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APPROXIMATION AND GEOMETRIC PROPERTIES OF SOME COMPLEX BERNSTEIN-STANCU POLYNOMIALS IN COMPACT DISKS*

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Abstract. In this paper, the order of simultaneous approximation, convergence results of the iterates and shape preserving properties, for complex Bernstein-Stancu polynomials (depending on one parameter) attached to analytic functions on compact disks are obtained.

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1. INTRODUCTION

Concerning the convergence of Bernstein polynomials in the complex plane, Bernstein proved (see e.g. [6, p. 88]) that if $f : G \to \mathbb{C}$ is analytic in the open set $G \subset \mathbb{C}$, with $\overline{\mathbb{D}}_1 \subset G$ (where $\mathbb{D}_1 = \{z \in \mathbb{C} : |z| < 1\}$), then the complex Bernstein polynomials $B_n(f)(z) = \sum_{k=0}^{n} \binom{n}{k} z^k (1-z)^{n-k} f(\frac{k}{n})$, uniformly converge to f in $\overline{\mathbb{D}}_1$.

Estimates of order $O(\frac{1}{n})$ of this uniform convergence and, in addition, of the simultaneous approximation, were found in [2]. Also, in [2] it was proved that the complex Bernstein polynomials preserve (beginning with an index), the univalence, starlikeness, convexity and spirallikeness.

In [3], quantitative and qualitative approximation results for iterates of complex Bernstein polynomials were obtained.

The goal of this paper is to extend the above mentioned approximation results, to the following kind of complex Bernstein-Stancu polynomials:

$$S_n^{<\gamma>}(f)(z) = \sum_{k=0}^{n} p_{n,k}^{<\gamma>}(z) f(\tfrac{k}{n}), \quad \gamma \geq 0, z \in \mathbb{C},$$

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where γ may to depend on n and

$$p_{n,k}^{<\gamma>}(z) = \binom{n}{k}\frac{z(z+\gamma)...(z+(k-1)\gamma)(1-z)(1-z+\gamma)...(1-z+(n-k-1)\gamma)}{(1+\gamma)(1+2\gamma)...(1+(n-1)\gamma)}.$$

For $\gamma = 0$, these polynomials become the classical complex Bernstein polynomials.

2. APPROXIMATION PROPERTIES

Concerning the approximation orders by the Bernstein-Stancu polynomials defined in Introduction, the main results are expressed by the following.

THEOREM 2.1. *Let $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ be with $R > 1$ and let us suppose that $f : \mathbb{D}_R \to \mathbb{C}$ is analytic in $\mathbb{D}_R$, that is we can write $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$.*

Let $0 \le \gamma$ which can be dependent on n and $1 \le r < R$. Then, for all $|z| \le r$ and $n \in \mathbb{N}$, we have

$$|S_n^{<\gamma>}(f)(z) - f(z)| \le M_{2,r,n}^{<\gamma>}(f),$$

where

$$0 < M_{2,r,n}^{<\gamma>}(f) = \frac{2}{n}\sum_{j=2}^{\infty} j(j-1)|c_j|r^j + \frac{\gamma(r+1)}{6r}\sum_{j=2}^{\infty} j(j-1)(2j-1)|c_j|r^j < \infty.$$

Also, if $1 \le r < r_1 < R$, then for all $|z| \le r$ and $n, p \in \mathbb{N}$, we have

$$\left|[S_n^{<\gamma>}(f)]^{(p)}(z) - f^{(p)}(z)\right| \le \frac{M_{2,r_1,n}^{<\gamma>}(f)p!r_1}{(r_1 - r)^{p+1}}.$$

Proof. Since $S_n^{<\gamma>}(f)(z) = \sum_{k=0}^{\infty} c_k S_n^{<\gamma>}(e_k)(z)$, we get

$$|S_n^{<\gamma>}(f)(z) - f(z)| \le \sum_{k=0}^{\infty} |c_k| \cdot |S_n^{<\gamma>}(e_k)(z) - e_k(z)|.$$

To estimate $|S_n^{<\gamma>}(e_k)(z) - e_k(z)|$ for any fixed $n \in \mathbb{N}$, we will consider two possible cases: 1) $0 \le k \le n$; 2) $k > n$.

