REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
**Rev. Anal. Numér. Théor. Approx., vol. 36 (2007) no. 1, pp. 51–66**
ictp.acad.ro/jnaat

# TOTAL POSITIVITY: AN APPLICATION TO POSITIVE LINEAR OPERATORS AND TO THEIR LIMITING SEMIGROUPS

ANTONIO ATTALIENTI* and IOAN RAŞA†

**Abstract.** Some shape-preserving properties of positive linear operators, involving higher order convexity and Lipschitz classes, are investigated from the point of view of weak Tchebycheff systems and total positivity in the sense of Karlin [8]. The same properties are shown to be fulfilled by the strongly continuous semigroup $(T(t))_{t\geq 0}$, if any, generated by the iterates of the relevant operators, in the spirit of Altomare's theory.

**MSC 2000.** 41A36, 47D06.
**Keywords.** Weak Tchebycheff systems, total positivity, positive linear operators, strongly continuous semigroups.

## 1. INTRODUCTION

As it is well-known, the notion of total positivity, as described and developed in great details in [8], has a wide range of applications inside and outside mathematics.

In the present paper we want to emphasize its role in the field of approximation theory, where it may be employed successfully while investigating some shape-preserving properties of positive linear operators, namely the preservation of higher order convexity and higher order Lipschitz classes.

All the analysis carried out hereby adopts the following result as a starting point. In a very general setting, let us consider the transformation $T$ such that

$$Tf(x) := \int_Y K(x,y)f(y)\mathrm{d}\sigma(y), \quad f \in D(T),\ x \in X,$$

where, according to our purposes, $X$ is a real interval, $Y$ is a real interval or a set of positive integers, $K$ is a function defined in $X \times Y$, $\mathrm{d}\sigma(y)$ is a $\sigma$-finite measure on $Y$, the domain $D(T)$ of $T$ is a linear space of real functions defined on a real interval $I \supseteq Y$, and assume that the integral in the right-hand side is absolutely convergent.

*Department of Economics-University of Bari, Via Camillo Rosalba 53, 70124 Bari, Italy, e-mail: `attalienti@matfin.uniba.it`.
†Department of Mathematics-Technical University of Cluj-Napoca, Str. C.Daicoviciu 15, 3400 Cluj-Napoca, Romania, e-mail: `Ioan.Rasa@math.utcluj.ro`.

Following Karlin [8, Chapter 6, pp. 284–285], the total positivity of the kernel $K(x,y)$ yields, as a consequence, that if $\{\varphi_1, \varphi_2, \cdots, \varphi_n\}$ $(n \geq 1)$ is a weak Tchebycheff system of functions in $D(T)$, so is $\{T\varphi_1, T\varphi_2, \ldots, T\varphi_n\}$, too.

Since, as we shall see, the notion of $m$-th order convexity $(m \geq 1)$ (which coincides with the standard definition of convexity if $m = 2$) may be expressed in terms of the Tchebycheff system $\{e_0, e_1, \cdots, e_{m-1}\}$ (here $e_i(x) := x^i$, $0 \leq i \leq m-1$), one is naturally led to ask whether or under which conditions the transformation $T$, endowed with a totally positive kernel, preserves also $m$-th order convexity.

In Theorem 2.3 we give a positive answer in this respect: after imposing some reasonable assumptions over $T$, we gain the preservation, under $T$, not only of $m$-th order convexity, but also of higher order Lipschitz classes, the two notions being however intimately correlated to each other, due to Proposition 2.1.

It goes without saying that a similar result becomes meaningful if applicable to known positive linear operators playing the role of $T$: in this respect, we observe that the expression of $T$ is so much general to cover most of the classical definitions of operators frequently occurring in approximation theory. Furthermore, all the assumptions in Theorem 2.3 (including the total positivity of the kernel) are commonly fulfilled in concrete cases.

Summing up, as testified in Section 3, we may conclude that several classes of positive linear operators (with the Bernstein ones at the top of the list, but including also Beta, Bernstein-Durrmeyer, Kantorovich, Post-Widder, Szasz-Mirakjan and Baskakov operators) share totally positive kernels and reproduce $m$-th order convex functions as well as higher order Lipschitz classes, which, in our opinion, justifies, to some extent, their nice approximating behaviour.

Lastly, in Theorem 2.7 we prove that the same shape-preserving properties may be transferred, with some necessary but slight changes somewhere, from a sequence of positive linear operators to the strongly continuous semigroup $(T(t))_{t\geq 0}$, if any, generated by the iterates of the operators themselves, in the spirit of Altomare's theory (see, for instance, [1, Chapter VI], [2], [3], [4], [5], [10], [13]).

In this framework one gets, in this way, some qualitative properties of the solution of a Cauchy problem, if expressed in terms of $(T(t))_{t\geq 0}$.

## 2. NOTATION AND MAIN RESULTS

Let us start this section by introducing the basic definitions as they are used throughout the paper; in this respect we refer to the book of Karlin [8] and to the notation adopted therein.

Let $\varphi_1, \varphi_2, \ldots, \varphi_n$ be $n$ real functions defined on a real interval $X$; we say that $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ is a weak Tchebycheff system if, for any selection $x_1 < x_2 < \cdots < x_n$, $x_i \in X$, one has

$$
(2.1) \qquad \begin{vmatrix} \varphi_1(x_1) & \varphi_1(x_2) & \dots & \varphi_1(x_n) \\ \varphi_2(x_1) & \varphi_2(x_2) & \dots & \varphi_2(x_n) \\ \vdots & \vdots & \vdots & \vdots \\ \varphi_n(x_1) & \varphi_n(x_2) & \dots & \varphi_n(x_n) \end{vmatrix} \geq 0,
$$

where, as usual, the first member denotes the determinant of the square matrix $(\varphi_i(x_j))_{1\le i,j\le n}$.

If the inequality in (2.1) is strict, $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ is simply called a Tchebycheff system (see [8, p. 274]).

