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CONTINUOUS SELECTIONS OF BOREL MEASURES, POSITIVE OPERATORS AND DEGENERATE EVOLUTION PROBLEMS

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Abstract. In this paper we continue the study of a sequence of positive linear operators which we have introduced in [9] and which are associated with a continuous selection of Borel measures on the unit interval. We show that the iterates of these operators converge to a Markov semigroup whose generator is a degenerate second-order elliptic differential operator on the unit interval. Some qualitative properties of the semigroup, or equivalently, of the solutions of the corresponding degenerate evolution problems, are also investigated.

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INTRODUCTION

In the previous paper [9] we have undertaken the study of a new sequence $(C_n)_{n\geq 1}$ of positive linear operators acting on the space of Lebesgue functions on the unit interval. We have investigated their approximation and shape preserving properties, presenting some estimates of the rate of convergence by means of suitable moduli of smoothness.

In this paper we continue the study of these operators by establishing an asymptotic formula. This formula leads to a one-dimensional second-order differential operator of the form

$$Au(x) := \alpha(x)u''(x) + \left(\tfrac{d}{2} - x\right)u'(x) \quad (0 < x < 1) \tag{1}$$

defined on a suitable domain of $\mathcal{C}([0,1]) \cap \mathcal{C}^2(]0,1[)$. Here $0 < d \leq 2$ and α is a continuous function on $[0,1]$ such that $\alpha(0) = \alpha(1) = 0$ and $\alpha(x) > 0$ for $0 < x < 1$.

Under additional hypotheses on α, we show that the operator A defined on the subspace

$$D_M(A) := \{u \in \mathcal{C}([0,1]) \cap \mathcal{C}^2(]0,1[) \mid \lim_{x\to 0^+} Au(x) \in \mathbb{R},\ \lim_{x\to 1^-} Au(x) \in \mathbb{R}\}$$

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is the generator of a Markov semigroup $(T(t))_{t\geq 0}$ on $\mathcal{C}([0,1])$. Furthermore we prove that for every $t \geq 0$, for every sequence $(k(n))_{n\geq 1}$ of positive integers such that $k(n)/n \to t$ $(n \to \infty)$ and for every $f \in \mathcal{C}([0,1])$,

$$\lim_{n\to\infty} C_n^{k(n)}(f) = T(t)f \tag{2}$$

uniformly on $[0,1]$.

Thanks to formula (2), we obtain some qualitative properties of the semigroup $(T(t))_{t\geq 0}$ and hence of the solutions of the diffusion equations associated with the operator $(A, D_M(A))$.

In the last part of the paper we discuss a converse problem and we show that, given a differential operator of the form (1) generating a Markov semigroup there exists a continuous selection of Borel measures whose corresponding operators C_n represent the semigroup by means of their iterates.

1. THE OPERATORS C_N

In this section we recall the definition and the main properties of the sequence of the operators C_n, introduced in [9], whose iterates will be studied in the subsequent sections as we quoted in the Introduction.

As usual, we shall denote by $\mathcal{C}([0,1])$ the space of all real valued continuous functions on $[0,1]$ endowed with the sup-norm $\|\cdot\|_\infty$.

Let $\mathcal{B}([0,1])$ be the σ-algebra of all Borel subsets of $[0,1]$ and denote by $M^+([0,1])$ the cone of all (regular) Borel measures on $[0,1]$ endowed with the vague topology. For every $x \in [0,1]$ we shall denote by ε_x the point-mass measure concentrated at x, i.e.,

$$\varepsilon_x(B) := \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{if } x \notin B, \end{cases} \qquad \text{for every } B \in \mathcal{B}([0,1]).$$

The symbol $\mathbf{1}$ stands for the constant function 1 and, for every $n \geq 1$, $e_n \in \mathcal{C}([0,1])$ denotes the functions $e_n(t) := t^n$ $(0 \leq t \leq 1)$.

A *continuous selection* of probability Borel measures on $[0,1]$ is a family $(\mu_x)_{0\leq x\leq 1}$ of probability Borel measures on [0,1] such that for every $f \in \mathcal{C}([0,1])$ the function $x \longmapsto \int_0^1 f \mathrm{d}\mu_x$ is continuous on $[0,1]$. Such a function will be denoted by $T(f)$, i.e.,

$$T(f)(x) := \int_0^1 f \mathrm{d}\mu_x \quad (0 \leq x \leq 1). \tag{1.1}$$

The operator $T : \mathcal{C}([0,1]) \longrightarrow \mathcal{C}([0,1])$ is positive (hence continuous) and $\|T\| = 1$.

As in [9] we shall fix a continuous selection $(\mu_x)_{0\leq x\leq 1}$ of probability Borel measures on $[0,1]$ satisfying the following additional assumption:

$$\int_0^1 e_1 \mathrm{d}\mu_x = x \quad (0 \leq x \leq 1) \tag{1.2}$$

(i.e., $T(e_1) = e_1$).

Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two real sequences such that, for every $n \geq 1$, $0 \leq a_n < b_n \leq 1$. For every $n \geq 1$ consider the positive linear operator $C_n : \mathcal{L}^1([0,1]) \longrightarrow \mathcal{C}([0,1])$ defined, for every $f \in \mathcal{L}^1([0,1])$ and $x \in [0,1]$, as

$$\begin{aligned} C_n(f)(x) :=& \int_{[0,1]^n} \left[\tfrac{n+1}{b_n-a_n} \int_{\frac{x_1+\cdots+x_n+a_n}{n+1}}^{\frac{x_1+\cdots+x_n+b_n}{n+1}} f(t)\mathrm{d}t \right] \mathrm{d}\mu_x^n(x_1,\ldots,x_n) \\ =& \int_0^1 \cdots \int_0^1 \left[\tfrac{n+1}{b_n-a_n} \int_{\frac{x_1+\cdots+x_n+a_n}{n+1}}^{\frac{x_1+\cdots+x_n+b_n}{n+1}} f(t)\mathrm{d}t \right] \mathrm{d}\mu_x(x_1)\ldots\mathrm{d}\mu_x(x_n), \end{aligned} \tag{1.3}$$

where μ_x^n denotes the tensor product of n copies of μ_x.