We will use the well-known representation (see [8])

$$S_n^{<\gamma>}(f)(z) = \sum_{p=0}^{n} \binom{n}{p}\frac{z(z+\gamma)...(z+(p-1)\gamma)}{(1+\gamma)...(1+(p-1)\gamma)}\Delta_{1/n}^p f(0).$$

Denoting

$$D_{n,p,k} = \binom{n}{p}\Delta_{1/n}^p e_k(0) = \binom{n}{p}[0, \tfrac{1}{n}, ..., \tfrac{p}{n}; e_k]\frac{p!}{n^p},$$

since e_k is convex of any order, it follows that all $D_{n,p,k} \ge 0$ and

$$S_n^{<\gamma>}(e_k)(z) = \sum_{p=0}^{\min\{n,k\}} D_{n,p,k}\frac{z(z+\gamma)...(z+(p-1)\gamma)}{(1+\gamma)...(1+(p-1)\gamma)}.$$

Also, since $S_n^{<\gamma>}(f)(1)=f(1)$, we get $\sum\limits_{p=0}^{n} D_{n,p,k}=\sum\limits_{p=0}^{\min\{n,k\}} D_{n,p,k}=1$.

Note that since for any $j=0,1,...,$ we have $\frac{r+j\gamma}{1+j\gamma}\le r$, for all $0\le p\le \min\{n,k\}\le k$ and $|z|\le r$ we obtain

$$\frac{|z(z+\gamma)...(z+(p-1)\gamma)|}{(1+\gamma)...(1+(p-1)\gamma)}\le r\frac{r+\gamma}{1+\gamma}\cdot ...\frac{r+(p-1)\gamma}{1+(p-1)\gamma}\le r^p\le r^k,$$

which for all $|z|\le r$ and $n,k\in\mathbb{N}$, immediately implies

$$|S_n^{<\gamma>}(e_k)(z)|\le r^k\sum_{p=0}^{\min\{n,k\}} D_{n,p,k}=r^k.$$

Case 1). If $k=0$, then obviously we have $S_n^{<\gamma>}(e_k)(z)-e_k(z)=0$. Therefore, let us suppose that $1\le k\le n$. By using the representation in [8], we obtain

$$\begin{aligned}|S_n^{<\gamma>}(e_k)(z)-e_k(z)|&\le\left|\frac{n(n-1)...(n-(k-1))}{n^k}\cdot\frac{z(z+\gamma)...(z+(k-1)\gamma)}{(1+\gamma)...(1+(k-1)\gamma)}-z^k\right|\\&+\sum_{p=0}^{k-1} D_{n,p,k}\left|\frac{z(z+\gamma)...(z+(p-1)\gamma)}{(1+\gamma)...(1+(p-1)\gamma)}\right|\\&:=E_{n,k}^{<\gamma>}(z)+F_{n,k}^{<\gamma>}(z).\end{aligned}$$

For $|z|\le r$ it follows

$$F_{n,k}^{<\gamma>}(z)\le r^k\sum_{p=0}^{k-1} D_{n,p,k}=r^k[1-D_{n,k,k}]=r^k[1-\frac{n(n-1)...(n-(k-1))}{n^k}]\le r^k\frac{k(k-1)}{2n}.$$

Here we have applied the inequality $1-\prod x_i\le\sum(1-x_i)$, with all $0\le x_i\le 1$.

Also,

$$\begin{aligned}E_{n,k}^{<\gamma>}(z)&\le\left|\frac{n(n-1)...(n-(k-1))}{n^k}\frac{z(z+\gamma)...(z+(k-1)\gamma)}{(1+\gamma)...(1+(k-1)\gamma)}-\frac{z(z+\gamma)...(z+(k-1)\gamma)}{(1+\gamma)...(1+(k-1)\gamma)}\right|\\&+\left|\frac{z(z+\gamma)...(z+(k-1)\gamma)}{(1+\gamma)...(1+(k-1)\gamma)}-z^k\right|\\&\le\left|\frac{z(z+\gamma)...(z+(k-1)\gamma)}{(1+\gamma)...(1+(k-1)\gamma)}\right|\cdot\left|1-\frac{n(n-1)...(n-(k-1))}{n^k}\right|\\&+\left|\frac{z(z+\gamma)...(z+(k-1)\gamma)}{(1+\gamma)...(1+(k-1)\gamma)}-z^k\right|\le r^k\frac{k(k-1)}{2n}+\left|\frac{z(z+\gamma)...(z+(k-1)\gamma)}{(1+\gamma)...(1+(k-1)\gamma)}-z^k\right|.\end{aligned}$$

For any fixed $|z|\le r$, let us denote $g_k(\alpha)(z)=\frac{z(z+\alpha)...(z+(k-1)\alpha)}{(1+\alpha)...(1+(k-1)\alpha)}$, where $\alpha\ge 0$. Then, by the mean value theorem, there is $\xi\in[0,\gamma]$ such that

$$\left|\frac{z(z+\gamma)...(z+(k-1)\gamma)}{(1+\gamma)...(1+(k-1)\gamma)}-z^k\right|=|g_k(\gamma)(z)-g_k(0)(z)|\le\gamma\cdot\max\left|\frac{\mathrm{d}g_k(\xi)(z)}{\mathrm{d}\alpha}\right|.$$

