REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 36 (2007) no. 1, pp. 25–38
ictp.acad.ro/jnaat

A DUAL GENERALIZATION OF CONVEX FUNCTIONS*

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Abstract. As it is well known, the convexity property of a function may be described by the quasiconvexity property of all "the dual perturbations" of this function. If we consider the "dual perturbation" only in a subset $M \subset X^*$ we obtain a general class of functions called M-convex. In this paper we establish some special properties and a continuity theorem of this new type of functions.

MSC 2000. 52A41, 46B20, 26A51, 90C25.
Keywords. M-convexity, convex function, quasiconvex function, extrem point, local bounded function.

1. INTRODUCTIONS

Taking as starting point the Crouzeix characterization of convex function by quasiconvexity property of all "dual perturbation" (see [7]), in an earlier paper we introduced a new type of convexity, only the "dual perturbation" in a given subset $M \subset X^*$.

In the sequel, X denotes a real linear normed space and X^* its topological dual. The symbol $(\cdot,\cdot)$ will be used for the usual pairing between X and X^*, while $\langle\ \cdot,\cdot\ \rangle$ will be used for the associated bilinear functional, i.e. $\langle x, x^*\rangle = x^*(x)$, for all $x \in X$, $x^* \in X^*$.

We recall some well known concepts in convex analysis (see [4], [5], [6], [10]).

For a function $f : X \longrightarrow \mathbb{R}\cup\{+\infty\}$ we denote by

$$\mathrm{Dom}(f) = \{x \in X|\ f(x) < +\infty\}$$

its *domain*.

When $\mathrm{Dom}(f)$ is nonempty we say that f is *proper*.

A function $f : X \longrightarrow \mathbb{R}\cup\{+\infty\}$ is said to be *convex* if for every x_1, $x_2 \in X$ and for every $\lambda \in [0,1]$ we have:

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2).$$

A function f is called *quasiconvex* if for every x_1, $x_2 \in X$ and for every $\lambda \in [0,1]$ we have:

$$f(\lambda x_1 + (1-\lambda)x_2) \le \max\{f(x_1), f(x_2)\},$$

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*This work was supported by Grant ID 387 4/28.09.2007.

equivalently

$$f(\lambda x_1 + (1-\lambda)x_2) \leq f(x_2)$$

whenever $x_1, x_2 \in X$, $\lambda \in [0,1]$ such that $f(x_1) \leq f(x_2)$.

Also, it is well known that a function is quasiconvex if and only if its level sets

$$L(f,\alpha) = \{x \in X |\ f(x) \leq \alpha\}$$

are convex for every $\alpha \in \mathbb{R}$.

Now, we remind the definition of M-convex functions, introduced in [1].

If M is a nonempty subset of X^*, we say that the function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is *M-convex* if for each $x^* \in M$ the sets

$$L(f,\alpha,x^*) = \{x \in X |\ f(x) \leq \alpha + \langle x^*, x\rangle\}$$

are convex for every $\alpha \in \mathbb{R}$.

If $-f$ is M-convex we say that f is *M-concave.*

Throughout this paper, for a given nonempty subset $M \subset X^*$, we will denote by

$\mathcal{C}(M)$ the class of all M-convex functions.

From this definition, we observe that if the set M contains the origin then we obtain a new type of convexity which lies between quasiconvexity and convexity.

For the beginning, we recall some property of this functions proved in [1].

Proposition 1. (i) *If $f \in \mathcal{C}(M)$ then $f - x^* \in C\,(M - x^*)$ for every $x^* \in X^*$.*

(ii) *$f \in \mathcal{C}(M)$ if and only if $f - x^*$ is quasiconvex, for every $x^* \in M$.*

(iii) *If $M_1 \subset M_2$ then $\mathcal{C}(M_2) \subseteq \mathcal{C}(M_1)$.*

(iv) *$\mathcal{C}(M) = C(\overline{M}^{w^*})$, where $\overline{M}^{w^*}$ is the closure of M with respect to w^*-topology.*

(v) *If $f \in \mathcal{C}(M)$ then for each $\lambda > 0, \lambda f$ is λM-convex.*

(vi) *If $f_i \in \mathcal{C}(M_i)$ for every $i \in I$, then $f = \sup\limits_{i\in I} f_i \in C(\bigcap\limits_{i\in I} M_i)$.*

(vii) *The domain of a M-convex function f is a convex set.*

Similarly to the convex case ([4], [5], [10]), the M-convex functions can be characterized with the aid of its one dimensional restrictions.

Theorem 2. *If $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ and $\emptyset \neq M \subseteq X^*$, then f is a M-convex function if and only if for every $x, v \in X$, the associated function F, defined by*

$$F(t) = f(x + tv),\ t \in \mathbb{R}$$

is M_v-convex, where

$$M_v = \{\langle x^*, v\rangle / x^* \in M\}\,.$$

In the following result, proved in [1], we characterize the M-convex functions using only its values on the line segments, establishing a characteristic inequality of convex type.