A real function $f : X \longrightarrow \mathbb{R}$ is said to be convex with respect to the weak Tchebycheff system $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ if $\{\varphi_1, \varphi_2, \dots, \varphi_n, f\}$ is a weak Tchebycheff system; according to [8, p. 280] the convex cone of all such functions will be denoted by $\mathcal{C}(\varphi_1, \varphi_2, \dots, \varphi_n)$.

A straightforward computation assures that, if $m \geq 1$, the $m$ functions $e_i(x) := x^i$ $(0 \leq i \leq m-1,\ x \in X)$ form a Tchebycheff system; a function $f : X \longrightarrow \mathbb{R}$ belonging to $\mathcal{C}(e_0, e_1, \dots, e_{m-1})$ will be called $m$-convex or convex of order $m$.

In particular, by direct verification, one can easily check that first order convexity is synonymous of increasing, whereas second order convexity coincides with the usual assertion that $f$ is convex.

It may be proved (see, e.g., [12, p. 109]) that $f : X \longrightarrow \mathbb{R}$ is $m$-convex iff all its divided differences $[x_0, x_1, \dots, x_m; f]$ on $m+1$ points in $X$ are $\geq 0$ (if $f$ is continuous, due to a result by Popoviciu [11], in the above statement the $m+1$ points may be chosen equally spaced).

Moreover, if $f \in C^m(X)$ (i.e., if $f$ is $m$ times continuously differentiable), then $f$ is $m$-convex iff $f^{(m)}(x) \geq 0$ for all $x \in X$ (see, e.g., [12, pp. 109–110]).

This last characterization allows to extend the concept of $m$-convexity also to the case $m = 0 : f$ is 0-convex means $f(x) \geq 0$ for any $x \in X$. Finally, we point out that, if $m \geq 2$, an $m$-convex function enjoys regularity conditions similar to the well-known properties of ordinary convex functions: for further details in this respect we refer the reader to [8, Theorem 4.1, p. 26].

Now fix an integer $m \geq 0$ and $M > 0$; we say that $f : X \longrightarrow \mathbb{R}$ belongs to the Lipschitz class $\mathrm{Lip}_m(M)$ if

$$|\Delta_h^m f(x)| \leq M h^m$$

for all $x \in X$ and $h > 0$ such that $x + mh \in X$: here, as usual, $\Delta_h^m f(x)$ denotes the $m$-th order difference of $f$ with step $h$ at $x$, i.e.,

$$\Delta_h^m f(x) := \sum_{i=0}^{m} (-1)^i \binom{m}{i} f(x + (m-i)h).$$

Observe that $\Delta_h^m e_m(x) = m! h^m$ so that $e_m \in \mathrm{Lip}_m(m!)$.

A useful characterization of Lipschitz classes in terms of $m$-th order convexity is indicated below, where, as usual, $C(X)$ denotes the space of all continuous functions defined on $X$.

PROPOSITION 2.1. *If $f \in C(X)$, then $f \in \mathrm{Lip}_m(M)$ iff $\frac{M}{m!}e_m \pm f$ are $m$-convex.*

*Proof.* If $m = 0$, the assertion is trivial; so suppose $m \geq 1$ and recall that

$$\Delta_h^m f(x) = m!h^m[x, x+h, \ldots, x+mh; f]$$

for all $x \in X$ and $h > 0$ so that $x+mh \in X$ (see, e.g., [7, p. 121]); accordingly, $f \in \mathrm{Lip}_m(M)$ iff

$$-\frac{M}{m!} \leq [x, x+h, \ldots, x+mh; f] \leq \frac{M}{m!} \quad (x \in X, h > 0),$$

or, equivalently,

$$[x, x+h, \ldots, x+mh; \frac{M}{m!}e_m \pm f] \geq 0 \quad (x \in X, h > 0)$$

since $[x, x+h, \ldots, x+mh; e_m] = 1$, whence the result follows. $\square$

Turning back to general definitions, according to [8, p. 11], if $X$ and $Y$ are real intervals or sets of positive integers, a function $K : X \times Y \longrightarrow \mathbb{R}$ is called a totally positive kernel if

$$\begin{vmatrix} K(x_1,y_1) & K(x_1,y_2) & \ldots & K(x_1,y_m) \\ K(x_2,y_1) & K(x_2,y_2) & \ldots & K(x_2,y_m) \\ \vdots & \vdots & \vdots & \vdots \\ K(x_m,y_1) & K(x_m,y_2) & \ldots & K(x_m,y_m) \end{vmatrix} \geq 0 \tag{2.2}$$

for all $m \geq 1$ and any selections $x_1 < x_2 < \cdots < x_m,\ y_1 < y_2 < \cdots < y_m,\ x_i \in X,\ y_i \in Y$. Note that in particular $K(x,y) \geq 0$ for all $(x,y) \in X \times Y$.

For a general survey of the theory of totally positive kernels and its several applications in and outside mathematics, we refer the reader to [8]: here we confine ourselves to remark some worthy consequences of (2.2), mainly related to shape-preserving properties, in a sense that will be discussed later in this section.

Now we need to describe a basic binary operation which allows to build up a new totally positive kernel starting from two such kernels; more specifically, let us fix Borel measurable functions $K : X \times Y \longrightarrow \mathbb{R},\ L : Y \times Z \longrightarrow \mathbb{R}$ and $M : X \times Z \longrightarrow \mathbb{R}$ such that

$$M(x,z) = \int_Y K(x,y)L(y,z)\mathrm{d}\sigma(y), \quad (x,z) \in X \times Z, \tag{2.3}$$

where the integral in the right-hand side is supposed to be absolutely convergent: here $X, Y$ and $Z$ are real intervals or sets of positive integers and $\mathrm{d}\sigma(y)$ denotes a $\sigma$-finite measure on $Y$.

Of course, if $Y$ is a (finite or infinite) discrete set, the integral in (2.3) has to be interpreted as a (finite or infinite) sum.

