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WEAKER CONDITIONS FOR THE CONVERGENCE OF NEWTON-LIKE METHODS

IOANNIS K. ARGYROS*

Abstract. We provide a semilocal convergence analysis for a certain class of Newton-like methods for the solution of a nonlinear equation containing a non differentiable term. Our approach provides: weaker sufficient conditions; finer error bounds on the distances involved; a more precise information on the location of the solution than before, and under the same computational cost.

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1. INTRODUCTION

In this study we are concerned with the problem of approximating a locally unique solution x^* of the nonlinear equation

$$F(x) + G(x) = 0, \tag{1}$$

where F, G are operator defined on an open subset Q at a Banach space X with values in a Banach space Y. Operator F is Fréchet-differentiable on $\overline{U}(\mathbf{z}, R)$, while the differentiability of G is not assumed.

Recently, in [3], we used the Newton-like method

$$x_0 \in U(z, R), \ x_{n+1} = x_n - A(x_n)^{-1}[F(x_n) + G(x_n)]\,(n \geq 0) \tag{2}$$

to generate a sequence approximating x^*. Here, $A(v) \in L(X, Y)\,(v \in X)$, denotes the space of bounded linear operators from X into Y. If $A(x) = F'(x)$ $\left(x \in \overline{U}(z, R)\right)$, then method (2) reduces to the popular Newton's method

$$y_{n+1} = y_n - F'(y_n)^{-1} F(y_n) \quad \left(y_0 \in \overline{U}(z, R)\right) \quad (n \geq 0). \tag{3}$$

A survey on local as well as semilocal convergence theorems for Newton methods can be found in [2]–[18], and the references there.

*Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, U.S.A., e-mail: iargyros@cameron.edu.

Throughout this study we assume there exist $z \in X$, $R > 0$, $a \geq 0$, $b \geq 0$, $\bar{\eta} \geq 0$ with $A(z)^{-1} \in L(Y,X)$, and for any $x, y \in \overline{U}(z,r) \subseteq \overline{U}(z,R) = \{x \in X \,|\, \|x - z\| \leq R\} \subseteq Q$

$$\left\| A(z)^{-1} \left[A(x) - A(x_0) \right] \right\| \leq w_0 (\|x - x_0\|) + a, \tag{4}$$

$$\begin{aligned} \left\| A(z)^{-1} \left[F'(x + t(y - x)) - A(x) \right] \right\| &\leq \\ \leq w(\|x - z\| + t\|x - y\|) - w_1(\|x - z\| + b, t \in [0,1]) \end{aligned} \tag{5}$$

$$\left\| A(z)^{-1} \left[G(x) - G(y) \right] \right\| \leq w_2(r) \|x - y\|, \tag{6}$$

$$\left\| A(z)^{-1} \left[F(z) + G(z) \right] \right\| \leq \bar{\eta}, \tag{7}$$

where, $w_0(r)$, $w_1(z)$, $w_2(r)$, $w(r)$, $w(r + t) - w_1(r)$ $(t \geq 0)$ are non-decreasing, non-negative functions on $[0, R]$ with $w(0) = w_0(0) = w_1(0) = w_2(0) = 0$, and parameters a, b satisfy

$$a + b < 1. \tag{8}$$

Using (4)–(8) instead of the less flexible conditions considered in [5], [8]–[13], [18] we showed in [3] and under the same computational cost that the following can be obtained:

(a) weaker sufficient convergence conditions for method (2);

(b) finer error bounds on the distances

$$\|x_{n+1} - x_n\|, \quad \|x_n - x^*\| \quad (n \geq 0);$$

(c) more precise information on the location of the solution.

Here we continue the work in [3] to show how to improve even further on (a)–(c).

