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CONVERGENCE OF SZÁSZ-MIRAKYAN TYPE OPERATORS

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Abstract. We introduce certain modification of Szász-Mirakyan operators and we study approximation properties of these operators. The result is in a form convenient for applications.

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1. INTRODUCTION

Approximation properties of Szász-Mirakyan operators

$$S_n(f;x) := \mathrm{e}^{-nx} \sum_{k=0}^{\infty} \tfrac{(nx)^k}{k!} f\left(\tfrac{k}{n}\right), \tag{1}$$

$x \in \mathbb{R}_0 := [0, +\infty]$, $n \in \mathbb{N} := \{1, 2, \cdots\}$, in polynomial weighted spaces C_p were examined in [1]. The space C_p, $p \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$, considered in [1] is associated with the weighted function

$$w_0(x) := 1, \quad w_p(x) := (1 + x^p)^{-1}, \qquad \text{if} \qquad p \geq 1, \tag{2}$$

and consists of all real-valued functions f, continuous on $\mathbb{R}_0$ and such that $w_p f$ is uniformly continuous and bounded on $\mathbb{R}_0$. The norm on C_p is defined by the formula

$$\|f\|_p \equiv \|f(\cdot)\|_p := \sup_{x \in \mathbb{R}_0} w_p(x) \, |f(x)|. \tag{3}$$

In this section we shall give some properties of the above operators, which we shall apply to the proof of the theorem. Most of these can be found in [1].

A. $S_n f$ is a positive linear operator $C_p \to C_p$.
B. $S_n(1;x) = 1$, $S_n f$ preserves constants.
C. For $f \in C_p$, $p \in \mathbb{N}_0$,

$$w_p(x)|S_n(f;x) - f(x)| \leq K_1(p)\omega_2\left(f; C_p; \sqrt{\tfrac{x}{n}}\right), \quad x \in \mathbb{R}_0, \quad n \in \mathbb{N}, \tag{4}$$

where $\omega_2(f;\cdot)$ is the modulus of smoothness of the order 2 and $K_1(p)$ is a positive constant.

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D. For $f \in C_p^2 := \{f \in C_p : f', f'' \in C_p\}$, $p \in \mathbb{N}_0$,

$$\lim_{n\to\infty} n(S_n(f;x) - f(x)) = \tfrac{x}{2} f''(x), \quad x \in \mathbb{R}_0.$$

E. For every $2 \le q \in \mathbb{N}$ we have

$$S_n\left((t-x)^q; x\right) = \sum_{j=1}^{[q/2]} c_{j,q} \frac{x^j}{n^{q-j}}, \qquad x \in \mathbb{R}_0, \quad n \in \mathbb{N}, \tag{5}$$

where $c_{j,q}$ are positive numerical coefficients depending only on j and q ([y] denotes the integral part of $y \in \mathbb{R}_0$) (see [17, 20]).

From (4) it was deduced that

$$\lim_{n\to\infty} S_n(f;x) = f(x), \tag{6}$$

for every $f \in C_p$, $p \in \mathbb{N}_0$ and $x \in \mathbb{R}_0$. Moreover, the above convergence is uniform on every interval $[x_1, x_2]$, $x_1 \ge 0$.

The Szász-Mirakyan operators and their connections with different branches of analysis, such as convex and numerical analysis have been studied intensively. We refer the readers to P. Gupta and V. Gupta [8], N. Ispir and C. Atakut [15], V. Gupta, V. Vasishtha and M. K. Gupta [12], G. Feng [6], [7], A. Ciupa [3], J. Grof [14]. Their results improve other related results in the literature.

