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ON THE ASYMPTOTICS OF ORTHOGONAL POLYNOMIALS ON THE CURVE WITH A DENUMERABLE MASS POINTS

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Abstract. We investigate the asymptotic behavior of orthogonal polynomials with respect to a measure of the type $\sigma = \alpha + \gamma$, where α is a measure concentrated on a rectifiable Jordan curve and γ is an infinite discrete measure.

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1. INTRODUCTION

Let σ be a finite positive Borel measure on a compact set of the complex plane whose support contains an infinite set of points. Denote by $T_n(z)$ the monic polynomial of degree n with respect to the measure σ i.e.

$$\begin{aligned} T_n(z) &= z^n + ... \\ \int_E T_n(z)\, \overline{z}^m \mathrm{d}\sigma &= 0; \quad m = 0, 1, 2, ..., n-1. \end{aligned}$$

Let P_n be the set of polynomials of degree n, it is well known that $T_n(z)$ satisfies the extremal properties:

$$\|T_n\|^2_{L_2(\sigma)} := \min_{Q \in P_{n-1}} \|z^n + Q\|^2_{L_2(\sigma)} = m_n(\sigma), \tag{1}$$

where as usual

$$\|f\|_{L_2(\sigma)} := \left(\int |f(\xi)|^2 \, \mathrm{d}\sigma(\xi) \right)^{1/2}.$$

One of the major areas of research in the study of orthogonal polynomials is to investigate the asymptotics behavior of $T_n(z)$ as $n \to \infty$. There exists different type of asymptotics behavior, in this context, we can mention the most frequent ones:

a) $n$ths-root (or weak) asymptotic behavior of orthogonal polynomials is the asymptotic of $\sqrt[n]{|T_n(z)|}$, $n \in \mathbb{N}$, it requires weakest assumptions and depends on the regularity properties of the measure σ. The main tool of study in this case is the logarithmic potential. Among the applications one can cite the

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location and asymptotics of zeros distribution. Extremely important results on this subject are given in the book of H. Stahl and V. Totik [8].

b) Ratio asymptotic is the asymptotic of $\frac{T_{n+1}(z)}{T_n(z)}$. The best condition to establish the ratio asymptotics for orthogonal polynomials is the strict positivity of the density of the measure, that is known as Rakhmanov condition.

c) Szegő (or strong) asymptotics is the uniform asymptotic of $T_n(z)$ outside the support of the measure. The essential condition imposed on the measure to obtain the strong asymptotic is the so-called Szegő condition, that permits to construct the associated Hardy spaces and to get the asymptotic from extremal properties of orthogonal polynomials.

It is easy to see that the strong asymptotic implies the two other types of asymptotics.

In the present work we establish the strong asymptotic of the orthogonal polynomials $T_n(z)$ associated with a measure σ supported on a rectifiable Jordan curve and perturbed by an infinite Blaschke sequence of point masses outside the curve.

For the case that $\sigma = \alpha$ is an absolutely continuous measure with respect to the Lebesgue measure $|\mathrm{d}\xi|$ on a rectifiable Jordan curve i.e.

$$\mathrm{d}\alpha(\xi) = \rho(\xi)\,|\mathrm{d}\xi|\,, \rho : E \to \mathbb{R}_+, \int_E \rho(\xi)\,|\mathrm{d}\xi| < +\infty, \tag{2}$$

Geronimus [1] has given such strong asymptotics. An extension of Geronimus results has been given by Kaliaguine [2] in the case when the measure $\sigma = \alpha + \gamma_l$, where α is the same as given in [1] and γ_l is a point measure supported on $\{z_k\}_{k=1}^{l}$ $(|z_k| > 1)$. In [3] Khaldi et al. presented an extension of Kaliaguine's results, where they studied the case of a measure of the form

$$\sigma = \alpha + \gamma,$$

where α is the same as given in [1] and γ is a point measure supported on a denumerable set of points $\{z_k\}_{k=1}^{\infty}$ in the region exterior to the curve E, i.e.

$$\gamma = \sum_{k=1}^{\infty} A_k \delta(z - z_k);\, A_k > 0, \sum_{k=1}^{\infty} A_k < \infty. \tag{3}$$

We note that, in this paper we generalize the results found in [3], more precisely in the proof of Theorem 6, we show that the condition (17) in [3, page 265] imposed on the points $\{z_k\}_{k=1}^{\infty}$ is redundant.

The structure of this paper is the following: In the next section we define Szegő function, Hardy space $\mathrm{H}^2(\Omega, \rho)$ and the extremal problem. In section 3, we give the main results, first we find the limit of a sequence of extremal values, then we prove the asymptotic formulas.