Theorem 3. *Let us consider* $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ *and* $\emptyset \neq M \subseteq X^*$. *Then* f *is* M*-convex if and only if*

$$f(\lambda x + (1-\lambda)y) \leq f(y) + \lambda \inf_{x^* \in M_{x,y}} \langle x^*, x - y \rangle \,, \tag{1}$$

for every $x, y \in X$, *and* $\lambda \in [0,1]$, *where*

$$M_{x,y} = \{x^* \in M |\, \langle x^*, x-y \rangle \geq f(x) - f(y)\} \,. \tag{2}$$

If f *is a* M*-convex function and*

$$\inf_{x^* \in M_{x,y}} \langle x^*, x-y \rangle = f(x) - f(y), \text{ for all } x, y \in X,$$

then f *is a convex function.*

In fact, by (2) we observe that

$$\inf_{x^* \in M_{x,y}} \langle x^*, x-y \rangle \geq f(x) - f(y), \text{ for all } x, y \in X.$$

On the other hand, if we have

$$\inf_{x^* \in M_{x,y}} \langle x^*, x-y \rangle \leq 0 \tag{3}$$

whenever $f(x) \leq f(y)$, then

$$f(\lambda x + (1-\lambda)y) \leq f(y),$$

i.e. f is quasiconvex.

Moreover, if (3) is fulfilled for all $x, y \in X$, then f is constant.

We recall that the *support functional* of a set $A \subset X^*$, σ_A is defined by

$$\sigma_A(x) = \sup_{x^* \in A} x^*(x), x \in X.$$

Thus, the inequality (1) can be rewritten as

$$f(\lambda x + (1-\lambda)y) \leq f(y) - \lambda \sigma_{M_{x,y}}(y - x).$$

Now, we consider some special cases for the set M. Thus, if M is a convex set, we have that

$$\inf_{x^* \in M_{x,y}} \langle x^*, x-y \rangle = \begin{cases} f(x) - f(y), & -\sigma_M(y-x) \leq f(x) - f(y) \leq \\ & \sigma_M(x-y), \\ -\sigma_M(y-x), & f(x) - f(y) \leq \; -\sigma_M(y-x), \\ +\infty, & f(x) - f(y) > \; \sigma_M(x-y). \end{cases} \tag{4}$$

If we take $M = S^*(0, r) = \{x^* \in X^* / \left\| x^* \right\| \leq r\}$, then

$$\sigma_M(y-x) = r \left\| y - x \right\|$$

and (4) becomes

$$\inf_{x^* \in M_{x,y}} \langle x^*, x-y \rangle = \begin{cases} f(x) - f(y), & -r \left\| y-x \right\| \leq f(x) - f(y) \leq \\ & r \left\| y - x \right\|, \\ -r \left\| y-x \right\|, & f(x) - f(y) \leq \; -r \left\| y - x \right\|, \\ +\infty, & f(x) - f(y) > r \left\| y - x \right\|, \end{cases}$$

or, equivalently, if $f(x) \leq f(y)$, then

$$\inf_{x^* \in M_{x,y}} \langle x^*, x - y \rangle = \begin{cases} f(x) - f(y), & f(y) - f(x) \leq r \|y - x\|, \\ -r \|x - y\|, & f(y) - f(x) \geq r \|y - x\|. \end{cases}$$

Considering this, it is easy to prove that if $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a Lipschitz function with constant L and f is $S^*(0, L)$-convex, then f is convex.

In [12], H. X. Phu and P. T. An introduce the notion of s-quasiconvex (s from "stable") functions, and he show that this functions are stable with respect to the following properties: "all lower level sets are convex", "each local minimum is a global minimum", "each stationary point is a global minimizer".

We specify that a function $f : D \subset X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is called *s-quasiconvex if there exists* $\sigma > 0$ such that

$$\frac{f(x_0) - f(x_1)}{\|x_0 - x_1\|} \leq \delta \text{ implies } \frac{f(x_\lambda) - f(x_1)}{\|x_\lambda - x_1\|} \leq \delta$$

for $|\delta| < \sigma$, $x_0, x_1 \in D$, $x_\lambda = (1 - \lambda)x_0 + \lambda x_1$ and $\lambda \in [0, 1]$.

The authors show that this functions can be characterized as follows

(5) a function f is s-quasiconvex if and only if there exists $\varepsilon > 0$ such that $f - x^*$ is quasiconvex, for every x^* with the property $\|x^*\| < \varepsilon$.

Remark 4. The relation (5) can be obtained (on the other way) from Theorem 3, taking M a ball with the center in the origin. In fact, as we see in the relation (5), the class of s-quasiconvex functions are the same with the class of M-convex functions, with $0 \in \operatorname{int}(M)$. □

Now we will consider the sets $M \subseteq X^*$ with the following property

(P) for every $x \in X \backslash \{0\}$, there exists a sequence $\{x_n^*\}_{n \in \mathbb{N}} \subseteq M$ such that $\langle x_n^*, x \rangle \searrow 0$.

It is easy to prove that if $0 \in \operatorname{int}(M)$, then the set M has the property (P).