The operator C_n is well-defined and maps the space $\mathcal{L}^1([0,1])$ into the space $\mathcal{C}([0,1])$. Moreover each C_n is continuous from $\mathcal{C}([0,1])$ into $\mathcal{C}([0,1])$ and its norm is equal to 1.

It is worth pointing out that to a given continuous selection $(\mu_x)_{0\leq x\leq 1}$ of probability Borel measures on $[0,1]$ it is possible to associate another sequence of positive linear operators, namely the Bernstein-Schnabl operators, which are defined as

$$\begin{aligned} B_n(f)(x) :=& \int_{[0,1]^n} f\left(\tfrac{x_1+\cdots+x_n}{n}\right) \mathrm{d}\mu_x^n(x_1,\ldots,x_n) \\ =& \int_0^1 \cdots \int_0^1 f\left(\tfrac{x_1+\cdots+x_n}{n}\right) \mathrm{d}\mu_x(x_1)\cdots\mathrm{d}\mu_x(x_n), \end{aligned}$$

for every $n \geq 1$, $f \in \mathcal{C}([0,1])$ and $0 \leq x \leq 1$. These operators have been extensively studied (see, e.g., [1], [7] and [10]).

There is a close relationship between the operators C_n and B_n. In [9, Remark 1.3] we showed that, for a given $f \in \mathcal{L}^1([0,1])$, considering the function $F \in \mathcal{C}([0,1])$ defined by $F(x) = \int_0^x f(t)\mathrm{d}t$ $(0 \leq x \leq 1)$ then, for every $n \geq 1$, the operator C_n can be written as

$$C_n(f) = \tfrac{n+1}{b_n-a_n} B_n(\sigma_n(F)), \tag{1.4}$$

where the mapping $\sigma_n : \mathcal{C}([0,1]) \longrightarrow \mathcal{C}([0,1])$ is defined by

$$\sigma_n(F)(x) := F\left(\tfrac{n}{n+1}x + \tfrac{b_n}{n+1}\right) - F\left(\tfrac{n}{n+1}x + \tfrac{a_n}{n+1}\right),$$

for every $F \in \mathcal{C}([0,1])$ and $x \in [0,1]$.

Another formula which relates the operators C_n to the operators B_n is given in [9, Remark 1.4]. It has been useful both for investigating the behaviour of the operators C_n on convex functions and for suggesting a possible generalization of our results replacing the interval $[0,1]$ with an arbitrary interval (not necessarily bounded) or with a convex subset of some locally convex space.

We recall it here:

$$C_n(f)(x)=\int_0^1\cdots\int_0^1 f\left(\tfrac{(b_n-a_n)s+a_n+x_1+\cdots+x_n}{n+1}\right)\mathrm{d}s\mathrm{d}\mu_x(x_1)\ldots\mathrm{d}\mu_x(x_n)$$
$$=\int_0^1\cdots\int_0^1 f\left(\tfrac{x_1+\cdots+x_{n+1}}{n+1}\right)\mathrm{d}\mu_n(x_1)\mathrm{d}\mu_x(x_2)\ldots\mathrm{d}\mu_x(x_{n+1}),$$

where μ_n denote the image measure of the Borel-Lebesgue measure λ_1 under the mapping $T_n(x)=(b_n-a_n)x+a_n$ $(0\le x\le 1)$.

Some examples of operators C_n can be found in [9, Examples 1.5]. In particular we point out that, if $\mu_x := x\varepsilon_1+(1-x)\varepsilon_0$, $x\in[0,1]$, then they can be rewritten as

$$C_n(f)(x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\left(\frac{n+1}{b_n-a_n}\int_{\frac{k+a_n}{n+1}}^{\frac{k+b_n}{n+1}} f(t)\mathrm{d}t\right) \tag{1.5}$$

and, by taking $a_n=0$, $b_n=1$ for each $n\ge 1$, they turn into the well-known Kantorovich operators ([21]; [7, pp. 333–335]).

Another example of operators C_n can be obtained by considering the continuous selection $(\nu_x^\lambda)_{0\le x\le 1}$ of the probability Borel measures ν_x defined by

$$\nu_x^\lambda := \lambda(x)\mu_x+(1-\lambda(x))\varepsilon_x \quad (0\le x\le 1),$$

where the measure μ_x is given by $\mu_x := x\varepsilon_1+(1-x)\varepsilon_0$ and $\lambda\in C([0,1])$ is a function satisfying $0\le\lambda\le 1$. The operators C_n associated with the continuous selection $(\nu_x^\lambda)_{0\le x\le 1}$ are given by

$$C_{n,\lambda}(f)(x)=$$
$$=\sum_{h=0}^{n}\sum_{k=0}^{n-h}\binom{n}{h}\binom{n-h}{k}\left(\frac{n+1}{b_n-a_n}\int_{\frac{k+hx+a_n}{n+1}}^{\frac{k+hx+b_n}{n+1}} f(t)\mathrm{d}t\right)x^k(1-x)^{n-h-k}\lambda(x)^{n-h}(1-\lambda(x))^h,$$

for every $f\in\mathcal{L}^1([0,1])$, $x\in[0,1]$ and $n\ge 1$.

In [9, Section 2] we investigated the approximation properties of the operators C_n in the space $\mathcal{C}([0,1])$ and, in some cases, in the space $\mathcal{L}^p([0,1])$. We also presented several estimates of the rate of convergence by means of suitable moduli of smoothness. Shape preserving properties of these operators were also discussed (see [9, Section 3]). In particular we proved that each operator C_n preserves both the class of Hölder continuous functions and the one of convex continuous functions.

We recall here some of these results which will be useful in Section 3. The first one shows that each operator C_n preserves the class of Hölder continuous functions. The next one gives information about the preservation of convex functions by the operators C_n. For more details see [9, Section 3].