But denoting $u_j(\alpha)(z)=\frac{z+j\alpha}{1+j\alpha}$, we have $g_k(\alpha)(z)=z\prod\limits_{j=1}^{k-1}u_j(\alpha)(z)$ and

$$\frac{\mathrm{d}g_k(\alpha)(z)}{\mathrm{d}\alpha}=z\sum_{j=1}^{k-1}\left(\frac{z+j\alpha}{1+j\alpha}\right)'_\alpha\cdot\prod_{i=1,i\ne j}^{k-1}\frac{z+i\alpha}{1+i\alpha}=z\sum_{j=1}^{k-1}\frac{j(1-z)}{(1+j\alpha)^2}\prod_{i=1,i\ne j}^{k-1}\frac{z+i\alpha}{1+i\alpha}.$$

Since $\frac{j}{(1+j\xi)^2} \le j^2$, passing to modulus (for $0 \le \xi \le \gamma$ and $|z| \le r$), we obtain

$$\left|\frac{\mathrm{d}g_k(\xi)(z)}{\mathrm{d}\alpha}\right| \le r(r+1)\sum_{j=1}^{k-1} j^2 r^{k-2} = (r+1)r^{k-1}\frac{k(k-1)(2k-1)}{6}.$$

It follows

$$E_{n,k}^{<\gamma>}(z) \le r^k \frac{k(k-1)}{2n} + \gamma(r+1)r^{k-1}\frac{k(k-1)(2k-1)}{6}.$$

Collecting all the above estimates, we get for all $|z| \le r$

$$\begin{aligned}|S_n^{<\gamma>}(e_k)(z) - e_k(z)| \le& r^k \frac{k(k-1)}{2n} + r^k \frac{k(k-1)}{2n} + \gamma(r+1)r^{k-1}\frac{k(k-1)(2k-1)}{6}\\ =& r^k\left[\frac{k(k-1)}{n} + \gamma \cdot \frac{r+1}{r} \cdot \frac{k(k-1)(2k-1)}{6}\right].\end{aligned}$$

Case 2). We have

$$\begin{aligned}|S_n^{<\gamma>}(e_k)(z) - e_k(z)| \le& |S_n^{<\gamma>}(e_k)(z)| + |e_k(z)|\\ \le& \sum_{p=0}^{n} D_{n,p,k}\left|\frac{z(z+\gamma)...(z+(p-1)\gamma)}{(1+\gamma)...(1+(p-1)\gamma)}\right| + |e_k(z)|.\end{aligned}$$

Reasoning as in the above Case 1), we get

$$|S_n^{<\gamma>}(e_k)(z) - e_k(z)| \le r^n + r^k \le 2r^k \le \frac{2(k-1)k}{n}r^k.$$

Collecting all the results in the Cases 1) and 2), we immediately obtain, for all $|z| < r$ and $k = 0, 1, 2, ...$,

$$|S_n^{<\gamma>}(e_k)(z) - e_k(z)| \le r^k\left[\frac{2k(k-1)}{n} + \gamma \cdot \frac{r+1}{r} \cdot \frac{k(k-1)(2k-1)}{6}\right],$$

which implies the corresponding estimate in statement.

For the simultaneous approximation, denoting by Γ the circle of radius $r_1 > r$ and center 0, since for any $|z| \le r$ and $v \in \Gamma$, we have $|v - z| \ge r_1 - r$, by the Cauchy's formulas it follows that for all $|z| \le r$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}|[S_n^{<\gamma>}(f)]^{(p)}(z) - f^{(p)}(z)| =& \frac{p!}{2\pi}\left|\int_\Gamma \frac{S_n^{<\gamma>}(f)(v) - f(v)}{(v-z)^{p+1}}\mathrm{d}v\right| \le M_{2,r_1,n}^{<\gamma>}(f)\frac{p!}{2\pi}\frac{2\pi r_1}{(r_1-r)^{p+1}}\\ =& M_{2,r_1,n}^{<\gamma>}(f)\frac{p! r_1}{(r_1-r)^{p+1}},\end{aligned}$$

which proves the theorem. □

Remark 2.2. For $\gamma = 0$ we get the results in [2]. □

3. ITERATES

Defining the m-th iterates by ${}^m S_n^{<\gamma>}(f)(z)$, first we prove the following qualitative result.