Then, by virtue of the basic composition formula [8, p. 98], the determinant corresponding to (2.2) for the kernel $M(x,z)$ (2.3) reads as follows: for any $m \geq 1, x_1 < x_2 < \cdots < x_m,\ z_1 < z_2 < \cdots < z_m,\ x_i \in X, z_i \in Z$ one has

$$\begin{vmatrix} M(x_1,z_1) & M(x_1,z_2) & \dots & M(x_1,z_m) \\ \vdots & \vdots & & \vdots \\ M(x_m,z_1) & M(x_m,z_2) & \dots & M(x_m,z_m) \end{vmatrix}$$

$$= \int \begin{vmatrix} K(x_1,y_1) & K(x_1,y_2) & \dots & K(x_1,y_m) \\ \vdots & \vdots & & \vdots \\ K(x_m,y_1) & K(x_m,y_2) & \dots & K(x_m,y_m) \end{vmatrix}$$

$$\times \begin{vmatrix} L(y_1,z_1) & L(y_1,z_2) & \dots & L(y_1,z_m) \\ \vdots & \vdots & & \vdots \\ L(y_m,z_1) & L(y_m,z_2) & \dots & L(y_m,z_m) \end{vmatrix} \mathrm{d}\sigma(y_1)\mathrm{d}\sigma(y_2)\dots\mathrm{d}\sigma(y_m),$$

where the integral is evaluated over the set $S := \{(y_1, y_2, \dots, y_m) \in Y^m : y_1 < y_2 < \quad \cdots < y_m\}$.

As a consequence, if $K$ and $L$ are totally positive kernels, so is the kernel $M$ in (2.3).

In the sequel, starting from a kernel $K : X \times Y \longrightarrow \mathbb{R}$, *$X$ being a real interval and $Y$ a real interval or a set of integers*, we shall be interested in the study of the properties of the general linear transformation (operator)

$$Tf(x) := \int_Y K(x,y)f(y)\mathrm{d}\sigma(y), \quad f \in D(T),\ x \in X, \tag{2.4}$$

where $\mathrm{d}\sigma(y)$ is a $\sigma$-finite measure on $Y$, the domain $D(T)$ of $T$ is a suitable linear space of real functions defined on a real interval $I \supseteqq Y$ and $K$ enjoys some regularity conditions so that the integral in the right-hand side is absolutely convergent for any bounded Borel-measurable function $f$ defined on $I$.

When $Y$ is a finite or infinite subset of $\mathbb{N}$, then (2.4) has to be meant as a finite or infinite sum; however, in order to cover many practical situations involving discrete-type operators, it is more convenient in this case to deal with the transformation

$$Tf(x) := \sum_{k\in Y} K(x,k)f(t_k), \quad f \in D(T),\ x \in X, \tag{2.5}$$

where the $t_k$'s belong to $I$ and $t_k < t_i$ if $k < i$ (see [8, example (ii), p. 287]).

Henceforth, by $T$ we shall mean (2.4) or (2.5) without distinction.

In connection with a totally positive kernel, the transformation $T$ enjoys some remarkable properties listed in the next theorem.

Theorem 2.2. (see [8, Chapter 6, Sect. 3]). *Let $K$ be a totally positive kernel and $\varphi_1, \varphi_2, \dots, \varphi_n$ in $D(T)$. Then we have:*

a) *If* $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ *is a weak Tchebycheff system, so is* $\{T\varphi_1, T\varphi_2, \ldots, T\varphi_n\}$.
b) *If* $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ *is a weak Tchebycheff system and* $\varphi \in D(T) \cap \mathcal{C}(\varphi_1, \varphi_2, \ldots, \varphi_n)$, *then* $T\varphi \in \mathcal{C}(T\varphi_1, T\varphi_2, \ldots, T\varphi_n)$, *i.e.,*
$$T(D(T) \cap \mathcal{C}(\varphi_1, \varphi_2, \ldots, \varphi_n)) \subset \mathcal{C}(T\varphi_1, T\varphi_2, \ldots, T\varphi_n).$$

Really, something more can be said under additional assumptions on $T$, as shown in the next theorem in which the concepts of higher order convexity and Lipschitz classes are heavily involved.

THEOREM 2.3. *Let $K$ be a totally positive kernel and suppose that the corresponding positive linear transformation $T$ satisfies the following:*
(i) $T(D(T) \cap C(I)) \subset C(X)$.
(ii) *There exists an integer $m \geq 2$ such that for each $r = 0, 1, \ldots, m$ the power function $e_r$ belongs to $D(T)$ and $Te_r$ is a polynomial of degree $r$ with leading coefficient $a_r > 0$.*

*Then we have:*
a) $T(D(T) \cap \mathcal{C}(e_0, e_1, \ldots, e_{m-1})) \subset \mathcal{C}(e_0, e_1, \ldots, e_{m-1})$.
b) $T(D(T) \cap C(I) \cap \mathrm{Lip}_m(M)) \subset \mathrm{Lip}_m(Ma_m)$ *for any* $M > 0$.
c) *If* $f \in D(T) \cap C^m(I)$ *has a bounded derivative of order $m$, i.e.,* $||f^{(m)}||_\infty := \sup_{x \in I} |f^{(m)}(x)| < +\infty$, *then* $Tf \in C^{m-2}(\overset{\circ}{X})$ *and* $(Tf)^{(m-2)}$ *has a right derivative which is right-continuous on* $\overset{\circ}{X}$ *and a left derivative which is left-continuous on* $\overset{\circ}{X}$. *Finally, if* $Tf \in C^m(X)$ *too, then* $||(Tf)^{(m)}||_\infty \leq a_m ||f^{(m)}||_\infty$.