For given $M>0$ and $0\le\alpha\le 1$, we shall denote by $\mathrm{Lip}_M\alpha$ the subset of all $f\in\mathcal{C}([0,1])$ such that

$$|f(x)-f(y)|\le M|x-y|^\alpha \quad \text{for every } x,y\in[0,1].$$

Moreover, for any $f \in \mathcal{C}([0,1])$ the symbol $\omega(f,\cdot)$ stands for the usual modulus of smoothness of the first order which is defined by

$$\omega(f,\delta) := \sup\{|f(x)-f(y)| : |x-y| \le \delta, x,y \in [0,1]\} \quad (\delta > 0).$$

THEOREM 1.1. *If $T(\mathrm{Lip}_1 1) \subset T(\mathrm{Lip}_c 1)$ for some $c \ge 1$, then for every $n \ge 1$, $f \in \mathcal{C}([0,1])$, $\delta > 0$, $M > 0$ and $0 < \alpha \le 1$*

$$\omega(C_n(f),\delta) \le (1+c)\,\omega(f,\delta) \quad \text{and} \quad C_n(\mathrm{Lip}_M\alpha) \subset \mathrm{Lip}_{c^\alpha M}\alpha.$$

In particular, if $T(\mathrm{Lip}_1 1) \subset \mathrm{Lip}_1 1$, then

$$\omega(C_n(f),\delta) \le 2\omega(f,\delta) \quad \text{and} \quad C_n(\mathrm{Lip}_M\alpha) \subset \mathrm{Lip}_M\alpha.$$

THEOREM 1.2. *Consider the operators C_n associated with the continuous selection of probability Borel measures $(\mu_x)_{0\le x\le 1}$ defined by (1.3). Suppose that:*

(c_1) *The operator T, given by (1.1), maps continuous convex functions into (continuous) convex functions;*

(c_2) *For every $x,y \in [0,1]$*

$$\int_{[0,1]^2} \varphi_f \,\mathrm{d}(\mu_x \otimes \mu_x + \mu_y \otimes \mu_y) \ge 2\int_{[0,1]^2} \varphi_f \,\mathrm{d}(\mu_x \otimes \mu_y),$$

where $\varphi_f(s,t) := f\left(\frac{s+t}{2}\right)$, $(s,t) \in [0,1]^2$ and the symbol $\otimes$ denotes the tensor product among measures.

Then each operator C_n maps continuous convex functions into (continuous) convex functions.

Examples of selections of measures satisfying (c_1) and (c_2) can be found in [10, Examples 2.7].

2. AN ASYMPTOTIC FORMULA

In this section we establish an asymptotic formula for the sequence of the operators C_n defined by (1.3) with respect to the uniform norm.

The usefulness of asymptotic formulae in the representation of $\mathcal{C}_0$-semigroups in terms of positive linear operators has been shown by many results in the last years, since the pioneer work of the first author ([1], [2], [3], [4]) (see also [6], [8], [11], [12], [13], [18], [19] and the references given there).

The first result about asymptotic formulae is due to Voronovskaja [24]. It states that, considering the sequence $(\mathcal{B}_n)_{n\ge 1}$ of the classical Bernstein operators defined on the unit interval [17] (see also, e.g., [7, pp. 218–220]), for any $f \in \mathcal{C}^2([0,1])$

$$\lim_{n\to\infty} n(\mathcal{B}_n(f)(x) - f(x)) = \tfrac{x(1-x)}{2} f''(x),$$

uniformly with respect to $x \in [0,1]$. Such a result shows that for Bernstein operators the convergence cannot be too fast, even if the approximating function is smooth.

In order to show an asymptotic formula for the sequence $(C_n)_{n\geq 1}$, we shall use a generalization of Voronovskaja's result due to Mamedov [22] (see also [5, Theorem 1]), which holds for an arbitrary sequence $(L_n)_{n\geq 1}$ of positive linear operators acting on $\mathcal{C}([0,1])$ and which is stated below.

As usual, for every $x \in [0,1]$ the symbol ψ_x stands for the function

$$\psi_x(t) := t - x \quad (0 \leq t \leq 1). \tag{2.1}$$

THEOREM 2.1. *Consider a sequence $(L_n)_{n\geq 1}$ of positive linear operators from $\mathcal{C}([0,1])$ into itself and let α, β and γ be functions defined on $[0,1]$. Assume that*

(i) $\lim\limits_{n\to\infty} n(L_n(\mathbf{1})(x) - 1) = \gamma(x)$ *uniformly on* $[0,1]$,
(ii) $\lim\limits_{n\to\infty} nL_n(\psi_x)(x) = \beta(x)$ *uniformly on* $[0,1]$,
(iii) $\lim\limits_{n\to\infty} nL_n(\psi_x^2)(x) = \alpha(x)$ *uniformly on* $[0,1]$,
(iv) $\lim\limits_{n\to\infty} nL_n(\psi_x^q)(x) = 0$ *uniformly on* $[0,1]$, *for some even positive integer* $q \geq 4$.

Then for every $f \in \mathcal{C}^2([0,1])$

$$\lim_{n\to\infty} n(L_n(f)(x) - f(x)) = \frac{\alpha(x)}{2} f''(x) + \beta(x) f'(x) + \gamma(x)$$

uniformly on $[0,1]$.

For a proof we refer the reader to [5, Theorem 1] where a more general result for not necessarily compact interval is presented.

Now we are in a position to state and prove the main result of this section.