THEOREM 3.1. *Let* $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ *be with* $R > 1$ *and let us suppose that* $f : \mathbb{D}_R \to \mathbb{C}$ *is analytic in* $\mathbb{D}_R$*, that is we can write* $f(z) = \sum_{k=0}^{\infty} c_k z^k$*, for all* $z \in \mathbb{D}_R$*. Let* $0 \le \gamma$*. Uniformly in* $|z| \le r$*, where* $1 \le r < R$*, we have*

$$\lim_{m\to\infty} {}^m S_n^{<\gamma>}(f)(z) = (1-z)f(0) + zf(1), \quad \forall n \in \mathbb{N}.$$

Proof. From [1], Remark 2 after Theorem 9, p. 165, for any $n \in \mathbb{N}$, we have $\lim_{m\to\infty} {}^m S_n^{<\gamma>}(f)(x) = (1-x)f(0) + xf(1)$, uniformly with respect to $x \in [0,1]$. From the classical Vitali's result, it suffices to show that for any fixed $n \in \mathbb{N}$, the sequence $({}^m S_n^{<\gamma>}(f)(z))_{m\in\mathbb{N}}$ is uniformly bounded for $|z| \le r$.

We have ${}^m S_n^{<\gamma>}(f)(z) = \sum_{k=0}^{\infty} c_k \cdot {}^m S_n^{<\gamma>}(e_k)(z)$. We will prove that for all $n, m, k \in \mathbb{N}$ and $|z| \le r$, we have $|{}^m S_n^{<\gamma>}(e_k)(z)| \le r^k$.

Indeed, for $m = 1$ it easily follows by (also see the proof of Theorem 2.1)

$$S_n^{<\gamma>}(e_k)(z) = \sum_{j=0}^{n} D_{n,j,k} \tfrac{z(z+\gamma)...(z+(j-1)\gamma)}{(1+\gamma)...(1+(j-1)\gamma)} = \sum_{j=0}^{\min\{n,k\}} D_{n,j,k} \tfrac{z(z+\gamma)...(z+(j-1)\gamma)}{(1+\gamma)...(1+(j-1)\gamma)}$$

with $D_{n,j,k} \ge 0$ and $\sum_{j=0}^{n} D_{n,j,k} = \sum_{j=0}^{\min\{n,k\}} D_{n,j,k} = 1$.

Denote $h_j(z) = z(z+\gamma)...(z+(j-1)\gamma) = \sum_{i=0}^{j} c_i^{(j)} e_i(z)$, where $c_i^{(j)} \ge 0$, $c_j^{(j)} = 1$ and $\sum_{i=0}^{j} c_i^{(j)} = h_j(1) = (1+\gamma)...(1+(j-1)\gamma)$.

By the linearity of $S_n^{<\gamma>}$, we get

$$\begin{aligned}
|{}^2 S_n^{<\gamma>}(e_k)| &= \left| \sum_{j=0}^{\min\{n,k\}} D_{n,j,k} \tfrac{1}{(1+\gamma)...(1+(j-1)\gamma)} \cdot \sum_{i=0}^{j} c_i^{(j)} S_n^{<\gamma>}(e_i)(z) \right| \\
&\le \sum_{j=0}^{\min\{n,k\}} D_{n,j,k} \tfrac{1}{(1+\gamma)...(1+(j-1)\gamma)} \cdot \sum_{i=0}^{j} c_i^{(j)} r^j \\
&\le r^k,
\end{aligned}$$

and by mathematical induction it follows that for all $n, m, k \in \mathbb{N}$ we have

$$|{}^m S_n^{<\gamma>}(e_k)(z)| \le r^k, \text{ for all } |z| \le r.$$

This implies that

$$|{}^m S_n^{<\gamma>}(f)(z)| \le \sum_{k=0}^{\infty} |c_k| \cdot |{}^m S_n^{<\gamma>}(e_k)(z)| \le \sum_{k=0}^{\infty} |c_k| r^k < \infty,$$

for all $m, n \in \mathbb{N}$, which proves the theorem. $\square$

We also have the following quantitative result.