*Proof.*
a) The assertion follows immediately from b) in Theorem 2.2 since, due to (ii), $\mathcal{C}(Te_0, Te_1, \ldots, Te_{m-1}) = \mathcal{C}(e_0, e_1, \ldots, e_{m-1})$.
b) Fix $f \in D(T) \cap C(I) \cap \mathrm{Lip}_m(M)$ $(M > 0)$; Proposition 2.1 yields that $\frac{M}{m!} e_m \pm f$ are $m$-convex, whence $\frac{M}{m!} Te_m \pm Tf$ are $m$-convex, too, by virtue of (a). Since, in view of (ii), $Te_m = a_m e_m +$ terms of lower degree, it follows that $\frac{M}{m!} a_m e_m \pm Tf$ are $m$-convex, which is equivalent to $Tf \in \mathrm{Lip}_m(Ma_m)$ on account of Proposition 2.1 and assumption (i).
c) Indeed, choose $m \geq 2$ and $f \in D(T) \cap C^m(I)$ with $||f^{(m)}||_\infty < +\infty$ and observe that, since
$$\Big(\frac{||f^{(m)}||_\infty}{m!} e_m \pm f\Big)^{(m)} = ||f^{(m)}||_\infty \pm f^{(m)} \geq 0,$$
$\frac{||f^{(m)}||_\infty}{m!} e_m \pm f$ are $m$-convex, whence the functions
$$\frac{||f^{(m)}||_\infty}{m!} a_m e_m \pm Tf \tag{2.6}$$

are $m$-convex as well (see the proof of statement b)). Applying now Theorem 1.4 of [8, p. 26] to the functions in (2.6) gives the regularity result for $(Tf)^{(m-2)}$ quoted in c), because $\frac{||f^{(m)}||_\infty}{m!} a_m e_m \in C^\infty(I)$. Finally, if $Tf \in C^m(X)$, the $m$-convexity of the functions in (2.6) is equivalent to

$$||f^{(m)}||_\infty \, a_m \pm (Tf)^{(m)} = \Big(\frac{||f^{(m)}||_\infty}{m!} a_m e_m \pm Tf\Big)^{(m)} \geq 0, \tag{2.7}$$

i.e., $||(Tf)^{(m)}||_\infty \leq a_m ||f^{(m)}||_\infty$ and the proof is complete.

□

Remark 2.4. Actually, a) and b) still hold true if $m \geq 1$ in (ii). Moreover, as an inspection of the proof shows, in order to preserve $m$-th order convexity one needs to require the preservation of the polynomials in (ii) up to the degree $m-1$.

Remark 2.5. As an improvement of Theorem 2.3, observe that under the same assumptions and notation, the transformation $T$ preserves $q$-th order convexity and maps $\mathrm{Lip}_q(M)$ into $\mathrm{Lip}_q(Ma_q)$ for any $q = 1, \ldots, m$. The proof runs very similarly with slight changes somewhere and is therefore omitted.

Remark 2.6. As the reader will quickly realize in the next section, most of classical positive linear operators occurring in approximation theory are already or may be cast in the form (2.4) or (2.5) with totally positive kernels; in addition, all the assumptions in Theorem 2.3 are commonly fulfilled so that the shape-preserving properties described in a) and b) are verified.

Before stating the next theorem, we need some more preliminaries and notation. Henceforth we shall denote by $(E, ||\cdot||)$ a Banach space of continuous functions on a real interval $I$ and assume that the convergence in the norm $||\cdot||$ implies pointwise convergence.

If $(L_n)_{n\geq 1}$ is a sequence of positive linear operators from $E$ to $E$, for any $m \geq 1$ $L_n^m$ stands for the iterate of $L_n$ of order $m$.

We shall also deal with strongly continuous semigroups $(T(t))_{t\geq 0}$ on $E$: for the general definitions and main results we refer, for instance, to [1, Chapter 1].

Now, we are in a position to state the following result, which extends Theorems 2.2 and 2.3 to a strongly continuous semigroup arising, in some way, from the iterates of positive linear operators.

Theorem 2.7. *Let $(L_n)_{n\geq 1}$ be a sequence of (positive) linear operators from $E$ to $E$ matching* (2.4) *or* (2.5) *with totally positive kernels, and assume the following:*

(i) *There exist two integers $m \geq 2$ and $n_0 \geq 1$ such that for each $r = 0, 1, \ldots, m$ and any $n \geq n_0$ the power function $e_r \in E$ and $L_n e_r$ is a polynomial of degree $r$ with leading coefficient $a_{n,r} > 0$.*

(ii) *The limit*

$$l_m := \lim_{n\to+\infty} (a_{n,m})^n \tag{2.8}$$

*exists and is finite.*

(iii) $(L_n)_{n\geq 1}$ *is a positive approximation process on* $E$*, i.e.,*

$$\lim_{n\to+\infty} L_n f = f \quad \textit{in} \quad (E, ||\cdot||) \tag{2.9}$$

*for all* $f \in E$.

(iv) *For every* $f \in E$ *and* $t \geq 0$ *the limit*

$$T(t)f := \lim_{n\to+\infty} L_n^{[nt]} f \tag{2.10}$$

*exists in* $(E, ||\cdot||)$ *and* $(T(t))_{t\geq 0}$ *is a strongly continuous semigroup on* $E$ *(here* $[nt]$ *stands for the integer part of* $nt$*).*

*Then we have:*

(a) *If* $\varphi_1, \varphi_2, \ldots, \varphi_n \in E$ $(n \geq 1)$ *and* $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ *is a weak Tchebycheff system, so is* $\{T(t)\varphi_1, T(t)\varphi_2, \ldots, T(t)\varphi_n\}$ *for all* $t \geq 0$*; in addition, for all* $t \geq 0$,

$$T(t)(E \cap \mathcal{C}(\varphi_1, \varphi_2, \ldots, \varphi_n)) \subset \mathcal{C}(T(t)\varphi_1, T(t)\varphi_2, \ldots, T(t)\varphi_n). \tag{2.11}$$

(b) *For all* $t \geq 0$,

$$T(t)(E \cap \mathcal{C}(e_0, e_1, \ldots, e_{m-1})) \subset \mathcal{C}(e_0, e_1, \ldots, e_{m-1}). \tag{2.12}$$