THEOREM 2.2. *Consider the sequence $(C_n)_{n\geq 1}$ of the operators C_n defined by* (1.3) *and assume that the sequence $(a_n + b_n)_{n\geq 1}$ is convergent. Then for every $f \in \mathcal{C}^2([0,1])$*

$$\lim_{n\to\infty} n(C_n(f)(x) - f(x)) = \frac{T(e_2)(x) - x^2}{2} f''(x) + \left(\frac{d}{2} - x\right) f'(x),$$

uniformly with respect to $x \in [0,1]$, where $T(e_2)(x) = \int_0^1 e_2 \mathrm{d}\mu_x$ (see (1.1)*) and $d := \lim\limits_{n\to\infty} (a_n + b_n)$.*

Proof. We shall apply Theorem 2.1 with $q = 4$. Observe that condition (i), (ii) and (iii) of the above result are satisfied with $\gamma = 0$, $\beta(x) = \frac{d}{2} - x$ and $\alpha(x) = T(e_2)(x) - x^2$ $(0 \leq x \leq 1)$ because, for every $0 \leq x \leq 1$,

$$C_n(\mathbf{1})(x) = 1, \quad C_n(\psi_x)(x) = \frac{1}{n+1}\left(\frac{a_n + b_n}{2} - x\right)$$

and

$$C_n(\psi_x^2)(x) = \frac{1-n}{(n+1)^2} x^2 + \frac{n}{(n+1)^2} T(e_2)(x) - \frac{a_n + b_n}{(n+1)^2} x + \frac{a_n^2 + a_n b_n + b_n^2}{3(n+1)^2},$$

(see [9, formulae (2.3), (2.4)]). In order to verify condition (iv), we shall explicitly determine the function $C_n(\psi_x^4)$.

Let $x \in [0,1]$. Since $\psi_x^4 = e_4 - 4xe_3 + 6x^2e_2 - 4x^3e_1 + x^4\mathbf{1}$, we have

$$C_n(\psi_x^4) = C_n(e_4) - 4xC_n(e_3) + 6x^2C_n(e_2) - 4x^3C_n(e_1) + x^4C_n(\mathbf{1}).$$

The expression of $C_n(\mathbf{1})$, $C_n(e_1)$ and $C_n(e_2)$ are the following

$$C_n(\mathbf{1})(x) = \tfrac{n+1}{b_n-a_n}\,\tfrac{b_n-a_n}{n+1} = 1, \quad C_n(e_1)(x) = \tfrac{n}{(n+1)}\,x + \tfrac{a_n+b_n}{2(n+1)}$$

and

$$C_n(e_2)(x) = \tfrac{n(n-1)}{(n+1)^2}\,x^2 + \tfrac{n}{(n+1)^2}\,T(e_2)(x) + \tfrac{n(a_n+b_n)}{(n+1)^2}\,x + \tfrac{a_n^2+a_nb_n+b_n^2}{3(n+1)^2},$$

(see (2), (3) and (4) of [9, Theorem 2.1]). Therefore we proceed to evaluate C_n on the functions e_3 and e_4, by using relation (1.4) between our operators C_n and the corresponding Bernstein-Schnabl operators B_n. A simply but laborious computations shows

$$\begin{aligned} C_n(e_3)(x) =&\tfrac{n^3}{(n+1)^3}\,B_n(e_3)(x)+\tfrac{3}{2}\,\tfrac{n^2(a_n+b_n)}{(n+1)^3}\,B_n(e_2)(x)\\ &+\tfrac{n(a_n^2+a_nb_n+b_n^2)}{(n+1)^3}\,x + \tfrac{(a_n+b_n)(a_n^2+b_n^2)}{4(n+1)^3} \end{aligned}$$

and

$$\begin{aligned} C_n(e_4)(x)=&\tfrac{n^4}{(n+1)^4}\,B_n(e_4)(x)+\tfrac{2n^3(a_n+b_n)}{(n+1)^4}\,B_n(e_3)(x)+\tfrac{2n^2(a_n^2+a_nb_n+b_n^2)}{(n+1)^4}\\ &\times B_n(e_2)(x)+\tfrac{n(a_n+b_n)(a_n^2+b_n^2)}{(n+1)^4}\,x+\tfrac{b_n^4+b_n^3a_n+b_n^2a_n^2+b_na_n^3+a_n^4}{5(n+1)^5}. \end{aligned}$$

Therefore

$$\begin{aligned} nC_n(\psi_x^4)(x)=&\tfrac{n^2}{(n+1)^4}\,T(e_3)(x)+\tfrac{4n^2(n-1)}{(n+1)^4}\,x\,T(e_3)(x)+\tfrac{3n^2(n-1)}{(n+1)^4}\,T(e_2)^2(x)\\ &+\tfrac{6n^2(5-n)}{(n+1)^4}\,x^2\,T(e_2)(x) + \tfrac{3n^3-22n^2+n}{(n+1)^4}\,x^4 + \tfrac{2n^2(a_n+b_n)}{(n+1)^4}\,T(e_3)(x)\\ &+\tfrac{6n^2(n-1)(a_n+b_n)}{(n+1)^4}x\,T(e_2)(x) + \tfrac{2n(5n-1)(a_n+b_n)}{(n+1)^4}\,x^3 + \tfrac{2n^2(n-1)}{(n+1)^4}\\ &\times(a_n^2+a_nb_n+b_n^2)x^2+\tfrac{2n^2(a_n^2+a_nb_n+b_n^2)}{(n+1)^4}\,T(e_2)(x)+\tfrac{n^2(a_n+b_n)(a_n^2+b_n^2)}{(n+1)^4}\,x\\ &+\tfrac{n\left(b_n^4+b_n^3a_n+b_n^2a_n^2+b_na_n^3+a_n^4\right)}{5(n+1)^5}-\tfrac{4n^2}{(n+1)^3}\,x\,T(e_3)(x)-\tfrac{6n^2}{(n+1)^3}x\,T(e_2)(x)\\ &-\tfrac{4n^2(a_n^2+a_nb_n+b_n^2)}{(n+1)^3}\,x^2 - \tfrac{n(a_n+b_n)(a_n^2+b_n^2)}{(n+1)^3}\,x + \tfrac{6n^2(n-1)}{(n+1)^2}\,x^4\\ &+\tfrac{6n^2}{(n+1)^2}\,x^2\,T(e_2)(x) + \tfrac{2n\left(a_n^2+a_nb_n+b_n^2\right)}{(n+1)^3}\,x^2 \end{aligned}$$

and so condition (iv) follows. $\square$

3. MARKOV SEMIGROUPS ASSOCIATED WITH A CLASS OF ONE-DIMENSIONAL DIFFUSION EQUATIONS AND THEIR APPROXIMATION

The main aim of this section is to discuss some one-dimensional diffusion equations on the unit interval by means of the theory of $\mathcal{C}_0$-semigroups of operators and to represent the relevant solutions (or the corresponding semigroups) by iterates of the operators C_n. For more details about the theory of $\mathcal{C}_0$-semigroups we refer the reader to [16], [20], [23].