2. SEMILOCAL CONVERGENCE ANALYSIS FOR METHOD (2)

It is convenient to define scalar iteration $\{t_n\}$ for some $r_0 \in [0,r]$, $r \in [0,R]$, $\eta \geq 0$

$$\begin{aligned} &t_0 = r_0, \quad t_1 = r_0 + \eta, \\ &t_{n+2} = t_{n+1} + \tfrac{1}{1 - a - w_0(t_{n+1})} \left\{ \left[\int_0^1 w\left[t_n + \theta\left(t_{n+1} - t_n\right)\right] \mathrm{d}\theta - w_1(t_n) + b \right] \right. \\ &\qquad \left. \cdot \left(t_{n+1} - t_n\right) + \int_{t_n}^{t_{n+1}} w_2(\theta) \mathrm{d}\theta \right\} \quad (n \geq 0). \end{aligned} \tag{9}$$

Iteration $\{t_n\}$ plays a crucial role in the study of the convergence of Newton-like method (2). It turns out that under certain conditions $\{t_n\}$ is a majorizing sequence for $\{x_n\}$, [3], [5]. Here we try to weaken the earlier conditions and further improve estimates on the error bounds and location of the solution x^*.

Clearly, if

$$t_n < w_0^{-1}(1-a) \quad (n \geq 0), \tag{10}$$

then it follows from (9) that sequence $\{t_n\}$ is nondecreasing, bounded above by $w_0^{-1}(1-a)$, and as such it converges to some $t^* \in \left[0, w_0^{-1}(1-a)\right]$.

We can provide stronger but more manageable conditions which imply (10).

We need the following general result on majorizing sequences for Newton-like method (2).

LEMMA 1. *Assume there exist constant $d \geq 0$, sequences $a_n \in [0,1)$, $b_n \geq 0$, $c_n \geq 0$, and $\overline{d}_n \geq 0$ such that for*

$$a_n = a + w_0(t_n), \quad b_n = (1-a_n)^{-1},$$
$$c_n = \left\{ \int_0^1 w\left[t_n + \theta(t_{n+1}-t_n)\right] \mathrm{d}\theta - w_1(t_n) + b + w_2(t_{n+1}) \right\} b_n, \tag{11}$$

$$\overline{d}_0 = d_0 = r_0, \quad \overline{d}_1 = d_1 = r_0 + \eta,$$
$$\overline{d}_n = r_0 + \eta + c_1(t_1 - t_0) + c_2(t_2 - t_1) + \cdots + c_{n-1}(t_{n-1} - t_{n-2}) \quad (n \geq 2), \tag{12}$$

the following conditions hold for all $n \geq 0$:

$$w_0\left(\overline{d}_n\right) \leq w_0(d_n) \leq w_0(d) < 1-a. \tag{13}$$

Then sequence $\{t_n\}$ generated by iteration (9) is well defined, nondecreasing bounded above by $w_0^{-1}(1-a)$, and converges to some t^*.

Moreover the following error bounds hold:

$$t_n \leq \overline{d}_n \leq d_n \leq d \quad (n \geq 0) \tag{14}$$

and

$$t_{n+1} - t_n = c_n(t_n - t_{n-1}) \quad (n \geq 1). \tag{15}$$

Proof. It suffices to show hypotheses of the Lemma imply condition (10). Indeed using (9), (11)–(13) we can have in turn for all $n \geq 2$ (since (10) holds for $n = 0, 1$ by the initial conditions):

$$t_{n+2} = t_{n+1} + c_{n+1}(t_{n+1} - t_n) = t_n + c_n(t_n - t_{n-1}) + c_{n+1}(t_{n+1} - t_n)$$
$$= \cdots + r_0 + \eta + c_1(t_1 - t_0) + \cdots + c_{n+1}(t_{n+1} - t_n) = \overline{d}_{n+2} \leq d_{n+2}, \tag{16}$$

which shows (14) for all $n \geq 0$. Moreover, by (13) we obtain

$$w_0(t_n) \leq w_0(d_n) < 1-a \text{ for all } n \geq 0, \tag{17}$$

which shows (10). □

That completes the proof of the Lemma.

