n \to \infty$, *is satisfied, respectively conditions* (8) *and* (17) *are satisfied. Then* (1) $\Rightarrow$ (2), *respectively* (2) $\Rightarrow$ (1), *where:*

(1) *the Krasnoselskij iteration* (2) *converges to the fixed point of* T,
(2) *then the Ishikawa iteration* (18) *converges to the fixed point of* T.

Remark 2. *In above results, if* B *is considered unbounded then* $\{x_n\}$ *bounded, if suppose that* $T(B)$ *is bounded.*

If X is a real smooth Banach space and conditions (8) and (17) are satisfied, then the following remarks are true.

Remark 3.

(1) *The operator* T *is a strongly pseudocontractive map if and only if* $(I - T)$ *is a strongly accretive map.*
(2) *Let* $T, S : X \to X$, *and* $f \in X$ *be given. A fixed point for the map* $Tx = f + (I - S)x, \forall x \in X$ *is a solution for* $Sx = f$.
(3) *Consider Krasnoselskij, Mann and Ishikawa iterations with* $Tx = f + (I - S)x$ *to obtain a similar result to Corollary 9 for such* T.

Remark 4.

(1) *Let* $f \in X$ *be given. If* $(f - T)$ *is a strongly accretive map, then* T *is a strongly pseudocontractive map.*
(2) *Let* $T, S : X \to X$, *and* $f \in X$ *be given. A fixed point for the map* $Tx = f - Sx, \forall x \in X$ *is a solution for* $x + Sx = f$.
(3) *Consider Krasnoselskij, Mann and Ishikawa iterations with* $Tx = f - Sx$ *to obtain a similar result to Corollary 9 for such* T.

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