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REMARKS ON INTERPOLATION IN CERTAIN LINEAR SPACES (IV)*

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Abstract. In the papers [5], [6], [7] we shall study a way of extending the model of interpolation the real functions, with simple nodes, to the case of the functions defined between linear spaces, specially between linear normed spaces.

In order to keep as many characteristics as possible from the case of the interpolation of real functions, in this paper we present a model of construction of the abstract interpolation polynomials and the divided differences based on the properties of multilinear mappings.

The aim of the present paper is the study of the conduct of the abstract interpolation polynomial, in the case when that the function for interpolation is a abstract polynomial. In the lest part we will construct the abstract interpolation polynomial and the divided differences, in the case in which the spaces X and Y have finite dimensions.

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1. INTRODUCTION

In the papers [2], [3], [4] and [6] we have defined the abstract interpolation polynomial attached to the function $f : E \to Y$, where $E \subseteq X$ and X is a linear space, Y is an algebra with a special structure. At the same time we have presented an example in which our construction is realized, different from the case of the real function's interpolation.

In order to emphasize some properties of these interpolation polynomials we will recall the elements of the construction from the aforementioned paper.

Let us consider the real or complex linear spaces X and Y; we note by $\mathcal{L}(X, Y)$ the set of the linear mappings from X to Y and for $n \geq 2$ we introduce:

$$\mathcal{L}_n(X, Y) = \mathcal{L}(X, \mathcal{L}_{n-1}(X, Y)),$$

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with $\mathcal{L}_1(X,Y) = \mathcal{L}(X,Y)$. We notice that $\mathcal{L}_n(X,Y)$ represents the set of $n-$ linear mappings from $\underbrace{X \times ... \times X}_{n \text{ times}}$ to Y.

Particularly $\mathcal{L}_2(X,Y)$ represents the set of the bilinear mappings from $Y \times Y$ to Y.

Let be θ_X and θ_Y the null elements of the space X and Y respectively. We will note by Θ_n the null element of the space $\mathcal{L}_n(X,Y)$. For $n = 1$ we will use the notation Θ.

Let us consider now $U \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}_2(Y,Y)$. Using these elements we introduce the sequence $(A_n)_{n\in\mathbb{N}}$ where for any $n \in \mathbb{N}$, we have $A_n \in \mathcal{L}_n(X,Y)$ through $A_1(u) = U(u)$ for $u \in X$ and:

$$A_n(u_1, ..., u_n) = B(A_{n-1}(u_1, ..., u_{n-1}), U(u_n)), \tag{1}$$

for $(u_1, ..., u_n) \in X^n$ and $n \in \mathbb{N},\ n \geq 2$.

We now suppose the next properties:

I) the mapping $B \in \mathcal{L}(Y,Y)$ determines in Y a commutative algebra, therefore:
  a) for any $u, v \in Y$ we have $B(u,v) = B(v,u)$;
  b) for any $u, v, w \in Y$ we have $B(B(u,v), w) = B(u, B(v,w))$;

II) there exists $Y_0 \subseteq U(X) \subseteq Y$ so that (Y_0, B) is an abelian group and the mapping $U : U^{-1}(Y_0) \to Y_0$ is a bijective mapping.

Let now be the set $D \subseteq X$ and a sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$. Using $k, n \in \mathbb{N}$ we introduce the non-linear mappings:

$$w_{k,n} : X \to Y;\ w_{k,n}(x) = A_{n+1}(x - x_k, x - x_{k+1}, ..., x - x_{k+n})$$

and for any $x \in X$, the mapping $w'_{k,n}(x) \in \mathcal{L}(X,Y)$ by:

$$w'_{k,n}(x)\, h = \sum_{i=k}^{k+n} A_{n+1}(x - x_k, ..., x - x_{i-1}, x - x_{i+1}, ..., x - x_{k+n}),$$

having evidently for any $i \in \{k, k+1, ..., k+n\}$ the equality:

$$w'_{k,n}(x_i)\, h = A_{n+1}(x_i - x_k, ..., x_i - x_{i-1}, x_i - x_{i+1}, ..., x_i - x_{k+n}, h) \tag{2}$$

and evidently $w'_{k,n}(x_i) \in \mathcal{L}(X,Y)$.

In the papers [2], [6] and [7] we have shown that for certain values $k, n \in \mathbb{N}$ and for any $i, j \in \{k, k+1, ..., k+n\}$ with $i \neq j$ we have:

$$x_i - x_j \in U^{-1}(x_0),$$

then for $i \in \{k, k+1, ..., k+n\}$ the restrictions at $U^{-1}(Y_0)$ of the mappings defined through (2), denoted by:

$$\left[w'_{k,n}(x_i)\right]_0 : U^{-1}(Y_0) \to Y_0$$

as bijective, thus there exist the mappings:

$$\left[w'_{k,n}(x_i)\right]_0^{-1} : Y_0 \to U^{-1}(Y_0).$$

Considering the set $\mathrm{sp}(Y_0)$ representing the linear cover of the set Y_0, the aforementioned mapping will prolong through linearity at $\mathrm{sp}(Y_0)$, obtaining the mapping $\left[w'_{k,n}(x_i)\right]_*^{-1} \in \mathcal{L}(\mathrm{sp}(Y_0), X)$ with the restriction to Y_0 being $\left[w'_{k,n}(x_i)\right]_0^{-1}$ itself.

Let us consider $n \in \mathbb{N}$, $D \subseteq X$ m and the elements $x_0, x_1, ..., x_n \in D$, supposing that they satisfy the aforementioned hypothesis for the spaces X, Y and for the mappings $U \in \mathcal{L}(X, Y)$, $B \in \mathcal{L}_2(Y, Y)$.

After that we suppose that for any $i, j \in \{0, 1, ..., n\}$ with $i \neq j$ we have $x_i - x_j \in U^{-1}(x_0)$ and so the mapping $\left[w'_{0,n}(x_i)\right]_*^{-1} \in \mathcal{L}(\mathrm{sp}(Y_0), X)$ exists.