THEOREM 3.2. *Let $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ be with $R > 1$ and let us suppose that $f : \mathbb{D}_R \to \mathbb{C}$ is analytic in $\mathbb{D}_R$, that is we can write $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$. Let $0 \leq \gamma$, $1 \leq r < R$ and $D_{n,k,k} = \frac{n(n-1)...(n-(k-1))}{n^k}$. Then, for all $|z| \leq r$ we have*

$$\begin{aligned}
&|^m S_n^{<\gamma>}(f)(z) - f(z)| \leq \\
&\leq m \sum_{k=2}^{\infty} |c_k| \left[\tfrac{2k(k-1)}{n} + \left(1 - \tfrac{D_{n,k,k}}{(1+\gamma)...(1+(k-1)\gamma)}\right) + \gamma(k-1)^2\right] r^k.
\end{aligned}$$

Proof. From the proof of Theorem 3.1, it follows that for all $n, m, k \in \mathbb{N}$ and $|z| \leq r$, we have $|^m S_n^{<\gamma>}(e_k)(z)| \leq r^k$. Also $|^m S_n^{<\gamma>}(f)(z) - f(z)| \leq \sum_{k=2}^{\infty} |c_k| \cdot |^m S_n^{<\gamma>}(e_k)(z) - e_k(z)|$. We have two possibilities: 1) $2 \leq k \leq n$; 2) $k > n$.

Case 1). With the notations for $g_j(\alpha)(z)$ in the proof of Theorem 2.1 and for $h_j(z), c_i^{(j)}$ in the proof of Theorem 3.1, we can write

$$\begin{aligned}
&|^m S_n^{<\gamma>}(e_k)(z) - e_k(z)| = \\
&= \left| \sum_{p=0}^{m-1} {}^p S_n^{<\gamma>}[S_n^{<\gamma>}(e_k)(z) - e_k(z)] \right| \\
&= \left| \sum_{p=0}^{m-1} {}^p S_n^{<\gamma>} \left[\sum_{j=1}^{k} D_{n,j,k} \cdot g_j(\gamma)(z) - e_k(z) \right] \right| \\
&= \left| \sum_{p=0}^{m-1} \left[\sum_{j=1}^{k} D_{n,j,k} \cdot {}^p S_n^{<\gamma>}(g_j(\gamma))(z) -^p S_n^{<\gamma>}(e_k)(z) \right] \right| \\
&\leq \sum_{p=0}^{m-1} \sum_{j=1}^{k-1} D_{n,j,k} |^p S_n^{<\gamma>}(g_j(\gamma))(z)| \\
&\quad + \sum_{p=0}^{m-1} |D_{n,k,k} \cdot^p S_n^{<\gamma>}(g_k(\gamma))(z) -^p S_n^{<\gamma>}(e_k)(z)|
\end{aligned}$$

$$= \sum_{p=0}^{m-1} \sum_{j=1}^{k-1} D_{n,j,k} |{}^p S_n^{<\gamma>}(g_j(\gamma))(z)|$$

$$+ \sum_{p=0}^{m-1} \left| \tfrac{D_{n,k,k}}{(1+\gamma)...(1+(k-1)\gamma)} \cdot {}^p\, S_n^{<\gamma>} \left[\sum_{i=0}^{k} c_i^{(k)} e_i(z) \right] - {}^p\, S_n^{<\gamma>}(e_k)(z) \right|$$

$$\leq \sum_{p=0}^{m-1} \sum_{j=1}^{k-1} D_{n,j,k} |{}^p S_n^{<\gamma>}(g_j(\gamma))(z)|$$

$$+ \sum_{p=0}^{m-1} \left| \tfrac{D_{n,k,k}}{(1+\gamma)...(1+(k-1)\gamma)} \cdot {}^p\, S_n^{<\gamma>}(e_k)(z) - {}^p\, S_n^{<\gamma>}(e_k)(z) \right|$$

$$+ \sum_{p=0}^{m-1} \tfrac{D_{n,k,k}}{(1+\gamma)...(1+(k-1)\gamma)} \left| \sum_{i=0}^{k-1} c_i^{(k)} \cdot {}^p\, S_n^{<\gamma>}(e_i)(z) \right|$$

$$:= T_1 + T_2 + T_3.$$

Reasoning exactly as in the proof of Theorem 3.1, we easily get for all j, p and $|z| \leq r$ that

$$|{}^p S_n^{<\gamma>}(g_j(\gamma))(z)| \leq r^j.$$

Taking into account the formula for $1 - D_{n,k,k}$ in the proof of Theorem 2.1, we get

$$T_1 \leq \sum_{p=0}^{m-1} \sum_{j=1}^{k-1} r^k D_{n,j,k} = m r^k [1 - D_{n,k,k}] \leq m r^k \tfrac{k(k-1)}{2n}.$$

Also,

$$T_2 = \sum_{p=0}^{m-1} |{}^p S_n^{<\gamma>}(e_k)(z)| \left[1 - \tfrac{D_{n,k,k}}{(1+\gamma)...(1+(k-1)\gamma)} \right]$$
$$\leq m r^k \left[1 - \tfrac{D_{n,k,k}}{(1+\gamma)...(1+(k-1)\gamma)} \right].$$