Example 5. Let $X = l^1$ and $M = \{\alpha_n e_n \mid \text{where } \alpha_n \in (-\frac{1}{n}, \frac{1}{n}),\ n \in \mathbb{N},\ \{e_i\}_{i \in \mathbb{N}^*}\text{-canonical bases in } l^1\}$. It is easily to prove that the set M has the property (P). The function defined by

$$f(x) = \begin{cases} \sup_{x^* \in M} \langle x^*, x \rangle, & x \neq 0 \\ -1, & x = 0, \end{cases}$$

is M-convex, but she is not s-quasiconvex. □

In the following proposition we present a sufficient condition from property (P).

Proposition 6. *Let X be a normed linear spaces such that* $\dim(X) \geq 2$ *and $M \subseteq X^*$ a bounded set, with the property* $\overline{\operatorname{con}(M)}^{w^*} = X^*$. *Then the set M has the property (P).*

Proof. We start with $x \in X\backslash\{0\}$. Since $\dim(X) \geq 2$ then there exists x^*, $y^* \in X^*\backslash\{0\}$ such that $\langle x^*, x\rangle = 0$ and $\langle y^*, x\rangle > 0$. For every $n \in \mathbb{N}$ we consider $y_n^* = \frac{1}{n}y^* + (1-\frac{1}{n})x^*$. Taking into account that $\overline{\mathrm{con}(M)}^{w^*} = X^*$, we find $\lambda_n > 0$ and $\overline{y}_n^*$ such that $\langle \overline{y}_n^*, x\rangle > 0, \langle y_n^* - \overline{y}_n^*, x\rangle < \frac{1}{n}$ and $\lambda_n \overline{y}_n^* \in M$ for every $n \in \mathbb{N}$. To prove that the set M has the property (P) we passing to limit in the following relation

$$0 < \langle \lambda_n \overline{y}_n^*, x\rangle = \lambda_n \langle \overline{y}_n^* - y_n^*, x\rangle + \lambda_n \langle y_n^*, x\rangle < \frac{\lambda_n}{n}(1 + \langle y^*, x\rangle),$$

and we obtain that $\langle \lambda_n \overline{y}_n^*, x\rangle \searrow 0$, because M is bounded. □

LEMMA 7. *Let X be a linear normed space and $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ an $Fr(S^*(0;r))$-convex function, for some $r > 0$. If $\dim(X) \geq 2$ then f is $S^*(0;r)$-convex.*

Proof. It is easy to prove that $S^*(0;r)_v = [-r\|v\|, r\|v\|]$. Since $\dim(X) \geq 2$, $Fr(S^*(0;r))$ is a conex set, therefore we obtain that $Fr(S^*(0;r))_v$ is an interval, namely

$$Fr(S^*(0;r))_v = [\inf_{\|x^*\|=r} \langle x^*, v\rangle, \sup_{\|x^*\|=r} \langle x^*, v\rangle] = [-r\|v\|, r\|v\|].$$

Using the characterization given by Theorem 2, we obtain that f is $S^*(0;r)$-convex. □

THEOREM 8. *Let X be a linear normed space such that $\dim(X) \geq 2$. Then $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is s-quasiconvex if and only if there exists $r > 0$ such that f is $Fr(S^*(0;r))$-convex.*

Proof. The theorem is a consequence of Lemma 7 and the characterization of s-quasiconvex functions. □

THEOREM 9. *Let $M \subseteq X^*$ be a set such that $0 \in int(M)$. If f is M-convex then for every $x, y \in X$, with $f(x) \leq f(y)$, there exists $\alpha \in (0,1]$ such that:*

$$f(\lambda x + (1-\lambda)y) \leq \lambda\alpha f(x) + (1-\lambda\alpha)f(y) \text{ for every } \lambda \in [0,1].$$

Proof. Obviously, if $x = y$, we can take $\alpha = 1$. If $x \neq y$ and $f(x) \leq f(y)$ then we find $0 \neq x_0^* \in X^*$ such that $\langle x_0^*, x - \bar{x}\rangle = f(x) - f(y)$. Since $0 \in \mathrm{int}(M)$ there exists $\alpha \in (0,1]$ such that $\alpha x_0^* \in M$. Therefore $\alpha x_0^* \in M_{x,y}$ because

$$\langle \alpha x_0^*, x - y\rangle = \alpha \langle x_0^*, x - y\rangle \geq f(x) - f(y).$$

Now, taking into account that f is M-convex , for every $\lambda \in [0,1]$ we get

$$f(\lambda x + (1-\lambda)y) \leq f(y) + \lambda \inf_{x^* \in M_{x,y}} \langle x^*, x - y\rangle \leq f(y) + \lambda \langle \alpha x_0^*, x - y\rangle,$$

and the proof is complete. □

When $M \subset X^*$ is a cone, we denote

$$M^{\perp} = \{x \in X | \langle x^*, x\rangle = 0, \text{ for every } x^* \in M\}$$

and

$$M^{\lozenge} = \{x \in X|\text{ exists } x^* \in M \text{ such that } \langle x^*, x\rangle > 0\} \cup \{0\}.$$

Is easy to prove that $M^{\lozenge}$ is a convex cone.