Let $(C_n)_{n\geq 1}$ be the sequence of the operators on $\mathcal{C}([0,1])$ associated with the continuous selection $(\mu_x)_{0\leq x\leq 1}$ of probability Borel measures on $[0,1]$ defined by (1.3). By Jensen's inequality [15, Theorem 3.9] it follows that $x^2 \leq T(e_2)(x) \leq x$ $(0 \leq x \leq 1)$, where the operator T is defined by (1.1). Therefore

$$0 \leq T(e_2)(x) - x^2 \leq x - x^2 = x(1-x) \quad (0 \leq x \leq 1).$$

In particular, $T(e_2)(0) = 0$ and so $\operatorname{Supp}\mu_0 = \{0\}$, that is $\mu_0 = \varepsilon_0$. Here $\operatorname{Supp}\mu_0$ stands for the support of the measure μ_0 (see, e.g., [7, Section 1.2]).

Moreover $T(e_2)(1) = 1$, so $\int_0^1 (e_1 - e_2)\mathrm{d}\mu_1 = 0$ and $\operatorname{Supp}\mu_1 \subset \{0,1\}$. Then there exist $\alpha, \beta \in [0,1]$, $\alpha + \beta = 1$ such that $\mu_1 = \alpha\varepsilon_0 + \beta\varepsilon_1$, so that $1 = \int_0^1 e_1 \mathrm{d}\mu_1 = \beta$. In conclusion we obtain $\alpha = 0$ and $\mu_1 = \varepsilon_1$.

Finally we observe that, if $0 \leq x \leq 1$, then

$$T(e_2)(x) = x^2 \quad \text{if and only if} \quad \mu_x = \varepsilon_x.$$

Indeed, if $T(e_2)(x) = x^2$, considering the function ψ_x defined by (2.1), we have $\int_0^1 \psi_x^2 \mathrm{d}\mu_x = 0$, so $\operatorname{Supp}\mu_x = \{x\}$ and $\mu_x = \varepsilon_x$. The converse is trivial.

From now on we suppose that the family $(\mu_x)_{0\leq x\leq 1}$ satisfies the following further condition

$$\mu_x \neq \varepsilon_x \qquad \text{for every } 0 < x < 1. \tag{3.1}$$

Set

$$\alpha(x) := \tfrac{1}{2}\left(T(e_2)(x) - x^2\right) = \tfrac{1}{2}\left(\int_0^1 e_2 \mathrm{d}\mu_x - x^2\right) \quad (0 \leq x \leq 1).$$

Then $\alpha \in \mathcal{C}([0,1])$, $\alpha(0) = \alpha(1) = 0$ and

$$0 < \alpha(x) \leq \tfrac{x(1-x)}{2} \quad \text{for every } 0 < x < 1.$$

Suppose in addition that α is differentiable at 0 and 1 and

$$\alpha'(0) \neq 0 \neq \alpha'(1). \tag{3.2}$$

Then

$$\alpha(x) = \tfrac{x(1-x)}{2}\lambda(x) \qquad (0 \leq x \leq 1)$$

where

$$\lambda(x) = \begin{cases} 2\alpha'(0) & \text{if } x = 0, \\ \frac{2\alpha(x)}{x(1-x)} & \text{if } 0 < x < 1, \\ -2\alpha'(1) & \text{if } x = 1, \end{cases} \tag{3.3}$$

and $\lambda \in \mathcal{C}([0,1])$, $0 < \lambda(x) \leq 1$ for every $x \in [0,1]$.

Suppose that there exists $d := \lim_{n\to\infty}(a_n+b_n) > 0$ and consider the differential operator A defined by setting for every $u \in \mathcal{C}^2(]0,1[)$

$$Au(x) := \alpha(x)u''(x) + \left(\frac{d}{2} - x\right)u'(x) \quad (0 < x < 1).$$

Set

(3.4)

$$D_M(A) := \left\{u \in \mathcal{C}([0,1]) \cap \mathcal{C}^2(]0,1[) \mid \lim_{x\to 0^+} A(u)(x) \in \mathbb{R}, \lim_{x\to 1^-} A(u)(x) \in \mathbb{R}\right\}$$

and continue to denote with $A : D_M(A) \longrightarrow \mathcal{C}([0,1])$ the operator defined by setting for every $u \in D_M(A)$ and $x \in [0,1]$

$$A(u)(x) := \begin{cases} \lim_{x\to 0^+} A(u)(x) & \text{if } x = 0, \\ \alpha(x)u''(x) + \left(\frac{d}{2} - x\right)u'(x) & \text{if } 0 < x < 1, \\ \lim_{x\to 1^-} A(u)(x) & \text{if } x = 1. \end{cases} \tag{3.5}$$

We recall that a core for a linear operator $A : D(A) \longrightarrow E$ defined on a linear subspace $D(A)$ of a Banach space E, is a subspace D_0 of $D(A)$ which is dense in $D(A)$ for the graph norm $\|u\|_A := \|u\| + \|Au\|$ $(u \in D(A))$.

A Feller semigroup on $\mathcal{C}([0,1])$ is a strongly continuous semigroup of positive linear contractions on $\mathcal{C}([0,1])$. A Markov semigroup on $\mathcal{C}([0,1])$ is a strongly continuous positive semigroup $(T(t))_{t\geq 0}$ on $\mathcal{C}([0,1])$ satisfying $T(t)\mathbf{1} = \mathbf{1}$ for every $t \geq 0$. A strongly continuous positive semigroup on $\mathcal{C}([0,1])$ with generator $(A, D(A))$ is a Markov semigroup if and only if $\mathbf{1} \in D(A)$ and $A\mathbf{1} = 0$.