Finally,

$$T_3 \leq \sum_{p=0}^{m-1} \tfrac{D_{n,k,k}}{(1+\gamma)...(1+(k-1)\gamma)} [(1+\gamma)...(1+(k-1)\gamma) - 1] r^k$$
$$= m r^k D_{n,k,k} \left[1 - \tfrac{1}{(1+\gamma)...(1+(k-1)\gamma)} \right].$$

But, taking into account the inequalities $D_{n,k,k} \leq 1$ and

$$1 - \prod_{j=1}^{k-1} x_j \leq \sum_{j=1}^{k-1} (1 - x_j), 0 \leq x_j \leq 1, j = 1, ..., k-1,$$

applied for $x_j = \frac{1}{1+j\gamma}$, we obtain

$$D_{n,k,k}\left[1-\frac{1}{(1+\gamma)\ldots(1+(k-1)\gamma)}\right] \le \sum_{j=1}^{k-1}[1-1/(1+j\gamma)] = \sum_{j=1}^{k-1}\frac{j\gamma}{1+j\gamma}$$
$$\le (k-1)\cdot\frac{\gamma(k-1)}{1+\gamma(k-1)} \le \gamma(k-1)^2.$$

Collecting all these inequalities, we obtain

$$|^{m}S_n^{<\gamma>}(e_k)(z)-e_k(z)| \le mr^k\left[\frac{k(k-1)}{2n}+\left(1-\frac{D_{n,k,k}}{(1+\gamma)\ldots(1+(k-1)\gamma)}\right)+\gamma(k-1)^2\right].$$

Case 2). We get

$$|^{m}S_n^{<\gamma>}(e_k)(z)-e_k(z)| \le |^{m}S_n^{<\gamma>}(e_k)(z)|+|e_k(z)| \le 2r^k \le \frac{2k(k-1)}{n}r^k.$$

As a conclusion, from both Cases 1) and 2), we obtain

$$|^{m}S_n^{<\gamma>}(f)(z)-f(z)| \le \sum_{k=2}^{\infty}|c_k|\cdot|^{m}S_n^{<\gamma>}(e_k)(z)-e_k(z)| =$$
$$= \sum_{k=2}^{n}|c_k|\cdot|^{m}S_n^{<\gamma>}(e_k)(z)-e_k(z)| + \sum_{k=n+1}^{\infty}|c_k|\cdot|^{m}S_n^{<\gamma>}(e_k)(z)-e_k(z)|$$
$$\le \sum_{k=2}^{n}|c_k|mr^k\left[\frac{k(k-1)}{2n}+\left(1-\frac{D_{n,k,k}}{(1+\gamma)\ldots(1+(k-1)\gamma)}\right)+\gamma(k-1)^2\right]$$
$$+\sum_{k=n+1}^{\infty}|c_k|r^k\frac{2k(k-1)}{n}$$
$$\le m\sum_{k=2}^{\infty}|c_k|r^k\left[\frac{2k(k-1)}{n}+\left(1-\frac{D_{n,k,k}}{(1+\gamma)\ldots(1+(k-1)\gamma)}\right)+\gamma(k-1)^2\right]r^k,$$

which proves the theorem. □

REMARK 3.3. For $\gamma = 0$ we get the results in [3]. □

COROLLARY 3.4. (i) *Let $1 \le r < R$. For $\gamma := \gamma_n = 1/n$ and $|z| \le r$ we have the estimate*

$$|^{m}S_n^{<\gamma_n>}(f)(z)-f(z)| \le \frac{m}{n}\sum_{k=2}^{\infty}|c_k|\left[2k(k-1)+2(k-1)^3+(k-1)^2\right]r^k.$$

(ii) *If $\gamma := \gamma_n = 1/n$ and $\frac{m_n}{n} \to 0$ as $n \to \infty$, then ${}^{m_n}S_n^{<\gamma_n>}(f)(z) \to f(z)$, uniformly with respect $|z| \le r$.*

Proof. (i) Taking $\gamma = 1/n$ we obtain

$$1-\frac{D_{n,k,k}}{(1+\gamma)\ldots(1+(k-1)\gamma)} = 1-\prod_{j=1}^{k-1}\frac{n-j}{n+j} \le \sum_{j=1}^{k-1}[1-\frac{n-j}{n+j}] = 2\sum_{j=1}^{k-1}\frac{j}{j+n}$$
$$\le 2(k-1)\frac{k-1}{n+(k-1)} \le 2\frac{(k-1)^3}{n}, \quad k \ge 2,$$

which replaced in Theorem 3.2, gives

$$|^m S_n^{<\gamma_n>}(f)(z) - f(z)| \leq \frac{m}{n} \sum_{k=2}^{\infty} |c_k| \left[2k(k-1) + 2(k-1)^3 + (k-1)^2\right] r^k.$$