Thus, a function $f : A \to \mathbb{R} \cup \{+\infty\}$ is called increasing related to $M^{\lozenge}$ if

$$x - y \in M^{\lozenge} \Longrightarrow f(x) \geq f(y)$$

THEOREM 10. *Let $A \subset X$ be a convex set and $f : A \to \mathbb{R} \cup \{+\infty\}$ be a M-convex function such that $(A - A) \cap M^{\perp} = \{0\}$.*
If f is increasing related to $M^{\lozenge}$ on the set A, then f is a convex function on A.

Proof. Let x, y be two points from A and $\lambda \in [0, 1]$.
If $x - y \in \mathcal{C}(M^{\lozenge} \cup -M^{\lozenge})$, then $x \neq y$ and for every $x^* \in M$ we have $\langle x^*, x - y\rangle = 0$, i.e. $x - y \in (A - A) \cap M^{\perp}$. By hypothesis we find that $x = y$, which is a contradiction. Consequently, we have $x - y \in M^{\lozenge}$ or $y - x \in M^{\lozenge}$. Now, we suppose that $x - y \in M^{\lozenge}$. Then $f(x) \geq f(y)$ because f is increasing related to $M^{\lozenge}$ on A. Also, there exists $x^* \in M$ such that $\langle x^*, x - y\rangle > 0$.

Since M is a cone, then

$$\inf_{x^* \in M_{x,y}} \langle x^*, x - y\rangle = f(x) - f(y)$$

and taking into account that f is a M-convex function we obtain that

$$f(\lambda x + (1 - \lambda)y) \leq f(y) + \lambda \inf_{x^* \in M_{x,y}} \langle x^*, x - y\rangle = \lambda f(x) + (1 - \lambda)f(y).$$

We proceed similarly when $y - x \in M^{\lozenge}$.

This proved that f is a convex function on the set A. □

Now, let us consider the special case of linear subspaces of X^*.

THEOREM 11. *Let M be a proper linear subspace of X^* and let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a M-convex function. Then:*

(i) *$f - x^* \in \mathcal{C}(M)$, for every $x^* \in M$;*
(ii) *$\lambda f \in \mathcal{C}(M)$, for every $\lambda \geq 0$;*
(iii) *$f_{|Y}$ is convex, whenever Y is a linear subspace such that $Y \cap M^{\perp} = \{0\}$.*

Proof. Since M is a linear subspace then the properties (i) and (ii) follow immediately by Proposition 1 (properties (i) and (v)).
(iii) Let Y be a linear subspace such that $Y \cap M^{\perp} = \{0\}$. Let us take $x, y \in Y$, and $\lambda \in [0, 1]$. If $x \neq y$, then $x - y \notin M^{\perp}$. Since M is a proper linear subspace there exists $x^* \in M$ such that $\langle x^*, x - y\rangle = f(x) - f(y)$, and so

$$\inf_{x^* \in M_{x,y}} \langle x^*, x - y\rangle = f(x) - f(y)$$

for any $x, y \in Y$, i.e. $f_{|Y}$ is a convex function. □

COROLLARY 12. *Let M be a subset of X^* such that $\overline{\text{span}M} = X^*$ and $\lambda M \subseteq M$ for every $\lambda \geq 0$. Then every M-convex function is convex.*

Proof. If $\overline{\text{span}M} = X^*$ then $M^{\perp} = \{0\}$. Following the proof of above theorem (iii), we observe that if $\lambda M \subseteq M$ for every $\lambda \geq 0$, then $f_{|Y}$ is convex, whenever Y is a linear subspace. Taking $Y = X$, we obtain that f is a convex function. □

In Proposition 1 (iv), we see that if $\overline{M}^{w^*} = X^*$ then, every M-convex function is also a convex function. But this sufficient condition is not necessary. In an earlier paper we established one more general result concerning the equality $\mathcal{C}(M_1) = \mathcal{C}(M_2)$. It is obvious that if $M_1 \subset M_2$, then every M_2-convex function is also M_1-convex, but conversely is generally not true.

THEOREM 13 ([1]). *Let $M_2 \subset X^*$ be an open nonempty set. If $\emptyset \neq M_1 \subset M_2$, then $\mathcal{C}(M_1) = \mathcal{C}(M_2)$ if and only if*

$$\text{(6)} \qquad \begin{array}{l} \textit{for every } x^* \in M_2, \textit{ and } x \in X\setminus\{0\}, \textit{ there exists a sequence} \\ \quad (x_n^*)_{n\in\mathbb{N}} \subseteq M_1 \textit{ such that } \langle x_n^* - x^*, x\rangle \searrow 0. \end{array}$$

Taking now $M_2 = X^*$ the property (6) can be written in the following form

$$\text{(7)} \qquad \begin{array}{l} \text{for every } \alpha \in \mathbb{R}, \text{ and } x \in X\setminus\{0\}, \\ \text{exists a sequence } (x_n^*)_{n\in\mathbb{N}} \subseteq M \text{ such that } \langle x_n^*, x\rangle \longrightarrow \alpha. \end{array}$$

Consequently, we obtain a characterization of the special cases when M-convexity coincides with convexity.