2. EXTREMAL PROBLEM IN THE HARDY SPACE $H^2(\Omega,\rho)$

Let E be a rectifiable Jordan curve in the complex plane, $\Omega = \operatorname{Ext}(E)$, $G = \{z \in C, |z| > 1\}$ ($\infty \in \Omega$, $\infty \in G$), and $\Phi : \Omega \to G$ is the conformal mapping with $\Phi(\infty) = \infty$. We denote $\Psi = \Phi^{-1}$.

If the weight function ρ (which defines the absolutely part of the measure σ) satisfies the Szegő condition:

$$\int_E (\log \rho(\xi)) \left|\Phi'(\xi)\right| |\mathrm{d}\xi| > -\infty,$$

then, the Szegő function D associated with the curve E and the weight function ρ defined by

$$D(z) = \exp\left\{ -\frac{1}{4\pi}\int_0^{2\pi} \frac{\Phi(z)+\mathrm{e}^{\mathrm{i}\theta}}{\Phi(z)-\mathrm{e}^{\mathrm{i}\theta}} \log \frac{\rho(\xi)}{|\Phi'(\xi)|}\mathrm{d}\theta\right\}, \quad \xi = \Psi(\mathrm{e}^{\mathrm{i}\theta}),$$

satisfies the following properties:

(i) $D(z)$ is analytic in Ω, $D(z) \neq 0$ in Ω, and $D(\infty) > 0$.

(ii) $|D(\xi)|^{-2}\left|\Phi'(\xi)\right| = \rho(\xi)$, $\xi \in E$, where $D(\xi) = \lim\limits_{z\to\xi} D(z)$ (a.e. on E).

One says that a function f analytic in Ω is from $H^2(\Omega,\rho)$ space if the function $f\circ\Psi/D\circ\Psi$ is from the usual Hardy space $H^2(G)$. (Let's recall that a function f analytic in Ω is from $H^2(G)$ space if $\lim\limits_{R\to1^+} \frac{1}{R}\int_{C_R} |f(w)|^2 |\mathrm{d}w| < \infty$, where $C_R = \{w \in G : |w| = R\}$).

Each function f from $H^2(\Omega,\rho)$ has limit values on E and

$$\|f\|^2_{H^2(\Omega,\rho)} = \lim_{R\to1^+} \frac{1}{R}\int_{E_R} \frac{|f(z)|^2}{|D(z)|^2}\left|\Phi'(z)\,\mathrm{d}z\right| = \int_E |f(\xi)|^2 \rho(\xi)\,|\mathrm{d}\xi|\,,$$

where $E_R = \{z \in \Omega : |\Phi(z)| = R\}$.

Lemma 1. [2] *If $f \in H^2(\Omega,\rho)$ then for every compact set $K \subset \Omega$ there is a constant C_K such that:*

$$\sup\{|f(z)| : z \in K\} \le C_K \|f\|_{H^2(\Omega,\rho)}\,.$$

Now we define $\mu(\rho)$ as the extremal value of the following problem:

$$\mu(\rho) = \inf\left\{\|\varphi\|^2_{H^2(\Omega,\rho)} : \varphi \in H^2(\Omega,\rho), \varphi(\infty) = 1\right\}. \tag{4}$$

It is proved in [2] that the extremal function of the problem (4) is exactly the function $\varphi^* = D/D(\infty)$.

Lemma 2. [3] *The extremal function of the problem*

$$\mu^\infty(\rho) = \inf\left\{\|\varphi\|^2_{H^2(\Omega,\rho)}, \varphi \in H^2(\Omega,\rho), \varphi(\infty) = 1,\ \varphi(z_k) = 0, k = 1, 2, ...\right\}$$

is given by $\psi^{\infty} = \varphi^* B_\infty$, *in addition*

$$\mu^{\infty}(\rho) = \mu(\rho) \prod_{k=1}^{+\infty} |\Phi(z_k)|^2,$$

where the constant $\mu(\rho)$ *and the function* φ^* *are defined by the problem* (4) *and* B_∞ *is the Blaschke product:*

$$B_\infty(z) = \prod_{k=1}^{+\infty} \tfrac{\Phi(z)-\Phi(z_k)}{\Phi(z)\overline{\Phi(z_k)}-1} \tfrac{|\Phi(z_k)|^2}{\Phi(z_k)}. \tag{5}$$

3. MAIN RESULTS

DEFINITION 3. *A measure* $\sigma = \alpha + \gamma$ *is said to belong to a class* A, *if the absolutely continuous part* α *and the discrete part* γ *satisfy the conditions* (2), (3) *and the Blaschke's condition, i.e.*

$$\sum_{k=1}^{+\infty} (|\Phi(z_k)| - 1) < \infty. \tag{6}$$

REMARK 4. The condition (6) is natural and it guarantees the convergence of the Blaschke product (5). □

We denote by $\lambda_n = \Phi^n - \Phi_n$, where Φ_n is the polynomial part of the Laurent expansion of Φ^n in the neighborhood of infinity.