The actual construction of the Szász-Mirakyan operators requires estimations of infinite series which in a certain sense restrict their usefulness from the computational point of view. Thus the question arises, whether the Szász-Mirakyan operators and their generalizations cannot be replaced by a finite sum, provided this will not change the order of approximation. In connection with this question in the paper [16] were considered certain positive linear operators for function of one variable

$$S_n(f;a_n;x) := \mathrm{e}^{-nx} \sum_{k=0}^{[n(x+a_n)]} \tfrac{(nx)^k}{k!} f\left(\tfrac{k}{n}\right), \quad x \in \mathbb{R}_0,\ n \in \mathbb{N}, \tag{7}$$

where $(a_n)_1^\infty$ is a sequence of positive numbers such that $\lim_{n\to\infty} \sqrt{n} a_n = \infty$.

In [16] it was proved that if $f \in C_p$, $p \in \mathbb{N}_0$, then

$$\lim_{n\to\infty} S_n(f;a_n;x) = f(x) \tag{8}$$

for every $f \in C_p$, $p \in \mathbb{N}_0$ and $x \in \mathbb{R}_0$. Moreover, the above convergence is uniform on every interval $[x_1, x_2]$, $x_1 \ge 0$.

Similar results in exponential weighted spaces can be found in [23].

The construction introduced the operators (7) not change the order of approximation.

In the paper [21] it was examined similar approximation problems for the following operators

$$L_n(f;x) := \frac{1}{(1+(x+n^{-1})^2)^n} \sum_{k=0}^{n} \binom{n}{k} (x+n^{-1})^{2k} f\left(\frac{k(1+(x+n^{-1})^2)}{nx+1}\right),$$

$f \in C_p$, $p \in \mathbb{N}_0$, $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$.

Thus the new question arises, whether the order of approximation given in C and D cannot be improved.

In connection in this question we propose a new family of linear operators. This together with the form of the operator makes results, given in the present paper, more helpful from the computational point of view.

In this paper we shall denote the suitable positive constants depending only on α by $K_i(\alpha), i = 1, 2, ...$

2. MAIN RESULTS

Similarly as in the paper [20] (see also [22]) let D_p, $p \in \mathbb{N}$, be the set of all real-valued functions $f(x)$, continuous on $\mathbb{R}_0$ for which $w_p(x)x^k f^{(k)}(x)$, $k = 0, 1, 2, \ldots, p$, are continuous and bounded on $\mathbb{R}_0$ and $f^{(p)}(x)$ is uniformly continuous on $\mathbb{R}_0$. The norm on D_p, $p \in \mathbb{N}$, is given by (3).

Approximation properties of modified Szász-Mirakyan operators

$$(9) \qquad A_n(f;p;x) := \mathrm{e}^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \sum_{j=0}^{p} \frac{f^{(j)}\left(\frac{k}{n}\right)}{j!} \left(x - \frac{k}{n}\right)^j, \qquad x \in \mathbb{R}_0,\ p \in \mathbb{N},$$

in D_p were examined in [20].

In [20] it was obtained that if $f \in D_p$, $p \in \mathbb{N}$, then

$$(10) \qquad \|A_n(f;p;\cdot) - f(\cdot)\|_p = \mathcal{O}(n^{-p/2}).$$

The assertion (10) for the operators A_n and $f \in C_0^r := \{f \in C_0 : f', \ldots, f^{(r)} \in C_0\}$ is given in [17].

We introduce the following

Definition 1. *For functions $f \in D_p$, $p \in \mathbb{N}$, we define the operators B_n*

$$(11) \qquad B_n(f;a_n;p;x) := \mathrm{e}^{-nx} \sum_{k=0}^{[n(x+a_n)]} \frac{(nx)^k}{k!} \sum_{j=0}^{p} \frac{f^{(j)}\left(\frac{k}{n}\right)}{j!} \left(x - \frac{k}{n}\right)^j,\ x \in \mathbb{R}_0,$$

where $(a_n)_1^\infty$ is a sequence of positive numbers such that $\lim_{n\to\infty} \sqrt{n}a_n = \infty$.

In this paper we shall study a relation between the order of approximation by B_n and the smoothness of the function f.