It is easily to prove that if $\overline{M}^{w^*} = X^*$, then the set M has the property (7), but conversely is not always true, as we can see if we consider $X = l^1$ and

$$M = \left\{x^* = (x_n)_{n\in\mathbb{N}} \in l^\infty \mid \exists n_0 \in \mathbb{N} \text{ such that } x_{n_0} \neq 0, x_n = 0, \forall n \neq n_0\right\}.$$

2. THE EXTREME POINTS OF M- CONVEX FUNCTIONS

In the sequel we shall be concerned with a family of functions that lies between the family of strictly quasiconvex functions and the family of the semistrictly quasiconvex functions.

Let us recall that a function f is strictly quasiconvex [4] if

$$\begin{array}{l} f(x) \leq f(y) \text{ implies that } f(\lambda x + (1-\lambda)y) < f(y), \\ \text{for every } x \neq y, \text{ and } 0 < \lambda < 1. \end{array}$$

Similarly, the condition for the semistrictly quasiconvexity [4] can be written as

$$\text{(8)} \qquad \begin{array}{l} f(x) < f(y) \text{ implies that } f(\lambda x + (1-\lambda)y) < f(y) \\ \text{where } x, y \in X, \text{ and } 0 < \lambda < 1. \end{array}$$

An important property of the convex functions is that every local minimum is a global one. This property, however, holds for more general families of functions (for instance, the family of semistrict quasiconvex functions, see [4]). In this line we consider the sets $M \subseteq X^*$ which satisfy the property (P).

Considering a set M with the property (P), we want to see the relationships between the family of M-convex functions and the families of generalized convex functions above defined.

When $M \subset X^*$ has the property (P), a M-convex function is not necessary a strictly quasiconvex function as we see if we consider the function $f(x) = 1$.

The following theorem shows the relationship between M-convexity and semistrict quasiconvexity.

THEOREM 14. *If $M \subset X^*$ has the property (P), then every M-convex function is a semistrictly quasiconvex function.*

Proof. Let us consider $f \in \mathcal{C}(M)$, $x, y \in X$ such that $f(x) < f(y)$, and $0 < \lambda < 1$. Since $f(x) < f(y)$, by virtute of property (P) there exists $x_0^* \in M$ such that

$$f(x) - f(y) < \langle x_0^*, x - y \rangle < 0.$$

Thus, according to (1) we have

$$f(\lambda x + (1-\lambda)y) \le f(y) + \lambda \inf_{x^* \in M_{x,y}} \langle x^*, x - y \rangle \le f(y) + \lambda \langle x_0^*, x - y \rangle < f(y).$$

Therefore (8) is fulfilled, i.e. f is semistrictly quasiconvex. □

COROLLARY 15. *If $M \subset X^*$ has the property (P) and $f \in \mathcal{C}(M)$, then every locally extreme point from* Dom(f) *is a minimum global point. Moreover, the set of points at which f attains its global minimum is a convex set.*

Proof. If $f \in \mathcal{C}(M)$ then by Theorem 14 it follows that f is semistrictly quasiconvex, therefore every locally extreme point from Dom(f) is a global minimum point (see [4]). If x, y are two global minimum points then

$$\inf_{x^* \in M_{x,y}} \langle x^*, x - y \rangle = 0,$$

and so, by (1), we obtain that

$$f(\lambda x + (1-\lambda)y) \le f(y), \text{ for every } \lambda \in [0, 1].$$

This prove that the set of global minimum points is a convex set. □

REMARK 16. When $M \subset X^*$ has the property (P), the function f may not have the strict local maximum points; moreover, in every locally maximum point the function f is locally constant. If f is M-convex and attains its maximum on int(Dom(f)), then f is constant. When $M \subset X^*$ has the property (P), the main difference between semistrictly quasiconvex functions and M-convex functions is that for M-convex functions the set of minimum points is a convex set, property which is not true in the case of semistrictly quasiconvex functions. For example, the function f, defined on $\mathbb{R}$ by $f(x) = 0$ for $x \neq 0$ and $f(x) = 2$ for $x = 0$, is semistrictly quasiconvex but the set of its global minimum points is not convex. □

3. CONTINUITY OF M-CONVEX FUNCTIONS

In this section we will study the continuity property of M-convex functions. The following result will be needed later on.

LEMMA 17. *Let $M \subseteq X^*$ be an open set. If $f : X \to \mathbb{R} \cup \{+\infty\}$ is a M-convex function and $x, y, z \in \operatorname{dom} f$ such that $y = \lambda x + (1-\lambda) z$ with $\lambda \in (0,1)$ then*

$$\frac{f(y)-f(x)}{\|y-x\|} \leq \frac{1}{\|z-y\|} \inf_{x^* \in M_{z,y}} \langle x^*, z-y \rangle \tag{9}$$

Proof. Indeed, if $x^* \in M_{z,y}$, since f is M-convex then

$$L(f, f(y) - \langle x^*, y \rangle, x^*) = \{u \in X / f(u) \leq f(y) + \langle x^*, u-y \rangle\}$$

is convex and $y, z \in L(f, f(y) - \langle x^*, y \rangle, x^*)$.