Let now be a function $f : X \to Y$ supposing that:

$$f(x_1), f(x_2), ..., f(x_n) \in \mathrm{sp}(Y_0).$$

In this way we can define the mapping $\mathbf{L}(x_0, x_1, ..., x_n; f) : X \to Y$ defined by:

$$\begin{aligned} &\mathbf{L}(x_0, x_1, ..., x_n; f)(x) = \\ &= \sum_{i=0}^{n} A_{n+1}\left(x - x_0, ... \underset{i}{|} ..., x - x_n, \left[w'_{0,n}(x_i)\right]_*^{-1} f(x_i)\right), \end{aligned} \tag{3}$$

where:

$$\begin{aligned} &\left(x - x_0, ... \underset{i}{|} ..., x - x_n\right) = \\ &= (x - x_0, ...x - x_{i-1}, x - x_{i+1} ..., x - x_n) \end{aligned}$$

and we can easily show that for any $i \in \{0, 1, ..., n\}$ we have:

$$\mathbf{L}(x_0, x_1, ..., x_n; f)(x_i) = f(x_i).$$

At the same time there exists $D_0 \in Y$ and $D_k \in \mathcal{L}_k(X, Y)$ for any $k = \overline{1, n}$ such that:

$$\mathbf{L}(x_0, x_1, ..., x_n; f)(x) = D_n x^n + D_{n-1} x^{n-1} + ... + D_1 x + D_0,$$

here for any $k = \overline{1, n}$ we denote:

$$D_k x^k = D_k \underbrace{(x, ..., x)}_{k \text{ times}}.$$

Due to the aforementioned reasons the non-linear mapping defined through the equality (3) will be called **(U-B) abstract interpolation polynomial** of the function $f : X \to Y$ corresponding to nodes $x_0, x_1, ..., x_n$.

In the expression (3) of the abstract interpolation polynomial a very important element is the coefficient of the term in x^n, namely the mapping $D_n \in \mathcal{L}_n(X, Y)$, mapping that we will denote by $[x_0, x_1, ..., x_n; f]$, and that will be defined through:

$$[x_0, x_1, ..., x_n; f]\, h_1...h_n = \tag{4}$$
$$= \sum_{i=1}^{n} A_{n+1}\left(h_1, ..., h_n; \left[w'_{0,n}(x_i)\right]_*^{-1} f(x_i)\right).$$

This mapping is called **generalized divided difference of the order** n of the function $f : D \to Y$ on the nodes $x_0, x_1, ..., x_n$.

The main result of the papers [6], [7] on expressed through the following theorem:

THEOREM 1. *With the given facts and with the aforementioned hypotheses:*

a) *we have the equalities:*

$$[x_0, x_1, ..., x_n; f](x_n - x_0) = [x_1, ..., x_n; f] - [x_0, ..., x_{n-1}; f], \tag{5}$$

the equality being taken between the elements of the space $\mathcal{L}_{n-1}(X, Y)$;

b) *the (U-B) abstract interpolation polynomial verifies the recurrence relation:*

$$\mathbf{L}(x_0, x_1, ..., x_n; f)(x) = \mathbf{L}(x_0, x_1, ..., x_{n-1}; f)(x) + \tag{6}$$
$$+ [x_0, x_1, ..., x_n; f](x - x_0)(x - x_1) ... (x - x_{n-1});$$

c) *the (U-B) abstract interpolation polynomial can be written under Newton's form (using the abstract divided differences):*

$$\mathbf{L}(x_0, x_1, ..., x_n; f)(x) = \tag{7}$$
$$= f(x_0) + \sum_{i=1}^{n} [x_0, x_1, ..., x_i; f](x - x_0)(x - x_1) ... (x - x_{i-1});$$

d) *the (U-B) abstract interpolation polynomial verifies a relation of the Aitken-Steffensen's type:*

$$B(\mathbf{L}(x_0, x_1, ..., x_n; f)(x), U(x_n - x_0)) = \tag{8}$$
$$= B(\mathbf{L}(x_1, ..., x_n; f)(x), U(x - x_0)) -$$
$$-B(\mathbf{L}(x_0, ..., x_{n-1}; f)(x), U(x - x_n));$$

e) *for any* $n \in \mathbb{N}$ *and any* $x \in X$ *we have:*

$$f(x) = \mathbf{L}(x_0, x_1, ..., x_n; f)(x) + \tag{9}$$
$$+ [x_0, x_1, ..., x_n, x; f](x - x_0)(x - x_1) ... (x - x_n).$$

For the proof one can consult [2], [6], [7].

The aim of the present paper is the study of the conduct of the abstract interpolation polynomial in he case when that the function $f : D \to Y$ is a abstract polynomial.

In the last part we will construct the abstract interpolation polynomial and the divided differences, in the case in which the spaces X and Y have finite dimensions.

2. SOME PROPERTIES OF THE ABSTRACT INTERPOLATION POLYNOMIAL AND OF THE DIVIDED DIFFERENCES

For these properties it is necessary to introduce the mappings that we will define hereafter.

We consider the sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ and the numbers $k, n \in \mathbb{N}$. Let be afterwards $p \in \mathbb{N},\ p \leq n+1$ and $i_1, i_2, ..., i_p \in \mathbb{N}$ with the verification of the inequalities $k \leq i_1 < i_2 < ... < i_p \leq k+n$.

For $x \in X$ we introduce the mappings $w_{k,n}^{[i_1,i_2,...,i_p]}(x) \in \mathcal{L}(X,Y)$ defined through:

$$w_{k,n}^{[i_1,i_2,...,i_p]}(x)\,h = A_{n-p+2}(t_1, ..., t_{n-p+1}, h) \tag{10}$$

where:

$$\{t_1, ..., t_{n-p+1}\} = \{x - x_k, ..., x - x_{k+n}\} \setminus \{x - x_{i_1}, ..., x - x_{i_p}\}$$

keeping the order succession from the initial set.

We evidently have that for any $s \in \{k, k+1, ..., k+n\} \setminus \{i_1, i_2, ..., i_p\}$,

$$w_{k,n}^{[i_1,i_2,...,i_p]}(x_s) = \Theta,$$

Θ representing the null mapping of the space $\mathcal{L}(X,Y)$, as well as:

$$w_{k,n}^{[i]}(x_i) = w'_{k,n}(x_i),$$

for any $i \in \{k, k+1, ..., k+n\}$.

It is also easy to remark that the restrictions to the set $U^{-1}(Y_0)$ are bijective, so there exist the mappings:

$$\left[w_{k,n}^{[i_1,i_2,...,i_p]}(x)\right]_0^{-1} : Y_0 \to U^{-1}(Y_0),$$

representing the inverses of the mappings defined by (10).