(c) *For all* $t \geq 0$ *and* $M > 0$,

$$T(t)(E \cap \mathrm{Lip}_m(M)) \subset \mathrm{Lip}_m(M l_m^t) \tag{2.13}$$

*(see* (2.8)*).*

(d) *If* $f \in E \cap C^m(I)$ *has a bounded derivative of order* $m$*, i.e.,* $||f^{(m)}||_\infty := \sup_{x\in I} |f^{(m)}(x)| < +\infty$*, then for all* $t \geq 0$ $T(t)f \in C^{m-2}(\mathring{I})$ *and* $(T(t)f)^{(m-2)}$ *has a right derivative which is right-continuous on* $\mathring{I}$ *and a left derivative which is left-continuous on* $\mathring{I}$*. Finally, if* $T(t)f \in C^m(I)$ *for all* $t \geq 0$ *too, then*

$$||(T(t)f)^{(m)}||_\infty \leq l_m^t ||f^{(m)}||_\infty \quad (t \geq 0). \tag{2.14}$$

*Proof.* It follows easily from Theorems 2.2 and 2.3, passing to the pointwise limit, as suggested by (2.10). Specifically, (a) is a consequence of Theorem 2.2 and representation (2.10); the proof of (b) and (c) follows from (a) and (b) in Theorem 2.3, on account of (2.8) and (2.10). Statement (d) may be proved as (c) in Theorem 2.3. □

Remark 2.8. As in Remark 2.4, in order to get (b) and (c), it is sufficient to assume $m \geq 1$ in (i). Moreover, the preservation of $m$-th order convexity under each $T(t)$ in (b) may be achieved under the weaker assumption that

$L_n e_r$ is a polynomial of degree $r$ with $r$ running up to $m-1$, condition (ii) being unnecessary to this aim.

Remark 2.9. We point out that, on account of Remark 2.5, under the same assumptions and notation as in Theorem 2.7, each $T(t)$ preserves $q$-th order convexity and maps $\mathrm{Lip}_q(M)$ into $\mathrm{Lip}_q(Ml_q^t)$ for any $q = 1, \dots, m$.

Obviously, as far as this last property is concerned, we have tacitly improved (ii), requiring the stronger condition that the limit

$$l_q := \lim_{n\to+\infty} (a_{n,q})^n \tag{2.15}$$

exists and is finite for each $q = 1, \dots, m$.

Remark 2.10. The reader has surely realized that condition (iii) and the fact that $(T(t))_{t\geq 0}$ is a strongly continuous semigroup on $E$ in (iv) are employed nowhere in the proof of the theorem. Actually, as it will be clear in the next section, (iii) and (iv) are commonly verified by several classes of positive linear operators: moreover the relevant semigroup $(T(t))_{t\geq 0}$ is generated, according to a result of Trotter (see, e.g., [1, Theorem 1.6.7, p. 67]), by a suitable differential operator $(A, D(A))$ (acting on some domain $D(A) \subset E$), which is linked, in turn, to the sequence $(L_n)_{n\geq 1}$ by a Voronovskaja-type result (for a rather complete survey in this respect we refer to [1], [2], [3], [4], [5], [10], [13] and to many references quoted therein).

The general theory of strongly continuous semigroups assures that for any $u_0 \in D(A)$ the following Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = A(u(\cdot,t))(x), & (x \in I,\ t > 0), \\ u(x,0) = u_0(x), & (x \in I), \end{cases}$$

has a unique classical solution $u : I \times [0, +\infty[ \to \mathbb{R}$ such that $u(x,t) = T(t)u_0(x)$ $(x \in I,\ t \geq 0)$. In this framework, the properties (2.12) and (2.13) (see also Remark 2.9) may be recast in more suggestive terms as follows:

1) *If $u_0$ is $q$-convex, then the solution $u(\cdot,t)$ is $q$-convex for all $t \geq 0$.*
2) *If $u_0$ belongs to some $\mathrm{Lip}_q(M)$, then the solution $u(\cdot,t)$ belongs to $\mathrm{Lip}_q(Ml_q^t)$ for all $t \geq 0$.*

The total positivity of the kernel of positive linear operators $(L_n)_{n\geq 1}$ allows sometimes to gain some information about the asymptotic behaviour of the semigroup $(T(t))_{t\geq 0}$ described according to (2.10): this is mainly the content of the next proposition, in which we adopt the same notation of Theorem 2.7.

Proposition 2.11. *Suppose that the sequence $(L_n)_{n\geq 1}$ fulfills all the assumptions in Theorem* 2.7, *with* (i) *and* (ii) *replaced by*

$$(i)' \quad e_0, e_1 \in E \quad \textit{and} \quad L_n e_0 = e_0,\ L_n e_1 = e_1 \quad \textit{for all} \quad n \geq 1,$$

*and fix a 2-convex function $f \in E$. Then we have:*

a) *$(T(t)f)_{t\geq 0}$ is a family of 2-convex functions in $E$ with $f \leq T(t)f$ for all $t \geq 0$.*

b) *For each* $x \in I$ *the mapping* $t \in [0, +\infty[ \longrightarrow T(t)f(x)$ *is increasing and therefore the limit*

$$V f(x) := \lim_{t \to +\infty} T(t)f(x) \tag{2.16}$$

*exists in* $\mathbb{R} \cup \{+\infty\}$.

c) *If* $f$ *is bounded above by a constant or by a polynomial of degree* 1*, the function* $V$ *defined in* (2.16) *is real-valued and* 2*-convex.*

*Proof.* Indeed, each $T(t)f$ is 2-convex by virtue of (b), Theorem 2.7 and Remark 2.8. Moreover, by Jensen's inequality (see, e.g., [6]), $f(L_n e_1) \le L_n f$, i.e., $f \le L_n f$ for all $n \ge 1$, because of (i)$'$.

But now it is quite easy to show that $L_n^{[ns]} f \le L_n^{[nt]} f$ for all $n \ge 1$ and $0 \le s \le t$ which, in turn, leads to $T(s)f \le T(t)f$ on account of (2.10). In particular, $f = T(0)f \le T(t)f$ and the proof of a) and b) is complete.