Now we shall give approximation theorem for B_n.

The methods used in to prove Theorem are similar to those used in construction of modified Szász-Mirakyan operators [14, 16, 23].

THEOREM 2. *Fix $p \in N$. Then for B_n defined by* (11) *we have*

$$\lim_{n\to\infty} \{B_n(f; a_n; p; x) - f(x)\} = 0, \quad f \in D_p, \tag{12}$$

uniformly on every interval $[x_1, x_2]$, $x_2 > x_1 \geq 0$.

Proof. We first suppose that $f \in D_p$, $p \in \mathbb{N}$. From (11) and (9) we obtain

$$\begin{aligned}
&B_n(f; a_n; p; x) - f(x) = \\
&= \mathrm{e}^{-nx} \sum_{k=0}^{[n(x+a_n)]} \frac{(nx)^k}{k!} \sum_{j=0}^{p} \frac{f^{(j)}\left(\frac{k}{n}\right)}{j!} \left(x - \frac{k}{n}\right)^j - f(x) \\
&= \mathrm{e}^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \sum_{j=0}^{p} \frac{f^{(j)}\left(\frac{k}{n}\right)}{j!} \left(x - \frac{k}{n}\right)^j - f(x) \\
&\quad - \mathrm{e}^{-nx} \sum_{k=[n(x+a_n)]+1}^{\infty} \frac{(nx)^k}{k!} \sum_{j=0}^{p} \frac{f^{(j)}\left(\frac{k}{n}\right)}{j!} \left(x - \frac{k}{n}\right)^j \\
&= A_n(f; p; x) - f(x) - M_n(f; x), \quad x \in \mathbb{R}_0,\ n \in \mathbb{N}.
\end{aligned}$$

By our assumption, using the elementary inequality $(a+b)^k \leq 2^{k-1}(a^k + b^k)$, $a, b > 0$, $k \in \mathbb{N}_0$, we get

$$\begin{aligned}
|f^{(k)}(t)| &\leq K_2(1 + t^{p-k}) \\
&\leq K_2(1 + (|t-x| + x)^{p-k}) \\
&\leq K_2(1 + 2^{p-k-1}(|t-x|^{p-k} + x^{p-k})), \quad k = 0, 1, 2 \ldots, p.
\end{aligned} \tag{13}$$

From this and by (1) we get

$$\begin{aligned}
&|M_n(f; x)| \leq \\
&\leq \mathrm{e}^{-nx} \sum_{k=[n(x+a_n)]+1}^{\infty} \frac{(nx)^k}{k!} K_2 \sum_{j=0}^{p} \left(1 + 2^{p-j-1} \left(\left|\frac{k}{n} - x\right|^{p-j} + x^{p-j}\right)\right) \left|\frac{k}{n} - x\right|^j \\
&\leq K_2 \sum_{j=0}^{p} \Bigg((1 + 2^{p-j-1} x^{p-j}) \mathrm{e}^{-nx} \sum_{k=[n(x+a_n)]+1}^{\infty} \frac{(nx)^k}{k!} \left|\frac{k}{n} - x\right|^j + \\
&\quad + 2^{p-j-1} \mathrm{e}^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \left|\frac{k}{n} - x\right|^p \Bigg) \\
&= K_2 \sum_{j=0}^{p} \Bigg((1 + 2^{p-j-1} x^{p-j}) \mathrm{e}^{-nx} \sum_{k=[n(x+a_n)]+1}^{\infty} \frac{(nx)^k}{k!} \left|\frac{k}{n} - x\right|^j \\
&\quad + 2^{p-j-1} S_n\left(|t-x|^p; x\right) \Bigg).
\end{aligned}$$