Definition 2. *Let be $U \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}_2(Y,Y)$ such that B determines on Y a commutative algebra and the mappings sequence $(A_n)_{n\in\mathbb{N}}$ is introduced by* (1).

a) *The mapping:*

$$M_n : X \to Y,\ M_n(x) = A_n(\underbrace{x, ..., x}_{n\ times})$$

*is called **(U-B)monomial with the** n **degree**.*

b) *A mapping $P : X \to Y$ for which there exist the elements $a_0, a_1, ...$ $..., a_n \in Y$ such that:*

$$P(x) = a_0 + \sum_{k=1}^{n} B(a_k, M_k(x)),$$

*where for any $k = \overline{1,n}$ the mapping $M_k : X \to Y$ represents the monomial with the k degree, is called **(U-B) polynomial that** n **degree**.*

Let us consider $n, k \in \mathbb{N};\ k \geq n$ and for any $i \in \{0, 1, ..., n\}$ the elements:

$$u_i^{(k)} = \left[w'_{0,n}(x_i)\right]^{-1} M_k(x_i) \in X.$$

We have as follows:

Lemma 3. *For any $k, n \in \mathbb{N},\ k \geq n,\ p \leq n+1$ and $i_1, i_2, ..., i_p \in \mathbb{N}$ with $0 \leq i_1 < i_2 < ... < i_p \leq n$ we have:*

$$\sum_{j=1}^{p} w_{0,n}^{[i_1,...,i_p]}\left(x_{i_j}\right) u_{i_j}^{(k)} = \sum_{\alpha_1+...+\alpha_p=k-p+1} A_{k-p+1} x_{i_1}^{\alpha_1} ... x_{i_p}^{\alpha_p}. \tag{11}$$

In the second member we have been using the notation:

$$A_{k-p+1} x_{i_1}^{\alpha_1} ... x_{i_p}^{\alpha_p} = A_{k-p+1}(\underbrace{x_{i_1}, ..., x_{i_1}}_{\alpha_1 \ times}, ..., \underbrace{x_{i_p}, ..., x_{i_p}}_{\alpha_p \ times}).$$

Proof. We will use the mathematical induction according to p.

Because for any $i \in \{0, 1, ..., n\}$ we have:

$$u_i^{(k)} = \left[w'_{0,n}(x_i)\right]^{-1} M_k(x_i),$$

we deduce that:

$$w'_{0,n}(x_i)\, u_i^{(k)} = M_k(x_i) = A_k(\underbrace{x_i, ..., x_i}_{k \text{ times}}) = A_k x_i^k.$$

therefore the equality (11) is true for $p = 1$.

We suppose that this equality is true for $p = s$ and we follow how it is established for $p = s+1$.

From the hypothesis of the induction we have:

$$\sum_{j=1}^{s-1} w_{0,n}^{[i_1,...,i_{s-1},i_s]}\left(x_{i_j}\right) u_{i_j}^{(k)} + w_{0,n}^{[i_1,...,i_{s-1},i_s]}(x_{i_s})\, u_{i_s}^{(k)} =$$

$$= \sum_{\alpha_1+...+\alpha_s=k-s+1} A_{k-s+1} x_{i_1}^{\alpha_1} ... x_{i_s}^{\alpha_s}.$$

Adding now the index i_{s+1} to $i_1, ..., i_s \in \mathbb{N}$ so the inequalities $0 \leq i_1 < ... < i_s < i_{s+1} \leq n$ are true, form the same hypothesis of the induction, replacing i_s by i_{s+1}, and α_s by α_{s+1} we will have:

$$\sum_{j=1}^{s-1} w_{0,n}^{[i_1,...,i_{s-1},i_{s+1}]}\left(x_{i_j}\right) u_{i_j}^{(k)} + w_{0,n}^{[i_1,...,i_{s-1},i_{s+1}]}(x_{i_{s+1}})\, u_{i_{s+1}}^{(k)} =$$

$$= \sum_{\alpha_1+...+\alpha_s=k-s+1} A_{k-s+1} x_{i_1}^{\alpha_1} ... x_{i_{s-1}}^{\alpha_{s-1}} x_{i_{s+1}}^{\alpha_{s+1}}.$$

From the last two equalities, through substraction, we will have:

$$\begin{aligned}
&\sum_{j=1}^{s-1}\left[w_{0,n}^{[i_1,\ldots,i_{s-1},i_s]}\left(x_{i_j}\right)-w_{0,n}^{[i_1,\ldots,i_{s-1},i_{s+1}]}\left(x_{i_j}\right)\right]u_{i_j}^{(k)}+\\
&+w_{0,n}^{[i_1,\ldots,i_{s-1},i_s]}\left(x_{i_s}\right)u_{i_s}^{(k)}-w_{0,n}^{[i_1,\ldots,i_{s-1},i_{s+1}]}\left(x_{i_{s+1}}\right)u_{i_{s+1}}^{(k)}=\\
&=\sum_{\alpha_1+\ldots+\alpha_s=k-s+1}\left[A_{k-s+1}x_{i_1}^{\alpha_1}\ldots x_{i_s}^{\alpha_s}-A_{k-s+1}x_{i_1}^{\alpha_1}\ldots x_{i_{s+1}}^{\alpha_{s+1}}\right].
\end{aligned}\tag{12}$$

The first member of this equality can be written under the form:

$$B\Big(U\left(x_{i_s}-x_{i_{s+1}}\right),\sum_{j=1}^{s+1}w_{0,n}^{[i_1,i_2,\ldots,i_s,i_{s+1}]}\left(x_{i_j}\right)u_{i_j}^{(k)}\Big).\tag{13}$$

Indeed, because for $j=\overline{1,s-1}$ we have:

$$U\left(x_{i_s}-x_{i_{s+1}}\right)=U\left(x-x_{i_{s+1}}\right)-U\left(x-x_{i_s}\right)$$

we will have as well:

$$\begin{aligned}
&B\left(U\left(x_{i_s}-x_{i_{s+1}}\right),w_{0,n}^{[i_1,\ldots,i_{s+1}]}\left(x_{i_j}\right)u_{i_j}^{(k)}\right)=\\
&=B\left(U\left(x-x_{i_{s+1}}\right),A_{n-k+1}\left(t_1,\ldots,t_{n-s},u_{i_j}^{(k)}\right)\right)-\\
&\quad-B\left(U\left(x-x_{i_s}\right),A_{n-k+1}\left(t_1,\ldots,t_{n-s},u_{i_j}^{(k)}\right)\right)=\\
&=A_{n-s+2}\left(t_1,\ldots,t_{n-s},x_{i_j}-x_{i_{s+1}},u_{i_j}^{(k)}\right)-\\
&\quad-A_{n-s+2}\left(t_1,\ldots,t_{n-s},x_{i_j}-x_{i_s},u_{i_j}^{(k)}\right)=\\
&=\left[w_{0,n}^{[i_1,\ldots,i_{s-1},i_s]}\left(x_{i_j}\right)-w_{0,n}^{[i_1,\ldots,i_{s-1},i_{s+1}]}\left(x_{i_j}\right)\right]u_{i_j}^{(k)}.
\end{aligned}$$

The former reasoning can be used as well in the cases $j=s,\ j=s+1$.

But for $j=s$, we have $w_{0,n}^{[i_1,\ldots,i_{s-1},i_{s+1}]}\left(x_{i_s}\right)u_{i_j}^{(k)}=\theta_Y$, while for $j=s+1$ we have $w_{0,n}^{[i_1,\ldots,i_{s-1},i_s]}\left(x_{i_{s+1}}\right)=\theta_Y$, therefore indeed the first member from (12) can be written under the form (13).