THEOREM 3.1. *Under the assumptions* (3.1) *and* (3.2)*, if moreover* $\alpha'(0) \leq d/2 \leq 1 + \alpha'(1)$ *and the function* $r(x) := \frac{d/2-x}{\lambda(x)}$ $(0 \leq x \leq 1)$ *is Hölder continuous at* 0 *and* 1*, then the operator* $(A, D_M(A))$ *is the generator of a Markov semigroup* $(T(t))_{t\geq 0}$ *on* $\mathcal{C}([0,1])$ *and* $\mathcal{C}^2([0,1])$ *is a core for* $(A, D_M(A))$.

Proof. We introduce the auxiliary operator

$$Bu(x) = \frac{x(1-x)}{2}u''(x) + r(x)u'(x) \tag{1}$$

defined on the domain $D(B) := D_M(A)$. Thus, $B = \lambda A$ and

$$D(B) := \left\{u \in \mathcal{C}([0,1]) \cap \mathcal{C}^2(]0,1[) \mid \lim_{x\to 0^+} B(u)(x) \in \mathbb{R}, \lim_{x\to 1^-} B(u)(x) \in \mathbb{R}\right\}.$$

Then, $(B, D(B))$ is the generator of a Feller semigroup on $\mathcal{C}([0,1])$ (see [13, pp. 120–121]). Therefore, since $A = \lambda B$, the result follows by a well-known result about the generation of the multiplicative perturbation of generators (see [7, Theorem 1.6.11]). Since $\mathbf{1} \in D_M(A)$ and $A\mathbf{1} = 0$, the semigroup is a Markov semigroup. Finally in order to prove that $\mathcal{C}^2([0,1])$ is a core for $(A, D_M(A))$, we use Theorem 2.3 of [13] applied to the operator B defined by (1), obtaining that $\mathcal{C}^2([0,1]) \cap D(B)$ is a core for $(B, D(B))$. But, since

$\mathcal{C}^2([0,1]) \subset D(A) = D(B)$, $\mathcal{C}^2([0,1])$ is a core for $(B, D(B))$ and hence for $(A, D(A))$. $\square$

At this point we are in a position to obtain a result about the approximation of the semigroup above.

The p-th power ($p \geq 1$) of the operator $C_n : \mathcal{C}([0,1]) \longrightarrow \mathcal{C}([0,1])$ is defined as

$$C_n^p := \begin{cases} C_n & \text{if } p = 1, \\ C_n \circ C_n^{p-1} & \text{if } p \geq 2. \end{cases}$$

Clearly

$$\|C_n^p\| \leq 1 \quad \text{for every } n \geq 1 \text{ and } p \geq 1, \tag{3.6}$$

since $\|C_n\| \leq 1$.

Note that, if $u \in \mathcal{C}^2([0,1])$, then

$$Au(x) = \alpha(x)u''(x) + \left(\frac{d}{2} - x\right)u'(x) \quad \text{for every} \quad x \in [0,1]$$

and hence, by Theorem 2.2,

$$\lim_{n\to\infty} n(C_n(u) - u) = Au \quad \text{uniforlmy on} \quad [0,1]. \tag{3.7}$$

We finally need the following general result which can be obtained by Trotter's theorem on the approximation of semigroups (see, e.g., [20, Corollary 5.8] or [7, Theorem 1.6.7]).

Theorem 3.2. *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space E with generator $(A, D(A))$. Consider a sequence $(L_n)_{n\geq 1}$ of bounded linear operators on E and assume that*

(i) *There exists $M \geq 1$ and $\omega \in \mathbb{R}$ such that*

$$\|T(t)\| \leq M \exp(\omega t) \quad \textit{and} \quad \|L_n^p\| \leq M \exp\left(\omega \tfrac{p}{n}\right)$$

for every $t \geq 0$, $n \geq 1$ $p \geq 1$.

(ii) *There exists a core D_0 for $(A, D(A))$ such that*

$$\lim_{n\to\infty} n(L_n(u) - u) = Au \quad \textit{for every} \quad u \in D_0.$$

Then for every $t \geq 0$, for every sequence $(k(n))_{n\geq 1}$ of positive integers such that $k(n)/n \to t$ $(n \to \infty)$ and for every $f \in E$

$$T(t)f = \lim_{n\to\infty} L_n^{k(n)} f.$$

From (3.6) and (3.7) and from Theorems 3.1 and 3.2 the next result immediately follows.

Theorem 3.3. *Under the same assumptions of Theorem* 3.1*, considering the Markov semigroup $(T(t))_{t\geq 0}$ generated by $(A, D_M(A))$, for every $t \geq 0$, for every sequence $(k(n))_{n\geq 1}$ of positive integers such that $k(n)/n \to t$ $(n \to \infty)$ and for every $f \in \mathcal{C}([0,1])$,*

$$T(t)f = \lim_{n\to\infty} C_n^{k(n)}(f) \quad \textit{uniformly on } [0,1]. \tag{3.8}$$

By using the representation formula (3.8), it is possible to obtain some properties of the semigroup from the preservation properties of the operators C_n (see Theorems 1.1 and 1.2).

PROPOSITION 3.4. *Under the same assumptions of Theorem* 3.3*, considering the Markov semigroup* $(T(t))_{t\geq 0}$ *generated by the operator* $(A, D_M(A))$ *defined by* (3.5)*, the following statements hold true:*

(1) *If the operator* T*, defined by* (1.1)*, maps* $\mathrm{Lip}_1 1$ *into* $\mathrm{Lip}_1 1$*, then*
$$T(t)(\mathrm{Lip}_M \alpha) \subset \mathrm{Lip}_M \alpha$$
for every $M \geq 1$ *and* $0 < \alpha \leq 1$*.*

(2) *If the operator* T*, given by* (1.1)*, satisfies the hypotheses of Theorem* 1.2*, then for every* $t \geq 0$*,* $T(t)$ *maps continuous convex functions into (continuous) convex functions.*

It is possible to obtain a further property which holds for the semigroup $(T(t))_{t\geq 0}$. We need the following lemma.