For simplicity next, we provide some choices of functions and parameters defined above in the special case of Newton's method (3). That is we choose

$$A(x) = F'(x), \quad G(x) = 0 \quad (x \in U(z,R)), \ z = x_0, \text{ and } r_0 = 0. \tag{18}$$

Then we have $\eta = \bar{\eta}$.

Remark 2. Assume the Lipschitz choices:

$$w_0(r) = \ell_0 r, \quad w(r) = w_1(r) = \ell r \, (r \in [0, R]), \text{ and } a = b = 0, \tag{19}$$

where

$$0 \leq \ell_0 \leq \ell, \tag{20}$$

holds in general, and $\frac{\ell}{\ell_0}$ can be arbitrarily large [3].

(a) The Newton-Kantorovich case. Assume $\ell_0 = \ell$, and

$$h = 2\ell\eta \leq 1. \tag{21}$$

Note that (21) is the famous for its simplicity and clarity Newton-Kantorovich which is the sufficient condition for the convergence of Newton's method to x^* [2]–[18].

Define $d_n, d \ (n \geq 0)$ by

$$d_n = \eta + \frac{1}{2^1} h^{2^1-1}\eta + \cdots + \frac{1}{2^{n-1}} h^{2^n-1}\eta, \tag{22}$$

and

$$d = \frac{1-\sqrt{1-h}}{\ell} \quad (\ell \neq 0). \tag{23}$$

Then it follows from the proof of the Newton-Kantorowich's theorem [14] that

$$a_n < 1, \tag{24}$$

and condition (10) hold.

(b) Assume that the following conditions hold:

$$h_\delta = (\delta\ell_0 + \ell)\,\eta \leq \delta \text{ for } \delta \in [0, 1], \tag{25}$$

or

$$h_\delta \leq \delta, \quad \frac{2\ell_0\eta}{2-\delta} \leq 1, \quad \frac{\ell_0\delta^2}{2-\delta} \leq \ell \text{ for } \delta \in [0, 2], \tag{26}$$

or

$$h_\delta \leq \delta, \quad \ell_0\eta \leq 1 - \tfrac{1}{2}\delta \text{ for } \delta \in [\delta_0, 2), \tag{27}$$

where,

$$\delta_0 = \frac{-\frac{\ell}{\ell_0} + \sqrt{\left(\frac{\ell}{\ell_0}\right)^2 + 8\frac{\ell}{\ell_0}}}{2} \quad (\ell_0 \neq 0). \tag{28}$$

Then, by Theorem 3 in [3, p. 663], conditions (10), and (24) hold for

$$d_n = \left[1 + \frac{\delta}{2} + \cdots + \left(\frac{\delta}{2}\right)^{n-1}\right]\eta \quad (n \geq 1), \tag{29}$$

and

$$d = \frac{2\eta}{2-\delta}. \tag{30}$$

□

Moreover, other alternatives which imply condition (10) are given in Remarks 2, 3 and Lemma 2 can follow.

Remark 3. Assume there exist parameters $\alpha_1 \in [0, 1-a)$, $b \in [0,1]$, α_2 (depending on b and α_1) such that

$$w_0\left(r_0+\eta\right) \leq \alpha_1 < 1-a, \tag{31}$$

$$\alpha_1 \leq \alpha_2, \tag{32}$$

$$q\left(\alpha_2\right) \leq b \quad \text{for} \quad b \in [0,1) \tag{33}$$

or

$$q\left(\alpha_2\right) < 1 \quad \text{for} \quad b = 1, \tag{34}$$

where

$$q\left(\alpha\right) = \tfrac{1}{1-a-\alpha}\left\{ \int_0^1 w\left[w_0^{-1}\left(\alpha+\theta\eta\right)\mathrm{d}\theta\right] - w_1\left(w_0^{-1}\left(\alpha\right)\right) + b + w_2\left(w_0^{-1}\left(\alpha\right)\right)\right\}. \tag{35}$$

Then, function

$$d\left(b\right) = r_0 + \left(1+b+b^2+\cdots b^n+\cdots\right)\eta, \tag{36}$$

is well defined on interval $I_b = [\alpha_1, \alpha_2]$ $(b \neq 1)$.