To get c), if $f \le ae_1 + be_0$ $(a, b \in \mathbb{R})$, then for a fixed $x_0 \in I$ one has

$$f(x_0) \le T(t)f(x_0) \le ax_0 + b \quad \text{for all} \quad t \ge 0,$$

because $T(t)e_0 = e_0$ and $T(t)e_1 = e_1$. Passing to the limit as $t$ goes to $+\infty$ yields

$$f(x_0) \le V f(x_0) \le ax_0 + b$$

and therefore $V f(x_0) \in \mathbb{R}$. Since $x_0$ was arbitrarily chosen, $V f$ is real-valued, being, moreover, 2-convex because of (2.16) and the 2-convexity of every $T(t)f$. □

## 3. APPLICATIONS

This section is devoted to employ the results obtained so far in concrete cases: namely, it is our intention to show how most of the classical positive linear operators occurring in approximation theory resemble (2.4) or (2.5), sharing, moreover, totally positive kernels and preserving polynomials.

In view of Theorem 2.3, they enjoy therefore nice shape-preserving properties.

Whenever their iterates converge towards a semigroup, the semigroup itself enjoys essentially the same, thanks to Theorem 2.7 (see, in this respect, the last part of Remark 2.10).

In the following examples, even if sometimes not explicitly said, the sequences in matter are positive approximation processes on the corresponding spaces.

Example 3.1. Consider the classical Beta operators (see [9]), i.e., the positive linear operators $\mathcal{B}_n : C([0,1]) \longrightarrow C([0,1])$ defined as follows:

$$\mathcal{B}_n f(x) := \frac{1}{B(nx+1, n(1-x)+1)} \cdot \int_0^1 y^{nx}(1-y)^{n(1-x)} f(y)\mathrm{d}y \tag{3.1}$$

for any $n \ge 1, f \in C([0,1])$ and $x \in [0,1]$, $B(\cdot,\cdot)$ denoting the standard Beta function.

According to (2.4), for any fixed $n \geq 1$ the corresponding kernel is given by

$$(x, y) \in [0,1] \times ]0,1[ \longrightarrow \frac{y^{nx}(1-y)^{n(1-x)}}{B(nx+1, n(1-x)+1)},$$

or, equivalently, by

$$(x, y) \in [0,1] \times ]0,1[ \longrightarrow \frac{\mathrm{e}^{n\log(1-y)} \cdot \mathrm{e}^{nx(\log y - \log(1-y))}}{B(nx+1, n(1-x)+1)}$$

and therefore it is totally positive because of [8, Theorem 1.1, part (a), p. 99 and (1.5), p. 100].

A direct computation (see, e.g., [5]) yields for any $n, r \geq 1$

$$\mathcal{B}_n e_r = \prod_{k=0}^{r-1} \Big( \frac{n}{n+k+2}\, e_1 + \frac{k+1}{n+k+2}\, e_0 \Big), \tag{3.2}$$

which, together with $\mathcal{B}_n e_0 = e_0$, allows to conclude that for any $r \geq 0$ $\mathcal{B}_n e_r$ is a polynomial of degree $r$ with leading coefficient

$$a_{n,r} := \begin{cases} \prod\limits_{k=0}^{r-1} \frac{n}{n+k+2}, & \text{if} \quad r \geq 1, \\ 1, & \text{if} \quad r = 0. \end{cases}$$

In [5, Theorem 2.10] it is shown that there exists a strongly continuous semigroup $(T(t))_{t \geq 0}$ on $C([0,1])$, whose generator is the differential operator

$$Au(x) := \frac{x(1-x)}{2} u''(x) + (1-2x)u'(x) \quad (0 < x < 1), \tag{3.3}$$

acting on its maximal domain $D(A) := \{v \in C([0,1]) \cap C^2(]0,1[) | Av \in C([0,1])\}$, satisfying a relationship similar to (2.10), i.e.,

$$T(t)f = \lim_{n \to +\infty} \mathcal{B}_n^{[nt]} f \quad \text{in} \quad (C([0,1]), ||\cdot||_\infty)$$

for any $t \geq 0$ and $f \in C([0,1])$. Summing up, we are in a position to apply both Theorems 2.3 and 2.7; in particular, also in view of Remarks 2.5 and 2.9, the operators $\mathcal{B}_n$ and their limiting semigroup $(T(t))_{t \geq 0}$ preserve convexity of any order and map Lipschitz classes into Lipschitz classes. As for the semigroup, since for any $q \geq 1$, according to (2.15), we have

$$l_q = \mathrm{e}^{-\frac{q(q+3)}{2}},$$

the following inclusion

$$T(t)(C([0,1]) \cap \mathrm{Lip}_q(M)) \subset \mathrm{Lip}_q \left( M \mathrm{e}^{-\frac{q(q+3)t}{2}} \right) \tag{3.4}$$

holds true for any $t \geq 0, q \geq 1$ and $M > 0$.

Note that (3.4) is in agreement with the asymptotic behaviour of $(T(t))_{t\geq 0}$ as $t \to +\infty$; indeed

$$\lim_{t\to+\infty} T(t)f(x) = 6\int_0^1 s(1-s)f(s)\mathrm{d}s \ \text{ uniformly on } [0,1] \tag{3.5}$$

for any $f \in C([0,1])$.

Here we present a brief proof of (3.5): consider for each $n \geq 1$ the $n$-th Jacobi polynomial $J_n^{(1,1)}$ and recall that the sequence $(J_n^{(1,1)})_{n\geq 0}$ is orthogonal with weight $x(1-x)$ in $[0,1]$; in addition, for every $n \geq 0$, $J_n^{(1,1)}$ is a solution of the differential equation

$$x(1-x)y'' + 2(1-2x)y' + n(n+3)y = 0 \tag{3.6}$$

(see, e.g., [14, p. 62]). It follows, on account of (3.3), that

$$AJ_n^{(1,1)} = -\frac{n(n+3)}{2}J_n^{(1,1)} \quad (n \geq 0),$$

which, by means of standard semigroup techniques, leads to

$$T(t)J_n^{(1,1)} = J_n^{(1,1)}\mathrm{e}^{-\frac{n(n+3)t}{2}} \quad (t \geq 0,\ n \geq 0). \tag{3.7}$$

As a consequence, one may easily check that (3.5) holds true for any $J_n^{(1,1)}$ and hence, by a density argument, for any $f \in C([0,1])$.