In what concerns the second member of the relation (12) it has been ascertained that for any $\alpha_1,\ldots,\alpha_s$ with $\alpha_1+\ldots+\alpha_s=k-s+1$ we have:

$$\begin{aligned}
&A_{k-s+1}x_{i_1}^{\alpha_1}\ldots x_{i_{s-1}}^{\alpha_{s-1}}x_{i_s}^{\alpha_s}-A_{k-s+1}x_{i_1}^{\alpha_1}\ldots x_{i_{s-1}}^{\alpha_{s-1}}x_{i_{s+1}}^{\alpha_s}=\\
&=\sum_{r=1}^{\alpha_s}\left[A_{k-s+1}x_{i_1}^{\alpha_1}\ldots x_{i_{s-1}}^{\alpha_{s-1}}x_{i_s}^{\alpha_s-(r-1)}x_{i_{s+1}}^{r-1}-A_{k-s+1}x_{i_1}^{\alpha_1}\ldots x_{i_{s-1}}^{\alpha_{s-1}}x_{i_s}^{\alpha_s-r}x_{i_{s+1}}^{r}\right]=\\
&=\sum_{r=1}^{\alpha_s}A_{k-s+1}x_{i_1}^{\alpha_1}\ldots x_{i_{s-1}}^{\alpha_{s-1}}x_{i_s}^{\alpha_s-r}\left(x_{i_s}-x_{i_{s+1}}\right)x_{i_{s+1}}^{r-1}=\\
&=B\Big(U\left(x_{i_s}-x_{i_{s+1}}\right),\sum_{r=1}^{\alpha_s}A_{k-s}x_{i_1}^{\alpha_1}\ldots x_{i_{s-1}}^{\alpha_{s-1}}x_{i_s}^{\alpha_s-r}x_{i_{s+1}}^{r-1}\Big).
\end{aligned}$$

Thus the expression of the second member of the equality (12) will be written under the form:

$$B\left(U\left(x_{i_s}-x_{i_{s+1}}\right), \sum_{\alpha_1+\ldots+\alpha_s=k-s+1}\sum_{r=1}^{\alpha_s}A_{k-s}x_{i_1}^{\alpha_1}\ldots x_{i_{s-1}}^{\alpha_{s-1}}x_{i_s}^{\alpha_s-r}x_{i_{s+1}}^{r-1}\right). \tag{14}$$

But $x_{i_s}-x_{i_{s+1}}\in U^{-1}\left(Y_0\right)$, therefore $U\left(x_{i_s}-x_{i_{s+1}}\right)\in Y_0$, while $\left(Y_0,B\right)$ is an abelian group, so from (12), (13) and (14) we deduce that:

$$\begin{aligned}&\sum_{j=1}^{s+1}w_{0,n}^{[i_1,i_2,\ldots,i_s,i_{s+1}]}\left(x_{i_j}\right)u_{i_j}^{(k)}=\\&=\sum_{\alpha_1+\ldots+\alpha_s=k-s+1}\sum_{r=1}^{\alpha_s}A_{k-s}x_{i_1}^{\alpha_1}\ldots x_{i_{s-1}}^{\alpha_{s-1}}x_{i_s}^{\alpha_s-r}x_{i_{s+1}}^{r-1}.\end{aligned} \tag{15}$$

We introduce the new indexes $\beta_1,\beta_2,\ldots,\beta_s,\beta_{s+1}$ through:

$$\beta_1=\alpha_1,\ \ldots,\beta_{s-1}=\alpha_{s-1},\ \beta_s=\alpha_{s-r},\ \beta_{s+1}=\alpha_{r-1},$$

and evidently:

$$\begin{aligned}&\left\{\left(\beta_1,\ldots,\beta_s,\beta_{s+1}\right)\in\mathbb{N}^{s+1}\big/\,\beta_1+\ldots+\beta_s+\beta_{s+1}=k-s\right\}=\\&=\left\{\left(\alpha_1,\ldots,\alpha_s,r\right)\in\mathbb{N}^{s+1}\big/\,\alpha_1+\ldots+\alpha_s=k-s+1,\ 1\le r\le\alpha_s\right\}.\end{aligned}$$

Therefore the relation (15) will be written under the form:

$$\sum_{j=1}^{s+1}w_{0,n}^{[i_1,\ldots,i_{s+1}]}\left(x_{i_j}\right)u_{i_j}^{(k)}=\sum_{\beta_1+\ldots+\beta_s+\beta_{s+1}=k-s}A_{k-s}x_{i_1}^{\beta_1}\ldots x_{i_s}^{\beta_s}x_{i_{s+1}}^{\beta_{s+1}}, \tag{16}$$

which indicates that the equality (11) is true for $p=s+1$.

Therefore according to the principle of the mathematical induction this equality is true for any $p\in\mathbb{N}$.

The lemma is thus proven. □

Remark 4. On the track of the proof we have seen that for $p=1$, denoting $i_1=i$, the only value of j is $1,$ in the second member the only possibility is $\alpha_1=k$, therefore the equality (11) becomes:

$$\left[w_{0,n}^{[i]}\left(x_i\right)\right]u_i^{(k)}=A_kx_i^k.$$

Let us consider now the case $p=n+1$.

Because $0\le i_1<i_2<\ldots<i_{n+1}\le n$ the only possibility is $i_j=j-1$ for any $j=\overline{1,n+1}$, therefore the sum of the first member is:

$$\sum_{j=1}^{n+1}w_{0,n}^{[0,1,\ldots,n]}\left(x_{j-1}\right)u_{j-1}^{(k)}.$$

Evidently however $w_{0,n}^{[0,1,\ldots,n]}(x_{j-1})\,h = A_1(h) = U(h)$, therefore the equality (11) becomes:

$$\sum_{j=1}^{n} U\left(u_j^{(k)}\right) = \sum_{\alpha_0+\alpha_1+\ldots+\alpha_n=k-n} A_{k-n} x_0^{\alpha_0} x_1^{\alpha_1} \ldots x_n^{\alpha_n}, \tag{17}$$

for the summing indexes we have adapted this notation for symmetry reasons. □

We have the following:

THEOREM 5. *With the hypotheses and with the above mentioned conditions, we have:*

$$[x_0, x_1, \ldots, x_n; M_k] = \begin{cases} \Theta_n & \text{for } k < n, \\ A_n & \text{for } k = n, \\ \sum\limits_{\alpha_0+\ldots+\alpha_n=k-n} A_k x_0^{\alpha_0} x_1^{\alpha_1} \ldots x_n^{\alpha_n} & \text{for } k > n, \end{cases} \tag{18}$$

here Θ_n is the null mapping of the space $\mathcal{L}_n(X,Y)$. In the case $k > n$ the equality is understood using the elements of the space $\mathcal{L}_n(X,Y)$. More precisely, in this case we have:

$$[x_0, x_1, \ldots, x_n; M_k]\, h_1 \ldots h_n = \sum_{\alpha_0+\ldots+\alpha_n=k-n} A_k x_0^{\alpha_0} x_1^{\alpha_1} \ldots x_n^{\alpha_n} h_1 \ldots h_n,$$

the terms of the sum being the values:

$$A_k(\underbrace{x_0, \ldots, x_0}_{\alpha_0 \text{ times}}, \underbrace{x_1, \ldots, x_1}_{\alpha_1 \text{ times}}, \ldots, \underbrace{x_n, \ldots x_n}_{\alpha_n \text{ times}}, h_1, \ldots, h_n) \in Y.$$