Moreover assume there exists $\alpha^* \in I_b$ such that

$$w_0\left(d\left(\alpha^*\right)\right) \leq \alpha^*. \tag{37}$$ □

Then using induction on $n \geq 0$ we can show condition (6). Indeed (10) holds for $n = 0, 1$ by the initial conditions. By (9) we have

$$t_2 - t_1 \leq q\left(\alpha^*\right)\left(t_1 - t_0\right),$$

so

$$w_0\left(t_2\right) \leq w_0\left[t_1 + q\left(\alpha^*\right)\left(t_1-t_0\right)\right] \leq w_0\left(d\left(\alpha^*\right)\right) \leq \alpha^* < 1.$$

If

$$w_0\left(t_n\right) \leq \alpha^* < 1-a, \text{ then } t_{n+1} - t_n \leq q\left(\alpha^*\right)\left(t_n - t_{n-1}\right),$$

so

$$\begin{aligned} w_0\left(t_{n+1}\right) &\leq w_0\left[t_n + q\left(\alpha^*\right)\left(t_n - t_{n-1}\right)\right] \\ &\leq w_0\left[r_0 + \left(1+\alpha^*+\left(\alpha^*\right)^2+\cdots+\left(\alpha^*\right)^{n-1}\right), c\right] \\ &\leq w_0\left(d\left(\alpha^*\right)\right) \leq \alpha^* < 1-a, \end{aligned}$$

which completes the induction.

Hence, we showed:

Lemma 4. *Under the stated hypotheses:*

(a) *the condition* (10) *holds;*

(b) *the sequence* $\{t_n\}$ *is nondecreasing and converges to some* t^* *such that*

$$w_0(t_n) \le w_0(t^*) \le 1 - a; \tag{38}$$

(c) *the following error bounds hold for all* $n \ge 0$:

$$0 \le t_{n+2} - t_{n+1} \le q(\alpha^*)(t_{n+1} - t_n) \le b(t_{n+1} - t_n) \le b^{n+1}\eta, \tag{39}$$

and

$$0 \le t^* - t_n \le \tfrac{b^n \eta}{1-b}. \tag{40}$$

Remark 5.

(a) For $b = 1$, condition (34) together with (9) imply

$$0 \le t_{n+1} - t_n < t_n - t_{n-1} \quad (n \ge 1) \tag{41}$$

Hence, we deduce again $t^* = \lim_{n\to\infty} t_n$ exists.

Moreover, if we replace (10) by

$$w_0(t^*) < 1 - a \tag{42}$$

the conclusions (a) and (b) of Lemma 2 hold, where as for error bounds of the form (39) we use (41).

(b) It can easily be seen from (33)–(36) that conditions (33) and (34) can be replaced by

$$q_1(\alpha_2) \le b \quad \text{for} \quad b \in [0, 1) \tag{43}$$

or

$$q_1(\alpha_2) < 1 \quad \text{for} \quad b = 1, \tag{44}$$

respectively, where

$$q_1(\alpha) = q\left(\tfrac{\eta}{1-b}\right). \tag{45}$$

□

We provide the main semilocal convergence theorem for Newton-line method (2), which improves our earlier result (see Theorem 3, p. 663 in [3]).