Example 3.2. Given $n \geq 1$, let us set $p_{n,k}(x) := \binom{n}{k}x^k(1-x)^{n-k}$ $(0 \leq x \leq 1,\ k = 0,1,\dots,n)$ and consider the $n$-th Bernstein-Durrmeyer operator

$$D_n f(x) := \int_0^1 \Big((n+1)\sum_{k=0}^{n} p_{n,k}(x)\cdot p_{n,k}(y)\Big)f(y)\mathrm{d}y \tag{3.8}$$

for all $f \in C([0,1])$ and $x \in [0,1]$ (see, e.g., [1, p. 335]; the more general operators introduced by D.H. Mache are considered in [13]).

For any fixed $n \geq 1$ the kernels

$$(x,k) \longrightarrow p_{n,k}(x) \quad (0 \leq x \leq 1,\ k = 0,1,\dots,n),$$

$$(k,y) \longrightarrow p_{n,k}(y) \quad (k = 0,1,\dots,n,\ 0 \leq y \leq 1)$$

are totally positive thanks to [8, p. 287 and Theorem 1.1, part (a), p. 99]. Applying the basic composition formula (compare with (2.3)) yields that the kernel

$$(x,y) \in [0,1]\times[0,1] \longrightarrow (n+1)\sum_{k=0}^{n} p_{n,k}(x)\cdot p_{n,k}(y)$$

is totally positive as well. Since for any $n \geq 1$

$$D_n = B_n \circ \mathcal{B}_n \tag{3.9}$$

where $B_n$ is the $n$-th Bernstein operator and $\mathcal{B}_n$ the $n$-th Beta operator (3.1), it may be readily shown that, for any $r \geq 0$, $D_n e_r$ is a polynomial of degree $r$ with leading coefficient

$$a_{n,r} := \begin{cases} \prod_{k=0}^{r-1}(1 - \frac{k+2}{n+k+2})(1 - \frac{k}{n}), & \text{if} \quad r \geq 1, \\ 1, & \text{if} \quad r = 0 \end{cases}$$

and therefore Theorem 2.3 may be applied to each $D_n$.

Example 3.3. Choose $n \geq 1$ and set again $p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}$ $(0 \leq x \leq 1, k = 0, 1, \ldots, n)$; the $n$-th Kantorovich operator is defined as follows:

$$K_n f(x) := (n+1) \sum_{k=0}^{n} p_{n,k}(x) \cdot \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(y)\mathrm{d}y \tag{3.10}$$

for all $f \in C([0,1])$ and $x \in [0,1]$ (see, e.g., [1, p. 333]). To our purposes this may be rewritten as

$$K_n f(x) = \int_0^1 \left( (n+1) \sum_{k=0}^{n} p_{n,k}(x) \cdot q_{n,k}(y) \right) f(y)\mathrm{d}y,$$

where $q_{n,k} := \chi_{]\frac{k}{n+1}, \frac{k+1}{n+1}[}$ is the characteristic function of the interval $]\frac{k}{n+1}, \frac{k+1}{n+1}[$.

We have already observed in the previous example that the kernel $(x,k) \longrightarrow p_{n,k}(x)$ is totally positive; really, the same happens for the kernel

$$(k,y) \longrightarrow q_{n,k}(y) \quad (k = 0, 1, \ldots, n, \ 0 \leq y \leq 1)$$

arguing similarly as in [8, p. 16 and p. 101]: more specifically, an elementary examination shows that in this case the determinant (2.2) is equal to 1 iff all the entries on the principal diagonal are equal to 1, being equal to 0 in all the other cases. Again, the basic composition formula does the job and the whole kernel

$$(x,y) \in [0,1] \times [0,1] \longrightarrow (n+1) \sum_{k=0}^{n} p_{n,k}(x) \cdot q_{n,k}(y)$$

is totally positive.

Since for any $f \in C^1([0,1])$ $K_n(f') = (B_{n+1}f)'$ (see [1, p. 333, formula (5.3.35)]), we may conclude that for any $r \geq 0$ $K_n e_r$ is a polynomial of degree $r$ with leading coefficient

$$a_{n,r} := \prod_{k=0}^{r} \left(1 - \frac{k}{n+1}\right), \quad r \geq 0.$$

The conclusions of Theorem 2.3 are now available.

Example 3.4. Fix an integer $m \geq 2$, set $w_m(x) := (1 + x^m)^{-1}$ $(x \geq 0)$ and consider the weighted space

$$E_m^0 := \{f \in C([0, +\infty[) | \lim_{x \to +\infty} w_m(x) f(x) = 0\}$$

which turns out to be a Banach space if endowed with the norm $||f||_m := \sup_{x\geq 0} w_m(x)|f(x)|$.

Now for any $n \geq 1, f \in E_m^0$ and $x \geq 0$ define

$$P_n f(x) := \frac{1}{\Gamma(n)} \int_0^{+\infty} t^{n-1} \mathrm{e}^{-t} f\left(\frac{xt}{n}\right) \mathrm{d}t \tag{3.11}$$

as the $n$-th Post-Widder operator (here $\Gamma$ is the usual Gamma function),

$$M_n f(x) := \mathrm{e}^{-nx} \sum_{k=0}^{+\infty} \frac{(nx)^k}{k!} f\Big(\frac{k}{n}\Big) \tag{3.12}$$

as the $n$-th Szasz-Mirakjan operator and

$$\beta_n f(x) := \frac{1}{(1+x)^n} \cdot \sum_{k=0}^{+\infty} \binom{n+k-1}{k} \Big(\frac{x}{1+x}\Big)^k f\Big(\frac{k}{n}\Big) \tag{3.13}$$

as the $n$-th Baskakov operator.