Proof. From the definition of the divided difference we have:

$$[x_0, x_1, \ldots, x_n; M_k]\, h_1 \ldots h_n = A_n\left(h_1, \ldots, h_n, \sum_{i=0}^{n} \left[w_{0,n}'(x_i)\right]_*^{-1} M_k(x_i)\right).$$

Let us consider first the case $k \geq n$.
Because for any $i = \overline{0,n}$ we have:

$$u_i^{(k)} = \left[w_{0,n}'(x_i)\right]_*^{-1} M_k(x_i)$$

so:

$$[x_0, x_1, ..., x_n; M_k]\, h_1...h_n = A_n\left(h_1, ..., h_n, \sum_{i=0}^{n} u_i^{(k)}\right) =$$

$$= B\left(U\left(\sum_{i=0}^{n} u_i^{(k)}\right), A_n(h_1, ..., h_n)\right) =$$

$$= B\left(\sum_{\alpha_0+\alpha_1+...+\alpha_n=k-n} A_{k-n} x_0^{\alpha_0} x_1^{\alpha_1} ... x_n^{\alpha_n}, A_n(h_1, ..., h_n)\right) =$$

$$= \sum_{\alpha_0+\alpha_1+...+\alpha_n=k-n} B\left(A_{k-n} x_0^{\alpha_0} x_1^{\alpha_1} ... x_n^{\alpha_n}, A_n(h_1, ..., h_n)\right) =$$

$$= \sum_{\alpha_0+\alpha_1+...+\alpha_n=k-n} A_k x_0^{\alpha_0} x_1^{\alpha_1} ... x_n^{\alpha_n} h_1...h_n.$$

Because $h_1, ..., h_n \in X$ are arbitrary, we deduce that:

$$[x_0, x_1, ..., x_n; M_k] = \sum_{\alpha_0+\alpha_1+...+\alpha_n=k-n} A_k x_0^{\alpha_0} x_1^{\alpha_1} ... x_n^{\alpha_n}.$$

In the special case $k = n$, the only possibility for the choice of the summing indexes is $\alpha_0 = ... = \alpha_n = 0$, therefore:

$$[x_0, x_1, ..., x_n; M_n] = A_n.$$

Let us consider now the case $k < n$,

If we note $p = k + 1$ we deduce that $p \in \{1, ..., n\}$. For this p, due to the relation (11), we have:

$$\sum_{j=1}^{p} w_{0,n}^{[i_1,...,i_p]}\left(x_{i_j}\right) u_{i_j}^{(k)} = K \in Y, \tag{19}$$

for any $i_1, ..., i_p \in \mathbb{N}$, with the verification of the inequalities:

$$0 \leq i_1 < ...... < i_p \leq n\ .$$

We are in the framework of the relation (11) if we consider $\mathcal{L}_0(X, Y) = Y$. Therefore $A_0 \in \mathcal{L}_0(X, Y)$, so $A_0 = K \in Y$.

If we introduce a new index i_{p+1} with $0 \leq i_1 < ... < i_p < i_{p+1} \leq n$ the relation (19) will be true as well in the case when the indexes are changed in $i_1, ..., i_{p-1}, i_{p+1}$. We have therefore the relation:

$$\sum_{j=1}^{p-1} w_{0,n}^{[i_1,...,i_{p-1},i_p]}\left(x_{i_j}\right) u_{i_j}^{(k)} + w_{0,n}^{[i_1,...,i_{p-1},i_p]}\left(x_{i_p}\right) u_{i_p}^{(k)} =$$

$$= \sum_{j=1}^{p-1} w_{0,n}^{[i_1,...,i_{p-1},i_{p+1}]}\left(x_{i_j}\right) u_{i_j}^{(k)} + w_{0,n}^{[i_1,...,i_{p-1},i_{p+1}]}\left(x_{i_{p+1}}\right) u_{i_{p+1}}^{(k)} = K,$$

from which:

$$\sum_{j=1}^{p-1}\left[w_{0,n}^{[i_1,\dots,i_{p-1},i_p]}\left(x_{i_j}\right)-w_{0,n}^{[i_1,\dots,i_{p-1},i_{p+1}]}\left(x_{i_j}\right)\right]u_{i_j}^{(k)}+$$

$$+\,w_{0,n}^{[i_1,\dots,i_{p-1},i_p]}\left(x_{i_p}\right)u_{i_p}^{(k)}-w_{0,n}^{[i_1,\dots,i_{p-1},i_{p+1}]}\left(x_{i_{p+1}}\right)u_{i_{p+1}}^{(k)}=\theta_Y,$$

so:

$$B\left(U\left(x_{i_p}-x_{i_{p+1}}\right),\sum_{j=1}^{p+1}w_{0,n}^{[i_1,i_2,\dots i_{p+1}]}\left(x_{i_j}\right)u_{i_j}^{(k)}\right)=\theta_Y.$$

From $x_{i_p}-x_{i_{p+1}}\in U^{-1}\left(Y_0\right)$ we have $U\left(x_{i_p}-x_{i_{p+1}}\right)\in Y_0$ and similarly with the proof of the Lemma 3, we deduce that:

$$\sum_{j=1}^{p+1}w_{0,n}^{[i_1,i_2,\dots i_{p+1}]}\left(x_{i_j}\right)u_{i_j}^{(k)}=\theta_Y.$$

In the special case of $q=n+1$ we have $0\le i_1<i_2<\dots<i_{n+1}\le n$ we obtain $i_j=j-1$ for any $j=\overline{1,n+1}$, thus previous equality will be written:

$$\sum_{j=0}^{n}w_{0,n}^{[0,1,\dots,n]}u_j^{(k)}=\theta_Y,$$

so:

$$\sum_{j=0}^{n}U\left(u_j^{(k)}\right)=\theta_Y.$$

Because of the linearity of the mapping U, we have from here:

$$\sum_{j=0}^{n}u_j^{(k)}=\theta_X.$$

In this way for any $h_1,\dots,h_n\in X$ we have:

$$[x_0,x_1,\dots,x_n;M_k]\,h_1\dots h_n=A_{n+1}\left(h_1,\dots,h_n;\sum_{j=0}^{n}u_j^{(k)}\right)=$$

$$=A_{n+1}\left(h_1,\dots,h_n;\theta_X\right)=\theta_Y,$$

so

$$[x_0,x_1,\dots,x_n;M_k]=\Theta_n.$$

The theorem is proven. □

We establish now:

THEOREM 6. *Let us consider the previously introduced elements, a set* $D\subseteq X$, *the points* $x_0,x_1,\dots,x_n\in D$, *the function* $f:D\to Y$ *such that* $f\left(x_0\right),f\left(x_1\right),\dots,f\left(x_n\right)\in\operatorname{sp}\left(Y_0\right)$. *We consider* $a\in\operatorname{sp}\left(Y_0\right)$ *and the mapping* $g:D\to Y$, $g\left(x\right)=B\left(a,f\left(x\right)\right)$.