Theorem 6. *Assume:*
hypotheses (4)–(8), (10) *hold for*

$$r_0 \in [0, r], \quad \eta = r_1 - r_0, \quad r \in [0, R] \tag{46}$$

$$w_0^{-1}(1 - a) + r_0 \le r, \quad \overline{U}(z, r) \subseteq Q, \tag{47}$$

and

$$x_0 \in D(t^*), \tag{48}$$

where

$$t^* = \lim_{n\to\infty} t_n, \tag{49}$$

$\{t_n\}$ *is given by* (9) *above, and* r_1, $D(t^*)$ *are defined by* (12), (14) *from* [3], *respectively.*

Then, iteration $\{x_n\}\,(n\geq 0)$ *generated by Newton-like method* (2) *is well defined, remains in* $\overline{U}(z,t^*)$ *for all* $n\geq 0$*, and converges to a solution* x^* *of equation* $F(x)+G(x)=0$.

Moreover, the following error bounds hold for all $n\geq 0$:

$$\|x_{n+1}-x_n\|\leq t_{n+1}-t_n \tag{50}$$

and

$$\|x_n-x^*\|\leq t^*-t_n. \tag{51}$$

Furthermore, the solution x^* *is unique in* $\overline{U}(z,t^*)$ *if*

$$\int_0^1[w((2t+1)t^*)-w_1(t^*)]\,\mathrm{d}t+w_2(3t^*)+w_0(t^*)+a+b<1, \tag{52}$$

and in $\overline{U}(z,R_0)$ *for* $R_0\in(t^*,r]$, *if*

$$\int_0^1[w(t^*+t[t^*+R_0))-w_1(t^*)]\,\mathrm{d}t+w_2(2t^*+R_0)+w_0(t^*)+a+b<1. \tag{53}$$

Proof. Simply repeat the corresponding proof of Theorem 3 from [3] but use condition (10) above instead of (54)–(57) in [3] until the uniqueness part.

To show uniqueness in $\overline{U}(z,t^*)$, let y^* be a solution of equation (1) in $\overline{U}(z,t^*)$. Using (2), (4)–(6), (52) we obtain in turn:

$$\begin{aligned}\|y^*-x_{k+1}\|\leq&\left\|A(x_k)^{-1}A(z)\right\|\left\{\int_0^1\left[\left\|A(z)^{-1}[F'(x_n+t\,(y^*-x_n)-A(x_n)]\right\|\mathrm{d}t\right]\right.\\&\left.+\left\|A(z)^{-1}(G(x_k)-G(y^*))\right\|\right\}\\&\leq\frac{1}{1-a-w_0(t^*)}\left\{\int_0^1[w(\|x_k-z\|+t\|y^*-x_k\|)-w_1(\|x_k-z\|)]\,\|y^*-x_k\|\,\mathrm{d}t+\right.\\&\left.+b\|y^*-x_k\|+\int_{\|x_k-z\|}^{\|x_k-z\|+\|y^*-x_k\|}w_2(t)\,\mathrm{d}t\right\}\\&\leq\frac{1}{1-a-w_0(t^*)}\left\{\int_0^1[w((1+2t)t^*)-w_1(t^*)]\,\mathrm{d}t+b+w_2(3t^*)\right\}\|y^*-x_k\|\\&<\|y^*-x_k\|\to 0\quad\text{as}\quad k\to\infty.\end{aligned}$$

That is $x^*=y^*$. If $y^*\in\overline{U}(z,R_0)$ then, as above,

$$\begin{aligned}&\|y^*-x_{k+1}\|\leq\\&\leq\frac{1}{1-a-w_0(t^*)}\left\{\int_0^1[w(t^*+t(t^*+R_0))-w_1(t^*)]\,\mathrm{d}t+b+w_2(2t^*+R_0)\right\}\|y^*-x_k\|\\&<\|y^*-x_k\|\to 0\quad\text{as}\quad k\to\infty.\end{aligned}$$

Hence, again we get $x^*=y^*$. □

We now return back to conditions (18), (19) to show that there are finer choices for the sequences $\left\{\overline{d}_n\right\}$, $\{d_n\}$ than the ones given in [3] or [5].