We remark that

$$\begin{aligned}
&\mathrm{e}^{-nx} \sum_{k=[n(x+a_n)]+1}^{\infty} \frac{(nx)^k}{k!} \left|\frac{k}{n} - x\right|^j \\
&\leq \mathrm{e}^{-nx} \sum_{a_n<|k/n-x|}^{\infty} \frac{(nx)^k}{k!} \left|\frac{k}{n} - x\right|^j \leq \mathrm{e}^{-nx} \sum_{a_n<|k/n-x|}^{\infty} \frac{(nx)^k}{k!} \frac{\left|\frac{k}{n}-x\right|^{p+j}}{a_n^p} \\
&\leq \frac{1}{a_n^p} \mathrm{e}^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \left|\frac{k}{n} - x\right|^{p+j} = \frac{1}{a_n^p} S_n\left(|t-x|^{p+j}; x\right).
\end{aligned}$$

From this and in view of (5), the Hölder inequality and the property $S_n(1;x) = 1$, we further have

$$\begin{aligned}
&|M_n(f;x)| \leq K_3 \Bigg(\sum_{j=0}^{p} \frac{1+2^{p-j-1}x^{p-j}}{a_n^p} \left(S_n\left((t-x)^{2p+2j}; x\right)\right)^{1/2} \\
&\quad + 2^{p-1} \left(S_n\left((t-x)^{2p}; x\right)\right)^{1/2} \Bigg) \\
&= K_3 \left(\sum_{j=0}^{p} \frac{1+2^{p-j-1}x^{p-j}}{a_n^p} \left\{ \sum_{i=1}^{p+j} c_{i,2p+2j} \frac{x^i}{n^{2p+2j-i}} \right\}^{1/2} \right. \\
&\quad \left. +2^{p-1} \left\{ \sum_{i=1}^{p} c_{i,2p} \frac{x^i}{n^{2p-i}} \right\}^{1/2} \right) \\
&\leq \frac{K_4}{n^{p/2}} \left(\frac{1}{a_n^p} \sum_{j=0}^{p} (1+2^{p-j-1}x^{p-j}) \left\{ \sum_{i=1}^{p+j} c_{i,2p+2j} x^i \right\}^{1/2} \right. \\
&\quad \left. +2^{p-1} \left\{ \sum_{i=1}^{p} c_{i,2p} x^i \right\}^{1/2} \right),
\end{aligned}$$

where K_3, K_4 are some positive constants depending only on p. The properties of a_n

$$\lim_{n\to\infty} \sqrt{n} a_n = \infty$$

imply that

$$\lim_{n\to\infty} M_n(f;x) = 0$$

uniformly on every interval $[x_1, x_2]$, $x_2 > x_1 \geq 0$. From this and by (10) we obtain

$$\lim_{n\to\infty} \{B_n(f; a_n; p; x) - f(x)\} = 0,$$

uniformly on every interval $[x_1, x_2]$, $x_2 > x_1 \geq 0$. This ends the proof of (12). □

3. REMARKS

Applying (10), the equality

$$B_n(f;a_n;p;x) - f(x) = A_n(f;p;x) - f(x) - M_n(f;x)$$

and arguing as in the proof of Theorem it is easy verified that operators B_n, $n \in \mathbb{N}$, give better the order of approximation ($\mathcal{O}(n^{-p/2})$) function $f \in D_p$, $p \in \mathbb{N}$, than S_n and the operators examined in [16, 21, 23] ($\mathcal{O}(n^{-1})$).

Observe that analogous approximation properties hold for the following operator

$$C_n(f;p;x) := \mathrm{e}^{-nx} \sum_{k=0}^{n} \frac{(nx)^k}{k!} \sum_{j=0}^{p} \frac{f^{(j)}\left(\frac{k}{n}\right)}{j!} \left(x - \frac{k}{n}\right)^j , \tag{14}$$

$f \in C_{[0,1]}$, $x \in [0,1)$, $n \in \mathbb{N}$.

We may remark here that the operator C_n, $n \in \mathbb{N}$, obtained from (11) for $a_n = 1 - x$, $x \in [0,1)$.

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