If we suppose that

$$\frac{f(y)-f(x)}{\|y-x\|} > \frac{1}{\|z-y\|} \inf_{x^* \in M_{z,y}} \langle x^*, z-y \rangle ,$$

then

$$f(x) < f(y) - \frac{\|y-x\|}{\|z-y\|} \langle x^*, z-y \rangle = f(y) + \langle x^*, x-y \rangle .$$

Since M is an open set, there exist $\overline{\alpha} > 0$ and $x_0^* \in X^*$ such that $\langle x_0^*, x-y \rangle = -1$ and $x^* + \alpha x_0^* \in M$, for every $\alpha \in (0, \overline{\alpha})$. Now, taking α sufficiently small, we have

$$\begin{aligned} f(x) &< f(y) + \langle x^* + \alpha x_0^*, x-y \rangle , \\ f(y) &= f(y) + \langle x^* + \alpha x_0^*, y-y \rangle , \\ f(z) &< f(y) + \langle x^* + \alpha x_0^*, z-y \rangle . \end{aligned}$$

Denoted $x^* + \alpha x_0^*$ by $\overline{x}^*$ and taking

$$\epsilon = \frac{1}{2} \min\{f(x) - f(y) - \langle x^* + \alpha x_0^*, x-y \rangle , f(z) < f(y) + \langle x^* + \alpha x_0^*, z-y \rangle\},$$

it follows that

$$x, z \in L(f, f(y) - \langle \overline{x}^*, y \rangle - \epsilon, \overline{x}^*) \tag{10}$$

and

$$y \notin L(f, f(y) - \langle \overline{x}^*, y \rangle - \epsilon, \overline{x}^*). \tag{11}$$

From (10) and (11) we obtain that the set $L(f, f(y) - \langle \overline{x}^*, y \rangle - \epsilon, \overline{x}^*)$ is not convex, which is not true.

Therefore

$$f(x) \geq f(y) - \frac{\|y-x\|}{\|z-y\|} \langle x^*, z-y \rangle \text{ for every } x^* \in M_{z,y}$$

which proved (9). □

Now we will define a special type of radial convexity.

Definition 18. *We say that the function* $f : X \to \mathbb{R} \cup \{+\infty\}$ *is radial upper convex if for every* $x \in X$*, there exists* $\varepsilon_x > 0$ *such that for every* $v \in X$*, with* $\|v\| = 1$*, the function* $F(t) = f(x + tv)$ *is* $(-\infty, -\varepsilon_x) \cup (\varepsilon_x, +\infty)$*-convex.*

Example 19. The function $f : X \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{2}\|x\|^2, & \|x\| \le 1 \\ \frac{1}{2}, & \|x\| > 1. \end{cases}$$

is radial upper convex but is not convex. Is obvious that this function is not convex. To prove that she is radial upper convex, we show that for every $x, v \in X$, with $\|v\| = 1$, the function $F : \mathbb{R} \longrightarrow \mathbb{R}$, defined by

$$F(t) = f(x + tv) - \alpha t \ ,$$

is quasiconvex for every $\alpha \in (-\infty, -1) \cup (1, +\infty)$. Since F is continuous, is sufficiently to prove that ∂F is an quasimonotone operator for every $\alpha \in (-\infty, -1) \cup (1, +\infty)$. Taking into account that $|\langle v, J(x + tv)\rangle| \le 1$ for every t with the property $\|x + tv\| \le 1$, we obtain that ∂F is quasimonotone (see [2], [5]).

We denote by J the duality mapping between X and X^*, defined by:

$$J(x) = \left\{x^* \in X^* \mid \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\right\}, \ x \in X.$$

It is well known (see [5]) that this mapping is the subdifferential of the function $\frac{1}{2}\|\cdot\|^2$, i.e. $J(x) = \partial(\frac{1}{2}\|x\|^2)$, $x \in X$. □

In the sequel, we say that the set $M \subseteq X^*$ has property (P^*) if satisfy

(P^*) for every $x \in X\backslash\{0\}$ exist $\{x_n^*\}_{n\in\mathbb{N}} \subseteq M$ such that $\langle x_n^*, x\rangle \to \infty$.

Theorem 20. *Let* M *be a set with property* (P^*) *and let* $f : X \to \mathbb{R}\cup\{+\infty\}$ *be a* M*-convex function. Then* f *is radial continuous on* $ri(\text{dom } f)$.

Proof. Without loose our generality, considering Theorem 2, we can suppose that f is defined on $\mathbb{R}$, otherwise we take $F(t) = f(x+tv)$ for $x, v \in X$. Thus, we must to prove that f is continuous. If $x_0 \in ri(\text{dom } f)$ and we suppose that f is not lower semicontinuous in x_0, then there exist $\epsilon > 0$ and a net $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \to x_0$ such that

$$-\epsilon > f(x_n) - f(x_0), \text{ for every } n \in \mathbb{N}. \tag{12}$$

We can suppose that $x_n > x_0$, for every $n \in \mathbb{N}$ (analogue when $x_n < x_0$).