Remark 7. Assume: there exist parameters $\ell_0 > 0$, $\ell > 0$, $\eta > 0$, $\beta \geq 1$ such that

$$p_\beta = (\ell + 2\ell_0\beta)\,\eta < 2. \tag{54}$$

Then the interval

$$I = \left[1, \tfrac{1}{\ell_0\eta} - \tfrac{\ell}{2\ell_0}\right] \neq \emptyset, \tag{55}$$

and the function

$$c = c(\beta) = \tfrac{\ell}{2(1-\ell_0\beta\eta)} \tag{56}$$

is well defined on I and

$$0 \leq c\eta < 1. \tag{57}$$

Moreover, assume

$$t_{n+1} \leq \beta\eta, \quad \text{for all } n \geq 0. \tag{58}$$

It then follows

$$t_{n+2} - t_{n+1} = \tfrac{\ell}{2(1-\ell_0 t_{n+1})}(t_{n+1} - t_n)^2 \leq c(t_{n+1} - t_n)^2 \tag{59}$$

and

$$c(t_{n+2} - t_{n+1}) \leq [c(t_{n+1} - t_n)]^2 \leq \ldots \leq (c\eta)^{2^{n+1}}. \tag{60}$$

Let

$$d(\beta) = \eta + \tfrac{1}{c}\left[(c\eta)^{2^1} + \ldots + (c\eta)^{2^n} + \ldots\right]. \tag{61}$$

Then d is a well defined function for all $\beta \in I$.

Furthermore assume there exists $\gamma \in I$ such that:

$$d(\gamma) \leq \gamma\eta. \tag{62}$$

It follows by hypotheses (54), (58) and (62) that sequence $\{t_n\}$ is nondecreasing and bounded above by $\gamma\eta$ and it converges to some t^*. It turns out that hypothesis (58) can be dropped since it is implied by the other two. □

In particular using induction on $n \geq 0$ we can show condition (10). Indeed (10) holds for $n = 0, 1$ by the initial conditions. By (9) we can have in turn:

$$t_2 - t_1 \leq c(\gamma)(t_1 - t_0)^2, \tag{63}$$

$$\ell_0 t_2 \leq \ell_0[\eta + c(\gamma)(t_1 - t_0)^2] \leq \ell_0 d(\gamma) \leq \ell_0\gamma\eta < 1, \tag{64}$$

and, since

$$t_{n+1} - t_n \leq c(\gamma)(t_n - t_{n-1})^2,$$

we get

$$\ell_0 t_{n+1} \leq \ell_0\left[\eta + c(\gamma)(t_1 - t_0)^2 + \cdots + (c(\gamma))^{2^n}\right] \leq \ell_0 d(\gamma) \leq \ell_0\gamma\eta < 1, \tag{65}$$

which completes the induction.

Hence we showed:

LEMMA 8. *Under the stated hypotheses:*

(a) *condition* (10) *holds;*

(b) *sequence* $\{t_n\}$ *is nondecreasing and converges to some* t^* *such that*

$$t_n \le t^* \le \tfrac{1}{\ell_0} \quad (\ell_0 \ne 0); \tag{66}$$

(c) *the following error bounds hold for all* $n \ge 0$*:*

$$0 \le t_{n+2} - t_{n+1} \le c(\gamma)(t_{n+1} - t_n)^2, \tag{67}$$

$$0 \le t^* - t_n \le c(\gamma)^{-1}\bar{s}_n, \tag{68}$$

where

$$\bar{s}_n = \lim_{k\to\infty} \left\{ [c(\gamma)\eta]^{2^{n+k-1}} + \cdots + [c(\gamma)\eta]^{2^n} \right\} \tag{69}$$

$$\le \lim_{k\to\infty} \tfrac{[c(\gamma)\eta]^{2^n}[1-(c(\gamma)\eta)^{2k}]}{1-[c(\gamma)\eta]^2}$$

$$\le \tfrac{[c(\gamma)\eta]^{2^n}}{1-[c(\gamma)\eta]^2}. \tag{70}$$

REMARK 9.