We have the relation:

$$[x_0,x_1,\dots,x_n;g]\,h_1\dots h_n=B\left(a,[x_0,x_1,\dots,x_n;f]\,h_1\dots h_n\right).\tag{20}$$

Proof. From the definition of the divided difference, it results that:

$$[x_0, x_1, ..., x_n; g]\, h_1...h_n = \sum_{i=0}^{n} A_{n+1}\left(h_1, ..., h_n, \left[w'_{0,n}(x_i)\right]^{-1} g(x_i)\right). \tag{21}$$

For any $i \in \{0, 1, ..., n\}$ we have evidently:

$$\left[w'_{0,n}(x_i)\right]^{-1} g(x_i) = \left[w'_{0,n}(x_i)\right]^{-1} B(a, g(x_i)). \tag{22}$$

For any $i, j = \overline{0, n};\ i \neq j$ we have $x_i - x_j \in U^{-1}(Y_0)$; we deduce that for:

$$q = A_n(x_i - x_1, ..., x_i - x_{i-1}, x_i - x_{i+1}, ..., x_i - x_{n+1}) \in Y_0$$

there exists $q' \in Y_0$ such that $B(q, q') = u_0$ (u_0 being the neutral element of the group (Y_0, B)). As well for any $t \in Y_0$ we have $B(t, u_0) = t$, that is:

$$B(t, B(q, q')) = t \Leftrightarrow B(B(q, q'), t) = t \Leftrightarrow B(q, B(q', t)) = t,$$

namely:

$$B(A_n(x_i - x_1, ..., x_i - x_{i-1}, x_i - x_{i+1}, ..., x_i - x_{n+1}), B(q', t)) = t,$$

or:

$$A_{n+1}\left(x_i - x_1, ..., x_i - x_{i-1}, x_i - x_{i+1}, ..., x_i - x_{n+1}, U^{-1}B(q', t)\right) = t \Leftrightarrow$$

$$\Leftrightarrow \left[w'_{0,n}(x_i)\right] U^{-1}B(q', t) = t \Leftrightarrow B(q', t) = U\left[w'_{0,n}(x_i)\right]^{-1} t$$

From this relation we notice that for $b, z \in Y_0$ we have:

$$U\left[w'_{0,n}(x_i)\right]^{-1} B(b, z) = B\left(\left[w'_{0,n}(x_i)\right]^{-1} z, b\right). \tag{23}$$

Indeed, we have:

$$U\left[w'_{0,n}(x_i)\right]^{-1} B(b, z) = B(q', (b, z)) = B((q', z), b) =$$
$$= B\left(U\left[w'_{0,n}(x_i)\right]^{-1} z, b\right),$$

therefore the relation (23) is true.

The relation (23) will be extended as well to the case when the elements $b, z \in Y_0$ are replaced respectively by $a, y \in \mathrm{sp}(Y_0)$.

Indeed, if $a, y \in \mathrm{sp}(Y_0)$ then there exists:

$$p, q \in \mathbb{N};\ \alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q \in K;\ b_1, ..., b_p; z_1, ..., z_q \in Y_0$$

such that:

$$a = \sum_{k=1}^{p} \alpha_k b_k, \quad y = \sum_{j=1}^{q} \beta_j z_j,$$

so because of the linearity of the mappings U, $\left[w'_{0,n}(x_i)\right]^{-1}$ and B we have:

$$U\left[w'_{0,n}(x_i)\right]^{-1}B(a,y)=\sum_{k=1}^{p}\sum_{j=1}^{q}\alpha_k\beta_j U\left[w'_{0,n}(x_i)\right]^{-1}B(b_k,z_j)=$$

$$=\sum_{k=1}^{p}\sum_{j=1}^{q}\alpha_k\beta_j B\left(U\left[w'_{0,n}(x_i)\right]^{-1}z_j,b_k\right)=$$

$$=B\left(U\left[w'_{0,n}(x_i)\right]^{-1}\sum_{j=1}^{q}z_j,\sum_{k=1}^{p}b_k\right)=B\left(U\left[w'_{0,n}(x_i)\right]^{-1}y,a\right).$$

Now, for $i\in\{0,1,...,n\}$ we choose $y=f(x_i)$ and we have:

$$U\left[w'_{0,n}(x_i)\right]^{-1}B(a,f(x_i))=B\left(U\left[w'_{0,n}(x_i)\right]^{-1}f(x_i),a\right). \tag{24}$$

From the relations (22) and (24) we obtain for any $i\in\{0,1,...,n\}$ the equalities:

$$A_{n+1}\left(h_1,...,h_n,\left[w'_{0,n}(x_i)\right]^{-1}g(x_i)\right)=$$

$$=B\left(A_n(h_1,...,h_n),U\left[w'_{0,n}(x_i)\right]^{-1}B(a,f(x_i))\right)$$

$$=B\left(A_n(h_1,...,h_n),B\left(U\left[w'_{0,n}(x_i)\right]^{-1}f(x_i),a\right)\right)$$

$$=A_{n+2}\left(U_*^{-1}(a),h_1,...,h_n,\left[w'_{0,n}(x_i)\right]^{-1}f(x_i)\right)$$

$$=B\left(a,A_{n+1}\left(h_1,...,h_n,\left[w'_{0,n}(x_i)\right]^{-1}f(x_i)\right)\right).$$

In this relation U_*^{-1} is the prolongation trough linearity of the mapping U^{-1} to $\mathrm{sp}(Y_0)$.