DEFINITION 5. [1] *A rectifiable curve* E *is said to be of class* Γ *if* $\lambda_n(\xi) \to 0$ *(*$n \to \infty$*) uniformly on* E.

THEOREM 6. *Let a measure* $\sigma = \alpha + \sum\limits_{k=1}^{\infty} A_k \delta(z - z_k)$ *satisfy the conditions* (2) *and* (3), *then*

$$\lim_{l\to\infty} m_n(\sigma_l) = m_n(\sigma),$$

where the measure $\sigma_l = \alpha + \sum\limits_{k=1}^{l} A_k \delta(z - z_k)$ *and* $m_n(.)$ *are defined as in* (1) *i.e.*

$$\begin{aligned} m_n(\sigma) &= \int_E |T_n(\xi)|^2 \rho(\xi) |d\xi| + \sum_{k=1}^{\infty} A_k |T_n(z_k)|^2, \\ m_n(\sigma_l) &= \int_E \left|T_n^l(\xi)\right|^2 \rho(\xi) |d\xi| + \sum_{k=1}^{l} A_k \left|T_n^l(z_k)\right|^2. \end{aligned} \tag{7}$$

Proof. It is easy to see that the extremal property of $T_n^l(z)$ (see (7)) implies that the sequences $\{m_n(\sigma_l)\}_{l=1}^{\infty}$ is increasing and $m_n(\sigma_l) \le m_n(\sigma)$ for every $l \ge 1$, and so Theorem 6 tells us what the limit is.

According to the reproducing property of the kernel polynomial $K_n(\xi, z)$ (see [9]), we have:

$$T_n^l(z_j) = \int\limits_E T_n^l(\xi)\,\overline{K_{n+1}(\xi, z_j)}\rho(\xi)\,|\mathrm{d}\xi|\,.$$

The Schwarz inequality implies

$$\begin{aligned} \left|T_n^l(z_j)\right|^2 &\leq \int\limits_E \left|T_n^l(\xi)\right|^2 \rho(\xi)\,|\mathrm{d}\xi| \int\limits_E |K_{n+1}(\xi, z_j)|^2 \rho(\xi)\,|\mathrm{d}\xi| \\ &\leq m_n(\sigma_l) \sup_{\xi\in E} |K_{n+1}(\xi, z_j)|^2\,, \end{aligned} \tag{8}$$

the extremal property of $T_n(z)$ implies that

$$\begin{aligned} m_n(\sigma) &\leq m_n(\sigma_l) + \sum_{k=l+1}^{\infty} A_k \left|T_n^l(z_k)\right|^2 \\ &\leq m_n(\sigma_l)\left[1 + \sup_{\xi\in E, k\geq l+1} |K_{n+1}(\xi, z_k)|^2 \sum_{k=l+1}^{\infty} A_k\right]. \end{aligned}$$

This gives

$$m_n(\sigma) \leq \liminf_{l\to+\infty} m_n(\sigma_l) \leq \limsup_{l\to+\infty} m_n(\sigma_l) \leq m_n(\sigma)\,.$$

The proof of the Theorem is complete. □

Theorem 7. *Let E be a curve from the class Γ and the measure $\sigma \in A$. If*

$$m_n(\sigma_l) \leq \left(\prod_{k=1}^{l} |\Phi(z_k)|\right) m_n(\alpha)\,, \forall n, \forall l, \tag{9}$$

then the orthogonal polynomials $T_n(z)$ and the extremal value $m_n(\sigma)$ have the following asymptotic behavior ($n \to \infty$):

(i) $\lim \frac{m_n(\sigma)}{(C(E))^{2n}} = \mu^\infty(\sigma)$;

(ii) $\lim \left\|\frac{T_n}{[C(E)\Phi]^n} - \psi^\infty\right\|_{H^2(\Omega,\rho)} = 0$;

(iii) $T_n(z) = [C(E)\Phi(z)]^n [\psi^\infty(z) + \varepsilon_n(z)]$, $\varepsilon_n(z) \to 0$ *uniformly on the compact subsets of* Ω.

Where $\frac{1}{C(E)} = \lim\limits_{z\to\infty} \frac{\Phi(z)}{z} > 0$, *the constant* $\mu^\infty(\sigma)$ *and the function* ψ^∞ *are defined in Lemma* 2.