(ii) It is evident by passing to limit with $n \to \infty$ in the estimate of (i). □

REMARK 3.5. The results in Theorem 3.2 and Corollary 3.4, are new even for the case of real functions of one real variable, since they are not covered by those in [4] or [5], whose estimates one refer to the difference $|^m L_n(f)(x) - B_1(f)(x)|$, with $B_1(f)(x) = f(0) + [f(1) - f(0)]x$ and $^m L_n(f)$ representing the mth iterate of the positive linear operator $L_n(f)$. □

4. GEOMETRIC PROPERTIES

In this section we present the geometric properties of $S_n^{<\gamma>}(f)(z)$.

THEOREM 4.1. *Let us suppose that $G \subset \mathbb{C}$ is open, such that $\overline{\mathbb{D}}_1 \subset G$ and $f : G \to \mathbb{C}$ is analytic in G. Also, let us consider $(S_n^{<\gamma(n)>}(f)(z))_{n\in\mathbb{N}}$, where we suppose that $\lim_{n\to\infty} \gamma(n) = 0$.*

If $f(0) = f'(0) - 1 = 0$ and f is starlike (convex, spirallike of type η, respectively) in $\overline{\mathbb{D}}_1$, that is for all $z \in \overline{\mathbb{D}}_1$ (see e.g. [7]*)*

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad \left(\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right) + 1 > 0, \mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\eta}\frac{zf'(z)}{f(z)}\right) > 0, \ \textit{resp.}\right),$$

then there exists an index n_0 depending on f (and on η for spirallikeness), such that, for all $n \geq n_0$, $S_n^{<\gamma(n)>}(f)(z)$ are starlike (convex, spirallike of type η, respectively) in $\overline{\mathbb{D}}_1$.

If $f(0) = f'(0) - 1 = 0$ and f is starlike (convex, spirallike of type η, respectively) only in $\mathbb{D}_1$ (that is the corresponding inequalities hold only in $\mathbb{D}_1$), then, for any disk of radius $0 < r < 1$ and center 0 denoted by $\mathbb{D}_r$, there exists an index $n_0 = n_0(f, \mathbb{D}_r)$ (n_0 depends on η too in the case of spirallikeness), such that, for all $n \geq n_0$, $S_n^{<\gamma(n)>}(f)(z)$ are starlike (convex, spirallike of type η, respectively) in $\overline{\mathbb{D}}_r$ (that is, the corresponding inequalities hold in $\overline{\mathbb{D}}_r$).

Proof. By Theorem 2.1, it follows that we have $S_n^{<\gamma(n)>}(f)(z) \to f(z)$, uniformly for $|z| \leq 1$, which by the well-known Weierstrass's theorem implies $[S_n^{<\gamma(n)>}(f)]'(z) \to f'(z)$ and $[S_n^{<\gamma(n)>}(f)]''(z) \to f''(z)$, for $n \to \infty$, uniformly in $\overline{\mathbb{D}}_1$. In all what follows, denote $P_n(f)(z) = \frac{S_n^{<\gamma(n)>}(f)(z)}{[S_n^{<\gamma(n)>}(f)]'(0)}$, well defined for sufficiently large n. We easily get $P_n(f)(0) = 0$, $P_n'(f)(0) = 1$ for sufficiently large n, and $P_n(f)(z) \to f(z)$, $P_n'(f)(z) \to f'(z)$ and $P_n''(f)(z) \to f''(z)$, uniformly in $\overline{\mathbb{D}}_1$.

Suppose first that f is starlike in $\overline{\mathbb{D}}_1$. Then, by hypothesis, we get $|f(z)| > 0$ for all $z \in \overline{\mathbb{D}}_1$ with $z \neq 0$, which, from the univalence of f in $\mathbb{D}_1$, implies that we can write $f(z) = zg(z)$, with $g(z) \neq 0$, for all $z \in \overline{\mathbb{D}}_1$, where g is analytic in $\mathbb{D}_1$ and continuous in $\overline{\mathbb{D}}_1$.

Writing $P_n(f)(z)$ in the form $P_n(f)(z) = zQ_n(f)(z)$, obviously $Q_n(f)(z)$ is a polynomial of degree $\leq n-1$. Also, for $|z| = 1$ we have $|f(z) - P_n(f)(z)| = |z| \cdot |g(z) - Q_n(f)(z)| = |g(z) - Q_n(f)(z)|$, which by the uniform convergence in $\overline{\mathbb{D}}_1$ of $P_n(f)$ to f and by the maximum modulus principle, implies the uniform convergence in $\overline{\mathbb{D}}_1$ of $Q_n(f)(z)$ to $g(z)$.