On account of the relation (22) we will have:

$$[x_0,x_1,...,x_n;g]\,h_1...h_n=B\left(a,\sum_{i=0}^{n}A_{n+1}\left(h_1,...,h_n,\left[w'_{0,n}(x_i)\right]^{-1}f(x_i)\right)\right)=$$

$$=Ba,[x_0,x_1,...,x_n;f]\,h_1...h_n,$$

the theorem being in this way proven. □

We have now:

Corollary 7. *If for $k\in\mathbb{N}$, $M_k:X\to Y$ is a (U-B) monomial of the k degree and we consider the mapping:*

$$g:X\to Y,\ g(x)=B(a,M_k(x))$$

with $a \in \operatorname{sp}(Y_0)$ *and supposing that all the hypotheses of the previous theorems are fulfilled, then we have the relation:*

$$[x_0, x_1, ...x_n; g]\, h_1...h_n =$$

(25)
$$= \begin{cases} \theta_Y & \text{for } k < n, \\ B\left(a, A_n\left(h_1, ..., h_n\right)\right) & \text{for } k = n, \\ \sum\limits_{\alpha_0+...+\alpha_n=k-n} B\left(a, A_k x_0^{\alpha_0}...x_n^{\alpha_n} h_1...h_n\right) & \text{for } k > n. \end{cases}$$

Proof. The conclusion of this corollary is evident if we use the Theorems 5 and 6. □

We have now:

Theorem 8. *If* $P : X \to Y$ *is a (U-B) polynomial of the* k *degree, where* $k \leq n$ *with the coefficients in* $\operatorname{sp}(Y_0)$ *and supposing that all the hypotheses of the previous theorems are fulfilled, then for any* $x_0, x_1, ..., x_n \in X$ *we have:*

(26)
$$P = \mathbf{L}\left(x_0, x_1, ..., x_n; P\right).$$

Proof. The Theorem 1-d) indicates that for any $x \in X$ we have:

$$P(x) = \mathbf{L}\left(x_0, x_1, ..., x_n; P\right)(x) + [x_0, x_1, ..., x_n, x; P]\,(x - x_0)\,...\,(x - x_n).$$

For $i \in \{0, 1, ..., n\}$, if we introduce $g_i : X \to Y$ with $g_0(x) = a_0$ and $g_i(x) = B(a_i, M_i(x))$ for $i \geq 1$, we have:

$$P(x) = a_0 + \sum_{k=1}^{n} B\left(a_k, M_k(x)\right) = \sum_{k=0}^{n} g_k(x),$$

in this way:

$$[x_0, x_1, ..., x_n, x; P] = \sum_{k=0}^{n} [x_0, x_1, ..., x_n, x; g_k].$$

In the divided differences from the second member in the expression of the mappings g_k there appear monomials having a degree at least two units smaller than the number of the nodes, so for any $k = \overline{0, n}$ we have:

$$[x_0, x_1, ..., x_n, x; g_k] = 0,$$

therefore:

$$[x_0, x_1, ..., x_n, x; P] = \Theta_{n+1},$$

and the theorem is proven. □

Consequently from this result we have:

Theorem 9. *If all hypotheses of the Theorem* 1 *are fulfilled for any* $x \in X$ *that verifies the conditions* $x - x_i \in U^{-1}(Y_0)$ *for any* $i = \overline{0, n}$, *we have:*

$$\mathbf{L}\left(x_0, x_1, ..., x_n; f\right)(x) = \left(\sum_{i=0}^{n} d_i^{-1}\right)^{-1} \sum_{i=0}^{n} d_i^{-1} f\left(x_i\right),$$

where for any $i=\overline{0,n}$, *the mappings* $d_i : U_*^{-1}(\mathrm{sp}(Y_0)) \to Y$ *are defined through:*

$$d_i(h) = A_{n+2}(x_i - x_0, ..., x_i - x_{i-1}, x_i - x_{i+1}, ..., x - x_n, h)$$

for any $h \in U_*^{-1}(\mathrm{sp}(Y_0))$.

Proof. In the papers [6], [7] we have proven that for any $t \in Y_0$ and for any $i,j \in \{0,1,...,n-1\}$ with $i \neq j$ we have the equality:

$$A_2\left(x_i - x_j, \left[w'_{0,n}(x_i)\right]_0^{-1} t\right) = U\left(\left[w_{0,n}^{[i,j]}(x_i)\right]_0^{-1} t\right). \tag{27}$$

If for a fixed $x \notin \{x_0, x_1, ..., x_n\}$ we introduce the mapping:

$$W : X \to Y; \;\; W(t) = A_{n+2}(t - x_0, ..., t - x_n, t - x)$$

and we deduce that for any $h \in U^{-1}(\mathrm{sp}(Y_0))$ we have:

$$W'(x)h = A_{n+2}(x - x_0, ..., x - x_n; h),$$

$$\begin{aligned} &W'(x_i)h = \\ &= A_{n+2}(x_i - x_0, ..., x_i - x_{i-1}, x_i - x_{i+1}, ..., x_i - x_n x_i - x; h) = -d_i(h) \end{aligned}$$

and:

$$\begin{aligned} &W^{[i,n+1]}(x_i)h = w'_{0,n}(x_i h) = \\ &= A_{n+1}(x_i - x_0, ..., x_i - x_{i-1}, x_i - x_{i+1}, ..., x_i - x_n; h). \end{aligned}$$

From the relation (27), replacing n with $n+1$ and considering $x_{n+1} = x$, for any $y \in Y_0$ we have:

$$\begin{aligned} U\left[w'_{0,n}(x_i)\right]_0^{-1} y &= U\left(\left[W^{[i,n+1]}(x_i)\right]^{-1} y\right) = A_2\left(x_i - x, \left[W'(x_i)\right]^{-1} y\right) = \\ &= A_2\left(x - x_i, d_i^{-1}(y)\right). \end{aligned}$$

So for same $y \in Y_0$ we have:

$$\begin{aligned} &A_{n+1}\left(x - x_0, ..., x - x_{i-1}, x - x_{i+1}, ..., x - x_n, \left[w'_{0,n}(x_i)\right]_0^{-1} y\right) = \\ &= B\left(A_n(x - x_0, ..., x - x_{i-1}, x - x_{i+1}, ..., x - x_n), U\left[w'_{0,n}(x_i)\right]_0^{-1} y\right) \\ &= B\left(A_n(x - x_0, ..., x - x_{i-1}, x - x_{i+1}, ..., x - x_n), A_2\left(x - x_i, d_i^{-1}(y)\right)\right) \\ &= A_{n+2}\left(x - x_0, ..., x - x_{i-1}, x - x_{i+1}, ..., x - x_n, d_i^{-1}(y)\right) \\ &= B\left(w_{0,n}(x), \left(U d_i^{-1}\right)(y)\right). \end{aligned}$$

Evidently the relation will be extended through linearity to $\mathrm{sp}\,(Y_0)$, so for any $y \in \mathrm{sp}\,(Y_0)$ we have:

$$A_{n+1}\left(x-x_0,...,x-x_{i-1},x-x_{i+1},...,x-x_n,\left[w'_{0,n}\left(x_i\right)\right]_*^{-1}y\right)= \tag{28}$$
$$=B\left(w_{0,n}\left(x\right),\left(Ud_i^{-1}\right)\left(y\right)\right).$$

Let there be any $h \in U^{-1}(Y_0)$. We consider the mapping $\varphi_h : X \to Y$ defined through $\varphi_h(x) = U(h)$ for any $x \in X$.