(a) It follows from (61) that condition (62) can be replaced by the stronger but easier to check:

$$d^0(\gamma) \le \gamma\eta \tag{71}$$

or

$$d^1(\gamma) \le \gamma\eta, \tag{72}$$

$$d^0(\beta) = \tfrac{1}{c(\beta)\left[1-(c(\beta)\eta)^2\right]} \tag{73}$$

and

$$d^1(\beta) = \eta + \tfrac{c(\beta)\eta^2}{1-(c(\beta)\eta)^2}. \tag{74}$$

(b) A practical way of approximating γ can be: Set $s_0 = \ell_0\eta$, and define for each fixed $n > 1$ the function $d_n(a)$ by

$$d_n(a) = \eta + \tfrac{1}{c}\left[(c\eta)^2 + [c\eta]^{2^2} + \cdots + (c\eta)^{2^n}\right]. \tag{75}$$ □

We complete this study with a simple numerical example.

EXAMPLE 10. Let $X = Y = \mathbf{R}$, $x_0 = -.6$, $Q = [-1, 2]$ and define function f on q by

$$f(x) = \tfrac{1}{3}x^3 + .897462. \tag{76}$$

Using (4)–(7), (18), (19), and (76) we obtain

$$\eta = .049295,\ \ \ell_0 = 3.\bar{8}, \text{ and } \ell = 11.\bar{1}. \tag{77}$$

Condition (21) is violated, since

$$h = 1.095\bar{4} > 1. \tag{78}$$

Therefore the Newton-Kantorovich theorem [8] cannot guarantee that Newton's method (3) starting from $y_0 = x_0$ converges to $x^* = -.645722284$.

However, our condition (54) for, say, $\gamma = 1.5$ holds, since

$$p_\beta = 1.228305 < 2, \quad d(\gamma) = .0711047 < .0739425 = \gamma\eta. \tag{79}$$

That is our Theorem 6 guarantees the convergence of Newton's method (3) to x^* in this case.

Note that the weaker of the conditions (25) also holds, since

$$h_1 \left(\ell + \ell_0\right) \eta \le 1, \tag{80}$$

becomes,

$$h_1 = .739425 < 1. \tag{81}$$

However error bounds (67), (68) in this case are finer than corresponding (68) (or (61)) and (69) in [3, p. 662, 664], respectively. □

Remark 11. Condition (54) is weaker than (21), and in general also weaker than (25) or (26) or (27). Moreover our error bounds (15), (67), (68) are finer than the corresponding ones in Theorem 3 [3, p. 664] which in turn were shown in the same paper to be finer than the ones given by then and Yamamoto in [5]. Furthermore the information on the location of the solution x^* is more precise than the corresponding ones in [3] or [5]. □

Remark 12. Assume the Newton-Mysovskii-type conditions [1], [2], [14]:

$$\left\| A\left(w\right)^{-1} \left[F'\left(x+t\left(y-x\right)\right)-A\left(x\right)\right] \right\| \le \overline{w}\left(\|x-z\|+t\left\|y-x\right\|\right)-\overline{w}_1\left(\|x-z\|\right)+\overline{b} \tag{82}$$

and

$$\left\| A\left(w\right)^{-1} \left[G\left(x\right)-G\left(y\right)\right] \right\| \le \overline{w}_2\left(r\right)\left\|x-y\right\|,$$

for all

$$x, y, w \in \overline{U}\left(z, r\right) \subseteq \overline{U}\left(z, R\right) \subseteq D, \quad t \in [0, 1], \tag{83}$$

where parameter $\overline{b}$, functions $\overline{w}$, $\overline{w}_1$, and $\overline{w}_2$ are as b, w, w_1 and w_2, respectively. Replace conditions (4)–(6), by (82), (83), condition (8) by $b < 1$, and set $b_n = 1$ for all $n \ge 0$ $(a = 0)$. □

Then clearly all results obtained here hold in this stranger, but simpler, setting.

All the above justify the claims (a)–(c) made in introduction.

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