Since g is continuous in $\overline{\mathbb{D}}_1$ and $|g(z)| > 0$ for all $z \in \overline{\mathbb{D}}_1$, there exist an index $n_1 \in \mathbb{N}$ and $a > 0$ depending on g, such that $|Q_n(f)(z)| > a > 0$, for all $z \in \overline{\mathbb{D}}_1$ and all $n \geq n_0$. Also, for all $|z| = 1$, we have

$$\begin{aligned}
&|f'(z) - P_n'(f)(z)| = \\
&= |z[g'(z) - Q_n'(f)(z)] + [g(z) - Q_n(f)(z)]| \\
&\geq ||z| \cdot |g'(z) - Q_n'(f)(z)| - |g(z) - Q_n(f)(z)|| \\
&= ||g'(z) - Q_n'(f)(z)| - |g(z) - Q_n(f)(z)||\,,
\end{aligned}$$

which from the maximum modulus principle, the uniform convergence of $P_n'(f)$ to f' and of $Q_n(f)$ to g, evidently implies the uniform convergence of $Q_n'(f)$ to g'.

Then, for $|z| = 1$, we get

$$\begin{aligned}
\tfrac{zP_n'(f)(z)}{P_n(f)} =& \tfrac{z[zQ_n'(f)(z)+Q_n(f)(z)]}{zQ_n(f)(z)} = \tfrac{zQ_n'(f)(z)+Q_n(f)(z)}{Q_n(f)(z)} \\
&\to \tfrac{zg'(z)+g(z)}{g(z)} = \tfrac{f'(z)}{g(z)} = \tfrac{zf'(z)}{f(z)},
\end{aligned}$$

which again, from the maximum modulus principle, implies

$$\tfrac{zP_n'(f)(z)}{P_n(f)} \to \tfrac{zf'(z)}{f(z)}, \text{ uniformly in } \overline{\mathbb{D}}_1.$$

Since $\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)$ is continuous in $\overline{\mathbb{D}}_1$, there exists $\varepsilon \in (0,1)$, such that

$$\mathrm{Re}\left(\tfrac{zf'(z)}{f(z)}\right) \geq \varepsilon, \text{ for all } z \in \overline{\mathbb{D}}_1.$$

Therefore

$$\mathrm{Re}\left[\tfrac{zP_n'(f)(z)}{P_n(f)(z)}\right] \to Re\left[\tfrac{zf'(z)}{f(z)}\right] \geq \varepsilon > 0$$

uniformly on $\overline{\mathbb{D}}_1$, i.e., for any $0 < \rho < \varepsilon$, there is n_0 such that for all $n \geq n_0$ we have

$$\mathrm{Re}\left[\tfrac{zP_n'(f)(z)}{P_n(f)(z)}\right] > \rho > 0, \text{ for all } z \in \overline{\mathbb{D}}_1.$$

Since $P_n(f)(z)$ differs from $S_n^{<\gamma(n)>}(f)(z)$ only by a constant, this proves the starlikeness of $S_n^{<\gamma(n)>}(f)(z)$, for sufficiently large n.

If f is supposed to be starlike only in $\mathbb{D}_1$, the proof is identical, with the only difference that instead of $\overline{\mathbb{D}}_1$, we reason for $\overline{\mathbb{D}}_r$.

The proofs in the cases when f is convex or spirallike of order η are similar and follow from the following uniform convergences (on $\overline{\mathbb{D}}_1$ or on $\overline{\mathbb{D}}_r$)

$$\mathrm{Re}\left[\frac{zP_n''(f)(z)}{P_n'(f)(z)}\right]+1 \to \mathrm{Re}\left[\frac{zf''(z)}{f'(z)}\right]+1, \quad \mathrm{Re}\left[\mathrm{e}^{\mathrm{i}\eta}\frac{zP_n'(f)(z)}{P_n(f)(z)}\right] \to \mathrm{Re}\left[\mathrm{e}^{\mathrm{i}\eta}\frac{zf'(z)}{f(z)}\right],$$

as $n \to \infty$, which proves the theorem. □

REMARK 4.2. If f is univalent in $\overline{\mathbb{D}}_1$, then from the uniform convergence in Theorem 2.1 and a well-known result in complex analysis, concerning sequences of analytic functions converging locally uniformly to an univalent function, it is immediate that for sufficiently large n, the complex polynomials $S_n^{<\gamma(n)>}(f)(z)$ (where $\gamma(n) \to 0$, for $n \to \infty$), must be univalent in $\overline{\mathbb{D}}_1$. □

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