Since $x_0 \in ri(\text{dom } f)$, there exists $u_1 < x_0$ such that $u_1 \in \text{dom } f$. Taking into account that f is M-convex and the property (P^*) is fulfilled, then there exists $x^* \in M$ such that

$$f(u_1) < f(x_0) - \frac{\epsilon}{2} + \langle x^*, u_1 - x_0\rangle \,. \tag{13}$$

But, by (12), exists $u_2 > x_0$ such that

$$f(u_2) < f(x_0) - \frac{\epsilon}{2} + \langle x^*, u_2 - x_0\rangle \,. \tag{14}$$

Considering (13) and (14) we obtain that $L(f, f(x_0) - \frac{\epsilon}{2}, x^*)$ is not convex i.e. f is not M-convex. Therefore f must be lower semicontinuous in x_0. Now, if we suppose, by a contradiction, that f is not upper semicontinuous in $x_0 \in ri(\text{dom} f)$, then there exist a net $\{x_n\}_{n\in\mathbb{N}} \subset \text{dom}(f)$ and $\epsilon > 0$ such that $x_n \to x_0$ and

$$f(x_n) - f(x_0) > \epsilon, \text{ for every } n \in \mathbb{N}.$$

Following the same steps as in previous case we obtain that there exist $v_1, v_2 \in \text{dom}(f)$ and $x^* \in M$ such that

$$f(v_1) > f(x_0) + \frac{\epsilon}{2} + \langle x^*, v_1 - x_0 \rangle . \tag{15}$$

and

$$f(v_2) > f(x_0) + \frac{\epsilon}{2} + \langle x^*, v_2 - x_0 \rangle . \tag{16}$$

But, by (15) and (16) we find that $L(f, f(x_0) + \frac{\epsilon}{2}, x^*)$ is not convex. This ended the proof. □

Remark 21. In the above theorem the condition that f to be M-convex can be replaced by the following property: for every $x \in X$ and $v \in X$ there exist $M_{xv} \subset \mathbb{R}$ such that $F(t) = f(x+tv)$, $t \in \mathbb{R}$, is M_{xv}-convex and (P^*) is fulfilled. Particularly, this property holds for radial upper convex functions. □

Theorem 22. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a radial upper convex function. If f is bounded from above in a neighborhood of one point $x_0 \in \text{int}(\text{dom } f)$, then it is locally bounded, that is, for every $x \in \text{int}(\text{dom } f)$ there exists a neighborhood on which f is bounded.*

Proof. We first show that if f is bounded from above in a neighborhood of x_0, it is also bounded from below in the same neighborhood. Since f is radial upper convex, we can find $r_{x_0} > 0$ such that for every $v \in X$ with $\|v\| = 1$, the function $F(t) = f(x_0 + tv)$ is $(-\infty, -r_{x_0}) \cup (r_{x_0}, +\infty)$-convex. Let $\varepsilon > 0$, $K > 0$ such that $f(x) \le K$ for every $x \in S(x_0; \varepsilon) \subseteq \text{dom} f$.

For every $z \in S(x_0; \varepsilon)$, there exists $\lambda \ge 0$ and $y \in X$ such that $\|y\| = 1$ and $z = x_0 + \lambda y$.

Taking $x_0 = \frac{1}{2}(x_0 + \lambda y) + \frac{1}{2}(x_0 - \lambda y)$, since the function F is $(-\infty, -r_{x_0}) \cup (r_{x_0}, +\infty)$-convex, we obtain

$$\begin{aligned} F(0) &\le F(\lambda) + \frac{1}{2}\inf\{-2\lambda t \mid |t| > r_{x_0},\ F(\lambda) - F(-\lambda) \le -2\lambda t\} \\ &\le F(\lambda) + \lambda\inf\{|t| \mid F(\lambda) - F(-\lambda) \le -2\lambda t\} \le F(\lambda) + \max\{2K, r_{x_0}\}, \end{aligned}$$

therefore $f(z) \ge f(x_0) - \max\{2k, r_{x_0}\} \ge -K - \max\{2k, r_{x_0}\}$, which prove that f is bounded from below on $S(x_0, \epsilon)$.

Let $x \in \text{int}(\text{dom } f)$ and $x \ne x_0$. Then $x = x_0 + \lambda y$, where again $y \in X$, $\|y\| = 1$ and μ is a positive number.