LEMMA 3.5. *Under the same assumptions of Theorem* 3.1 *consider the Markov semigroup* $(T(t))_{t\geq 0}$ *generated by* $(A, D_M(A))$*. Then for every* $t \geq 0$
$$T(t)e_1 = \mathrm{e}^{-t} e_1 + \tfrac{d}{2}(1 - \mathrm{e}^{-t}).$$
Therefore, for every $t > 0$
$$e_1 \leq T(t)\, e_1 \quad \textit{if and only if} \quad d = 2.$$

Proof. We use the representation formula (3.8) in order to show the first part of the claim. For every $n \geq 1$ we have indeed
$$\begin{aligned} C_n(e_1) &= \tfrac{n}{n+1}\, e_1 + \tfrac{a_n + b_n}{2(n+1)}, \\ C_n^2(e_1) &= \left(\tfrac{n}{n+1}\right)^2 e_1 + \left(1 - \left(\tfrac{n}{n+1}\right)^2\right) \tfrac{a_n + b_n}{2} \end{aligned}$$
and, reasoning by induction, for every $p \geq 3$
$$C_n^p(e_1) = \left(\tfrac{n}{n+1}\right)^p e_1 + \left(1 - \left(\tfrac{n}{n+1}\right)^p\right) \tfrac{a_n + b_n}{2}.$$
Given $t \geq 0$ and considering a sequence $(k(n))_{n\geq 1}$ of positive integers such that $k(n)/n \to t$, we get
$$T(t)e_1 = \lim_{n\to\infty} C_n^{k(n)} e_1 = \mathrm{e}^{-t} e_1 + \tfrac{d}{2}(1 - \mathrm{e}^{-t}),$$
since $\left(\tfrac{n}{n+1}\right)^{k(n)} \to \mathrm{e}^t$. The second part of the statement follows from the first one, since $d = \lim\limits_{n\to\infty}(a_n + b_n) \leq 2$. $\square$

PROPOSITION 3.6. *Under the same hypotheses of Theorem* 3.1 *the following propositions are equivalent:*

(a) *For every increasing convex function* $f \in \mathcal{C}([0,1])$ *and for every* $t \geq 0$*,*
$$f \leq T(t)f.$$

(b) $d = 2$.

Proof. $(a) \Rightarrow (b)$ For $f = e_1$, we get $e_1 \le T(t)e_1$ for every $t \ge 0$ and so $d = 2$ by the previous lemma.
$(b) \Rightarrow (a)$ Fix $t > 0$. Since $d = 2$, $e_1 \le T(t)e_1$. Let $f \in \mathcal{C}([0,1])$ convex and increasing. Let $x \in]0,1[$ and let φ be an increasing affine function on $[0,1]$ such that $f(x) = \varphi(x)$ and $\varphi \le f$. Let $a, b \in \mathbb{R}$, $a \ge 0$ such that $\varphi = ae_1 + b$. Then $T(t)\varphi = aT(t)e_1 + b$ and so $\varphi \le T(t)\varphi$. Accordingly

$$f(x) = \varphi(x) \le T(t)\varphi(x) \le T(t)f(x).$$

By continuity the inequality $f \le T(t)f$ can be extended on the whole interval $[0,1]$ and so the result follows. $\square$

Remark 3.7. From the general theory of $\mathcal{C}_0$-semigroups of operators and from Theorem 3.1 it follows that for every $u_0 \in D_M(A)$ the following initial-boundary differential problem of diffusion type

$$\left\{\begin{array}{ll} \frac{\partial u}{\partial t}(x,t) = \alpha(x)\frac{\partial^2 u}{\partial x^2}(x,t) + \left(\frac{d}{2} - x\right)\frac{\partial u}{\partial x}(x,t) & 0 < x < 1\,,\ t \ge 0, \\ u(x,0) = u_0(x) & 0 \le x \le 1, \\ \lim\limits_{\substack{x \to 0^+ \\ x \to 1^-}} \alpha(x)\frac{\partial^2 u}{\partial x^2}(x,t) + \left(\frac{d}{2} - x\right)\frac{\partial u}{\partial x}(x,t) \in \mathbb{R} & t \ge 0, \end{array}\right. \tag{3.9}$$

has a unique solution given by

$$u(x,t) = T(t)u_0(x) \quad (0 \le x \le 1\,,\ t \ge 0).$$

Moreover $|u(x,t)| \le \|u_0\|$ $(0 \le x \le 1\,,\ t \ge 0)$ and $u(\cdot,t)$ is positive for every $t \ge 0$ provided that $u_0 \ge 0$.

Furthermore, by Theorem 3.3, the solutions can be approximate by means of iterates of the operators C_n. Finally, Propositions 3.4 and 3.6 give some qualitative properties of them as well. $\square$

We end the paper by considering a kind of converse problem. Let $d \in \mathbb{R}$, $0 < d \le 2$ and $\alpha \in \mathcal{C}([0,1])$, $\alpha(0) = \alpha(1) = 0$, $\alpha(x) > 0$ for every $0 < x < 1$. Moreover suppose that α is differentiable at 0 and 1, $\alpha'(0) \ne 0 \ne \alpha'(1)$ and the function

$$r(x) := \left\{\begin{array}{ll} \frac{d}{4\alpha'(0)} & \text{if } x = 0, \\ \frac{x(1-x)(d/2-x)}{2\alpha(x)} & \text{if } 0 < x < 1, \\ \frac{1-d/2}{2\alpha'(1)} & \text{if } x = 1, \end{array}\right.$$

is Hölder continuous at 0 and 1.

Consider the differential operator $(A, D_M(A))$, defined in (3.4) and (3.5), that is,

$$Au(x) := \alpha(x)u''(x) + \left(\frac{d}{2} - x\right)u'(x) \quad (0 < x < 1)$$

for every $u \in D_M(A) := \left\{ u \in \mathcal{C}([0,1]) \cap \mathcal{C}^2(]0,1[) \mid \lim_{\substack{x\to 0^+\\ x\to 1^-}} A(u)(x) \in \mathbb{R} \right\}$.