The sequences $(P_n)_{n\geq 1}, (M_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$ are positive approximation processes on $(E_m^0, ||\cdot||_m)$ (see [10], [2] and [3]).

A quick examination through the same techniques adopted so far in the list of examples (in (3.11) the change of variable $\frac{xt}{n} = u$ is suggested) enables us to say that we are dealing with positive linear transformations falling within the class (2.4) or (2.5) with totally positive kernels. Furthermore, for all $r = 0, 1, \ldots, m-1$ the powers $e_r$ belong to the underlying space $E_m^0$ and are mapped, by each $P_n, M_n$ and $\beta_n$, into polynomials of degree $r$ with corresponding leading coefficients $p_{n,r}, m_{n,r}$ and $b_{n,r}$ given by

$$p_{n,r} = b_{n,r} := \begin{cases} \prod_{k=0}^{r-1} (1 + \frac{k}{n}), & \text{if} \quad 1 \leq r \leq m-1, \\ 1, & \text{if} \quad r = 0, \end{cases} \tag{3.14}$$

$$m_{n,r} = 1 \quad \text{for any} \quad 0 \leq r \leq m-1.$$

This is straightforward to prove for $p_{n,r}$; in the other two cases one may argue by induction.

The application of Theorem 2.3 and of the subsequent Remark 2.5 is now available, yielding that the operators under examination preserve $q$-th order convexity for any $q = 0, 1, \ldots, m$.

The preservation of Lipschitz classes $\mathrm{Lip}_q(M)$ $(1 \leq q \leq m-1, M > 0)$ is guaranteed as well, in the terms of Remark 2.5, with the constant $a_q$ replaced, case by case, by $p_{n,q}$, $m_{n,q}$ and $b_{n,q}$.

Finally, we recall that in [10], [2] and [3] it has been proved that the iterates of $P_n, M_n$ and $\beta_n$ converge, in $(E_m^0, ||\cdot||_m)$, towards strongly continuous semigroups $(P(t))_{t\geq 0}$, $(M(t))_{t\geq 0}$ and $(B(t))_{t\geq 0}$ on $E_m^0$, according to (2.10).

The $q$-th order convexity preserving property $(0 \leq q \leq m)$ for each of the above semigroups follows immediately from Theorem 2.7 and Remark 2.9.

Moreover, by virtue of the same theorem and remark, since $\lim_{n\to+\infty}(p_{n,q})^n = \lim_{n\to+\infty}(b_{n,q})^n = \mathrm{e}^{\frac{(q-1)q}{2}}$ and $\lim_{n\to+\infty}(m_{n,q})^n = 1$ for all $q = 1, 2, \dots, m-1$, the following inclusions

$$P(t)(E_m^0 \cap \mathrm{Lip}_q(M)) \subset \mathrm{Lip}_q\left(M\mathrm{e}^{\frac{(q-1)qt}{2}}\right),$$

$$M(t)(E_m^0 \cap \mathrm{Lip}_q(M)) \subset \mathrm{Lip}_q(M),$$

$$B(t)(E_m^0 \cap \mathrm{Lip}_q(M)) \subset \mathrm{Lip}_q\left(M\mathrm{e}^{\frac{(q-1)qt}{2}}\right),$$

are satisfied for all $t \geq 0, M > 0$ and $q = 1, \dots, m-1$. Note that in particular $\mathrm{Lip}_1(M)$ is invariant in all the cases.

Example 3.5. In [4] the authors consider a particular modification of the classical Bernstein operators $B_n$; namely, let $a, b \geq -1$ and for any $n \geq n_0 := \max\{a+1, b+1\}$ define

$$L_n f := B_n\left(f \circ \left((1 - \tfrac{a+b+2}{2n})\, e_1 + \tfrac{a+1}{2n}\, e_0\right)\right) \tag{3.15}$$

for all $f \in C([0,1])$.

After writing down explicitly (3.15), one may easily realize that we fall in the class (2.5) with the totally positive kernel $\binom{n}{k}x^k(1-x)^{n-k}$ (see Example 3.2). Moreover, with the aid of the well-known properties of $B_n$, we see that for any $n \geq n_0$ and $r \geq 0$ $L_n e_r$ is a polynomial of degree $r$ with leading coefficient

$$a_{n,r} := \begin{cases} (1 - \frac{a+b+2}{2n})^r \cdot \prod_{k=0}^{r-1}(1 - \frac{k}{n}), & \text{if} \quad r \geq 1, \\ 1, & \text{if} \quad r = 0. \end{cases} \tag{3.16}$$

Like in the previous examples, we infer that each $L_n$ ($n \geq n_0$) preserves $q$-th order convexity for any $q \geq 0$ (improving, in this way, statement (i) in [4, Proposition 3.2]) and in addition

$$L_n(C([0,1]) \cap \mathrm{Lip}_q(M)) \subset \mathrm{Lip}_q(M a_{n,q}) \tag{3.17}$$

for all $n \geq n_0$, $q \geq 1$ and $M > 0$, with $a_{n,q}$ defined as in (3.16). In particular, for $q = 1$ we have

$$L_n(\mathrm{Lip}_1(M)) \subset \mathrm{Lip}_1\left(M(1 - \tfrac{a+b+2}{2n})\right),$$

which is the same as in [4, Proposition 3.2, (ii)] when $\alpha = 1$.

For the semigroup $(T(t))_{t\geq 0}$ generated by the iterates of $L_n$ (see [4, Theorem 3.4]) we get the preservation of the convexity of any order and the inclusion

$$T(t)(C([0,1]) \cap \mathrm{Lip}_q(M)) \subset \mathrm{Lip}_q\left(M\mathrm{e}^{-q(a+b+q+1)t}\right)$$

for all $t \geq 0, q \geq 1$ and $M > 0$ (compare with [4, Corollary 3.5]).

Finally, observe that when $a = b = -1$, then $L_n = B_n$: obviously, all the above properties still hold true with the necessary changes (for the representation of the semigroup through iterates of $B_n$, see, however [1, Chapter VI]).

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Received by the editors: May 26, 2006.