Since $x \in \mathrm{int}(\mathrm{dom}\ f)$, there exists $\alpha > \mu$ such that $v = x_0 + \alpha y \in \mathrm{int}(\mathrm{dom} f)$. Taking $\lambda = \frac{\mu}{\alpha}$, the set

$$V = \{u \in \mathrm{dom}\ f \mid u = (1-\lambda)z + \lambda v,\ z \in S(x_0;\varepsilon)\}$$

is a neighborhood of x ($V = S(x,\gamma)$, where $\gamma = (1-\lambda)\varepsilon$). Therefore we find $r_z > 0$ such that for every $u \in V$, the function defined by $F(\lambda) = f(z+\lambda(v-z))$, $\lambda \in \mathbb{R}$, is $(-\infty,-r_z) \cup (r_z,+\infty)$-convex, thus

$$F(\lambda) \le F(0) + \lambda \inf \{t \mid |t| > r_z,\ F(1) - F(0) \le t\}.$$

Since f is bounded on $S(x_0;\varepsilon)$, then

$$F(1) - F(0) \le 2K,$$

and

$$\inf \{t \mid |t| > r_z,\ F(1) - F(0) \le t\} \le \max\{2K, r_z\},$$

which proved that

$$f(u) \le K + \max\{2K, r_z r\},$$

for every $u \in V$, i.e. f is also bounded from above on V, as claimed. □

Now, we establish an extension of well known continuity theorem of convex functions (see [5], [6]).

Theorem 23. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a radial upper convex function. If f is bounded from above in a neighborhood of one point $x_0 \in \mathrm{int}(\mathrm{dom}\ f)$, then f is continuous on* $\mathrm{int}(\mathrm{dom}\ f)$.

Proof. By Theorem 22, for each $x_0 \in \mathrm{int}(\mathrm{dom}\ f)$ we can find a neighborhood $S(x_0, 2\epsilon)$ on which f is bounded, i.e. there exist $K_{x_0} > 0$ such that

$$|f(x)| \le K_{x_0} \text{ for every } x \in S(x_0, 2\epsilon).$$

For the begining we prove that for every $x \in \mathrm{int}(\mathrm{dom}\ f)$ we find a constant $\overline{K}_{x_0} > 0$ such that

$$\inf_{\substack{|t|>r_{x_0} \\ t\|x-x_0\| \ge f(x)-f(x_0)}} t \ge -\overline{K}_{x_0} \tag{17}$$

and

$$\inf_{\substack{|t|>r_{x_0} \\ t\|x-x_0\| \ge f(x_0)-f(x)}} t \ge -\overline{K}_{x_0} \tag{18}$$

for every $x \in S(x_0;\varepsilon)$. If we suppose, to the contrary, that (17) not holds, there exists $x \in S(x_0,\epsilon)$ such that

$$\inf_{\substack{|t|>r_{x_0} \\ t\|x-x_0\| \ge f(x)-f(x_0)}} t < -\frac{2K_{x_0}}{\varepsilon}.$$

Now, we take $y \in S(x_0;2\varepsilon)$ such that $x_0 = \lambda x + (1-\lambda)y$ and $\|y - x_0\| = \varepsilon$.

Since the function $F(t) = f(x_0 + t\frac{x-x_0}{\|x-x_0\|})$ is $(-\infty, -r_{x_0}) \cup (r_{x_0}, +\infty)$-convex, from Lemma 17, we obtain that

$$\frac{f(x_0)-f(y)}{\varepsilon} \leq \frac{1}{\|x-x_0\|} \inf_{\substack{|t|>r_{x_0} \\ t\|x-x_0\| \geq f(x)-f(x_0)}} t\,\|x-x_0\| < -\frac{2K_{x_0}}{\varepsilon}.$$

Therefore $f(x_0) - f(y) < -2K_{x_0}$ and hence $f(y) > K_{x_0}$, which is not possible because $y \in S(x_0; 2\varepsilon)$. Consequently, the inequality (17) is always true.

Analogue we proof relation (18), using in this case the function $F(t) = f(x + t\frac{x_0-x}{\|x-x_0\|})$, which is also a $(-\infty, -r_{x_0}) \cup (r_{x_0}, +\infty)$-convex function.

Now, if we suppose that f is not continuous in x_0, then we find $\epsilon > 0$ and a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq S(x_0; \varepsilon)$ such that $x_n \to x_0$,

$$f(x_n) - f(x_0) > \varepsilon \tag{19}$$

or

$$f(x_n) - f(x_0) < -\varepsilon. \tag{20}$$

From (19) and (20) we find $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$,

$$f(x_n) - f(x_0) = \inf_{\substack{|t|>r_{x_0} \\ t\|x-x_0\| \geq f(x_n)-f(x_0)}} t\,\|x_n - x_0\|$$

and

$$f(x_0) - f(x_n) = \inf_{\substack{|t|>r_{x_0} \\ t\|x-x_0\| \geq f(x_0)-f(x_n)}} t\,\|x_n - x_0\|.$$

From above relations we obtain that

$$|f(x_n) - f(x_0)| \leq \overline{K}_{x_0} \|x_n - x_0\|, \text{ for every } n \geq n_0,$$

which is not possible because is a contradiction with (19) or (20). □

REMARK 24. It is obvious that in the special case $M = X^*$ we obtain the usual continuity theorem of convex functions. □

ACKNOWLEDGEMENT. I would like to express my deep gratitude to Professor Teodor Precupanu, for his support, for the invaluable suggestions, and attention to my work.

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Received by the editors: July 31, 2007.