Remark 8. Note that in the case of the circle the condition (9) is satisfied (see [4, th. 5.2]). □

Proof. By passing to the limit when l tends to infinity in (9) and using Theorem 6, we obtain

$$\frac{m_n(\sigma)}{(C(E))^{2n}} \leq \left(\prod_{k=1}^{+\infty} |\Phi(z_k)|\right) \frac{m_n(\alpha)}{(C(E))^{2n}}. \tag{10}$$

It is proved in [2] that

$$\lim_{n\to\infty} \frac{m_n(\alpha)}{(C(E))^{2n}} = \mu(\alpha). \tag{11}$$

Using (10), (11) and Lemma 2, we get

$$\limsup_{n\to\infty} \frac{m_n(\sigma)}{(C(E))^{2n}} \leq \mu^\infty(\sigma). \tag{12}$$

Putting $\varphi_n^* = \frac{T_n}{[C(E)\Phi]^n}$, then from (7) and (12) we deduce that the products $|\varphi_n^*(z_k)|^2 \left|\Phi(z_k)^{2n}\right|$ are bounded for all $k \geq 1$, so $\varphi_n^*(z_k) \to 0, n \to \infty, (|\Phi(z_k)| > 1)$.

If we set $\psi_n = \frac{1}{2}[\varphi_n^* + \psi^\infty]$, then we can see that $\psi_n(\infty) = 1$ and $\psi_n(z_k) \to 0, n \to \infty$, therefore (see [2, page 234]),

$$\liminf_{n\to\infty} \|\psi_n\|_{H^2(\Omega,\rho)} \geq \mu^\infty(\sigma). \tag{13}$$

From the parallelogram identity, Lemma 2 and (7), it yields

$$\begin{aligned} \|\varphi_n^* - \psi^\infty\|^2_{H^2(\Omega,\rho)} &= 2\|\varphi_n^*\|^2_{H^2(\Omega,\rho)} + 2\|\psi^\infty\|^2_{H^2(\Omega,\rho)} - 4\left\|\tfrac{1}{2}[\varphi_n^* + \psi^\infty]\right\|^2_{H^2(\Omega,\rho)} \\ &\leq 2\frac{m_n(\sigma)}{(C(E))^{2n}} + 2\mu^\infty(\sigma) - 4\|\psi_n\|^2_{H^2(\Omega,\rho)}. \end{aligned} \tag{14}$$

Using (13) and the fact that the norm is non negative we obtain

$$\liminf_{n\to\infty} \frac{m_n(\sigma)}{(C(E))^{2n}} \geq \mu^\infty(\sigma).$$

This, with (12), proves (i) of the Theorem.

The inequalities (12), (13) and (14) imply

$$\limsup_{n\to\infty} \|\varphi_n^* - \psi^\infty\|^2_{H^2(\Omega,\rho)} \leq 2\mu^\infty(\sigma) + 2\mu^\infty(\sigma) - 4\mu^\infty(\sigma) = 0. \tag{15}$$

(ii) of the Theorem follows immediately from (15).

Now, to prove (iii) of the Theorem, we apply Lemma 1 for the function $\varphi_n^* - \psi^\infty$ which belongs to $H^2(\Omega,\rho)$, then, for all compact $K \subset \Omega$, we have

$$\begin{aligned} \sup_{z\in K}\left|\frac{T_n(z)}{[C(E)\Phi(z)]^n} - \psi^\infty(z)\right| &= \sup_{z\in K}|(\varphi_n^* - \psi^\infty)(z)| \\ &\leq C_K\|\varphi_n^* - \psi^\infty\|^2_{H^2(\Omega,\rho)} \underset{n\to\infty}{\to} 0. \end{aligned}$$

This achieves the proof of Theorem 7. □

REFERENCES

[1] Geronimus, L., On *some extremal problems in* $L_\sigma^{(p)}$ *spaces*, Sb. Math., **31**, pp. 3–23, 1952.

[2] Kaliaguine, V.A., *On asymptotics of* L_p *extremal polynomials on a complex curve* $(0 < p < \infty)$, J. Approx. Theory, **74**, pp. 226–236, 1993.

[3] Khaldi, R. and Benzine, R., *On a generalization of an asymptotic formula of orthogonal polynomials*, Int. J. Appl. Math., **3**, pp. 261–274, 2000.

[4] Khaldi, R. and Benzine, R., *Asymptotics for orthogonal polynomials off the circle,* J. Appl. Math., **1**, pp. 37–53, 2004.

[5] Korovkine, P.P., *On orthogonal polynomials on a closed curve*, Sb. Math., **9**, pp. 469–484, 1941.

[6] Li, X. and Pan, K., *Asymptotic behavior of class of orthogonal polynomials corresponding to measure with discrete part off the unit circle*, J. Approx. Theory, **79**, pp. 54–71, 1994.

[7] Rudin, W., *Real and complex analysis,* McGraw-Hill, New York, 1968.

[8] Stahl, H. and Totik, V., *General orthogonal polynomials,* Cambridge University Press, 1992.

[9] Szego, G., *Orthogonal polynomials*, 4th ed. American Mathematical Society, Colloquium Publications, Vol. **23**, American Mathematical Society, Rhode Island, 1975.

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