If $\alpha'(0) \leq d/2 \leq 1+\alpha'(1)$, then $(A, D_M(A))$ generates a Markov semigroup $(T(t))_{t\geq 0}$ on the space $\mathcal{C}([0,1])$, as the same proof of Theorem 3.1 shows.

The problem is then to find a continuous selection $(\nu_x)_{0\leq x\leq 1}$ of probability Borel measures on $[0,1]$, satisfying (1.2), and two sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ in $[0,1]$ such that the corresponding operators C_n represent the semigroup by means of their iterates, as in Theorem 3.3. To this respect we have the following result.

THEOREM 3.8. *Under the previous hypotheses, further suppose that* $\alpha(x) \leq \frac{x(1-x)}{2}$ $(0 \leq x \leq 1)$ *and for every* $x \in [0,1]$ *set*

$$\nu_x := \lambda(x)\mu_x + (1-\lambda(x))\varepsilon_x, \tag{3.10}$$

where $\mu_x := x\varepsilon_1 + (1-x)\varepsilon_0$ *and the function* $\lambda \in \mathcal{C}([0,1])$ *is defined by* (3.3). *For every* $n \geq 1$ *set*

$$b_n := d/2 \quad \textit{and} \quad a_n := \begin{cases} 0 & 1/n \geq d/2, \\ d/2 - 1/n & 1/n < d/2. \end{cases} \tag{3.11}$$

Consider the sequence $(C_n)_{n\geq 1}$ *associated with the selection* $(\nu_x)_{0\leq x\leq 1}$ *and the sequences* $(a_n)_{n\geq 1}$ *and* $(b_n)_{n\geq 1}$. *Then, for every* $t \geq 0$, *for each sequence* $(k(n))_{n\geq 1}$ *of positive integers such that* $k(n)/n \to t$ $(n \to \infty)$ *and for every* $f \in \mathcal{C}([0,1])$,

$$T(t)f = \lim_{n\to\infty} C_n^{k(n)}(f) \quad \textit{uniformly on } [0,1].$$

Proof. We preliminarily observe that $0 \leq a_n < b_n \leq 1$ and $a_n + b_n \to d$. Moreover the mapping $x \mapsto \nu_x$ is continuous, the conditions (1.2) and (3.1) are verified and finally

$$\tfrac{1}{2}\left(\int_0^1 e_2 \mathrm{d}\nu_x - x^2\right) = \tfrac{\lambda(x)x(1-x)}{2} = \alpha(x) \qquad (0 \leq x \leq 1).$$

From Theorem 3.3, the result follows. $\square$

We point out that, if $d \leq 1$, then one can consider the sequences $a_n = 0$ and $b_n = d$ $(n \geq 1)$ instead of the ones given by (3.11).

We also remark that, under the assumptions of Theorem 3.8, the operator T corresponding to the selection (3.10) via formula (1.1) is given by

$$T(f) = (1-\lambda)f + f(1)\lambda e_1 + f(0)\lambda(1-e_1) \quad (f \in \mathcal{C}([0,1])).$$

Therefore, if such an operator T maps $\mathrm{Lip}_1 1$ into $\mathrm{Lip}_1 1$ and/or if it satisfies conditions (c_1) and (c_2), then the semigroup generated by $(A, D_M(A))$ maps $\mathrm{Lip}_M\alpha$ into itself $(M > 0,\ 0 < \alpha \leq 1)$ and/or continuous convex functions into convex functions.

Moreover, if $d = 2$, property (a) of Proposition 3.6 holds true as well.

For instance, if λ is constant, i.e., $\alpha(x) = \lambda\frac{x(1-x)}{2}$ $(0 \le x \le 1)$, then T maps $\mathrm{Lip}_1 1$ into $\mathrm{Lip}_1 1$ and it satisfies (c_1) and (c_2).

We shall finally restrict our attention to the particular case $d = 1$ and $\alpha(x) = \frac{x(1-x)}{2}$ $(0 \le x \le 1)$.

In this case Theorem 3.8 and Proposition 3.4 can be fully applied and the operators C_n, $n \ge 1$, whose iterates converge to the semigroup $(T(t))_{t\ge0}$ generated by $(A, D_M(A))$, are the Kantorovich operators defined by (1.5) with $a_n = 0$ and $b_n = 1$, $n \ge 1$.

In this particular case we wish to point out another property of the semigroup (and hence of the solutions of problems (3.9)) which is concerned with functions of bounded variation on $[0, 1]$. If $f : [0, 1] \longrightarrow \mathbb{R}$ is such a function, we denote by $V_{[0,1]}(f)$ its total variation, i.e.,

$$V_{[0,1]}(f) := \sup\left\{\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \,\Big|\, (x_i)_{0\le i\le n} \text{ partition of } [0,1]\right\}.$$

We recall that, if $(f_n)_{n\ge1}$ is a sequence of functions of bounded variation on $[0, 1]$ pointwise convergent to a function $f : [0, 1] \longrightarrow \mathbb{R}$ and if, in addition, $\sup\limits_{n\ge1} V_{[0,1]}(f) < +\infty$, then f is a function of bounded variation on $[0, 1]$ and $V_{[0,1]}(f) \le \sup\limits_{n\ge1} V_{[0,1]}(f_n)$.

In [14, Proposition 3.3], it is shown that for every function $f : [0, 1] \longrightarrow \mathbb{R}$ of bounded variation on $[0, 1]$ and for any $n \ge 1$

$$V_{[0,1]}(C_n(f)) \le V_{[0,1]}(f).$$

An analogue inequality holds true for the iterates of the operators C_n.

Therefore from (3.8) and from the previous remark it follows that, for every continuous function of bounded variation $f : [0, 1] \longrightarrow \mathbb{R}$ and, for every $t \ge 0$,

$T(t)f$ is a (continuous) function of bounded variation

and

$$V_{[0,1]}(T(t)(f)) \le V_{[0,1]}(f).$$

We leave as an open problem the question whether the above inequality is still true for other classes of semigroups as stated in Theorem 3.1.

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