Because $\varphi_h : X \to Y$ is a constant function, namely a (U-B) polynomial of the degree 0, using the Theorem 8, we will have:

$$\mathbf{L}\left(x_0,x_1,...,x_n;\varphi_h\right)(x)=\varphi_h(x)=U(h).$$

From this last relation together with the equality (28), we have:

$$U(h)=\sum_{i=0}^{n}B\left(w_{0,n}(x),\left(Ud_i^{-1}U\right)(h)\right)= \tag{29}$$
$$=B\left(w_{0,n}(x),\sum_{i=0}^{n}\left(Ud_i^{-1}U\right)(h)\right)$$

Because (Y_0, B) is an abelian group, we deduce that for any $a \in Y_0$, there exists a $\overline{y} \in Y_0$ so that if $t \in Y_0$ we have:

$$B\left(a,B\left(\overline{y},t\right)\right)=t. \tag{30}$$

Let us fix an $i \in \{0,1,...,n\}$.

From the initial conditions imposed to the nodes to which we add the condition $x - x_i \in U_{-1}(Y_0)$, we deduce that the choice:

$$a=A_{n+1}\left(x_i-x_0,...,x_i-x_{i-1},x-x_i,x_i-x_{i+1},...,x_i-x_n\right)\in Y_0$$

is possible and so the relation (30) implicates:

$$B\left(A_{n+1}\left(x_i-x_0,...,x_i-x_{i-1},x-x_i,x_i-x_{i+1},...,x_i-x_n\right),B\left(\overline{y},t\right)\right)=t,$$

or

$$d_iU^{-1}B\left(\overline{y},t\right)=t,$$

namely:

$$B\left(\overline{y},t\right)=Ud_i^{-1}t$$

for any $t \in Y_0$.

From this we deduce that, for any $u, v \in Y_0$, replacing in the previous equality $t = B(u, v)$ we obtain:

$$Ud_i^{-1}B(u,v)=B\left(\overline{y},(u,v)\right)=B\left(B\left(\overline{y},u\right),v\right)=B\left(Ud_i^{-1}u,v\right).$$

We consider now in the previous relation $u = U(h)$, $v = w_{0,n}(x)$ and we obtain:

$$Ud_i^{-1}B\left(w_{0,n}(x),U(h)\right)=B\left(w_{0,n}(x),\left(Ud_i^{-1}U\right)(h)\right).$$

We use now the relation (2) and we obtain:

$$U(h) = B\Big(w_{0,n}(x), \sum_{i=0}^{n}\left(Ud_i^{-1}U\right)(h)\Big) = \sum_{i=0}^{n} Ud_i^{-1}B\left(w_{0,n}(x), U(h)\right)$$

and because of the injectivity of the mapping U, we will have:

$$h = \left(\sum_{i=0}^{n} d_i^{-1}\right) B\left(w_{0,n}(x), U(h)\right)$$

or

$$B\left(w_{0,n}(x), U(h)\right) = \left(\sum_{i=0}^{n} d_i^{-1}\right)(h).$$

In this way:

$$\mathbf{L}(x_1, x_2, ..., x_n; f)(x) =$$

$$= \sum_{i=0}^{n} A_{n+1}\left(x - x_0, ..., x - x_{i-1}, x - x_{i+1}, ..., x - x_n, \left[w'_{0,n}(x_i)\right]_*^{-1} f(x_i)\right)$$

$$= B\left(w_{0,n}(x), U\left(\sum_{i=0}^{n} d_i^{-1} f(x_i)\right)\right) = \left(\sum_{i=0}^{n} d_i^{-1}\right)^{-1} \sum_{i=0}^{n} d_i^{-1} f(x_i).$$

Obviously, the previous reasoning needs $f(x_i) \in Y_0,\ i = \overline{0,n}$, but through linearity the result extends to the case $f(x_i) \in \operatorname{sp}(Y_0),\ i = \overline{0,n}$; as well.

The theorem is thus proven. □

3. THE EFFECTIVE CONSTRUCTION OF THE ABSTRACT INTERPOLATION POLYNOMIAL AND OF THE DIVIDED DIFFERENCES IN THE CASE OF A FUNCTION BETWEEN TWO SPACES WITH FINITE DIMENSIONS

In the paper [7] we have shown that the construction and the properties of the abstract interpolation polynomial, as well as those of the divided differences are conditioned as follows:

a) it is necessary for a set $Y_0 \subseteq Y$ and a mapping $B \in \mathcal{L}_2(Y, Y)$ to exist such that (Y_0, B) to be an abelian group and $sp(Y_0) = Y$;
b) there exists a linear mapping $U : X \to Y$ such that $Y_0 \subseteq U(X)$;
c) the points $x_0, x_1, ..., x_n \in X$ verify the conditions $x_i - x_j \in U^{-1}(Y_0)$ for any $i, j \in \{0, 1, ..., n\}$ with $i \neq j$.

The condition b) implicates the fact that $\dim X \leq \dim Y$ (the inequality relation between transfinite numbers).

We will suppose now that $\dim X, \dim Y \in \mathbb{N}$. In this case if $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ we have $X = \mathbb{K}^p$ and $Y = \mathbb{K}^q$ where $p, q \in \mathbb{N} \setminus \{0\}$ and $p \leq q$.

We define:

$$Y_1 = \left\{ y = \left(y^i\right)_{i=\overline{1,q}} \Big/ \ y^i \in \mathbb{K};\ i = \overline{1,q};\ y^{p+1} = ... = y^q = 1 \right\},$$

and:

$$U:\mathbb{K}^{p}\to\mathbb{K}^{q};\;\; U(x)=y=\left(y^{1},...,y^{q}\right),$$

for $x=\left(x^{1},...,x^{q}\right)$, where:

$$y^{i}=\begin{cases} x^{i} & \text{for} \quad i\in\{1,2,...,p\}, \\ 1 & \text{for} \quad i\in\{p+1,...,q\}\end{cases}$$

and we have $U(X)=Y^{1}\subseteq Y$.

We mention that for the co-ordinates of the points form the spaces $\mathbb{K}^{p}$ and $\mathbb{K}^{q}$ respectively, we use superior indexes.

We now define:

$$Y_{0}=\left\{y=\left(y^{i}\right)_{i=\overline{1,q}}\Big/\, y^{i}\in\mathbb{K},\; y^{i}\neq 0,\; i=\overline{1,p};\; y^{p+1}=...=y^{q}=1\right\}.$$

Evidently $Y_{0}\subseteq Y_{1}$ and from the definition of U we deduce that:

$$U^{-1}(Y_{0})=\left\{x=\left(x^{i}\right)_{i=\overline{1,p}}\Big/\, x^{i}\in\mathbb{K},\; x^{i}\neq 0,\; i=\overline{1,p}\right\}.$$

We now define the bilinear mapping $B\in\mathcal{L}_{2}\left(\mathbb{K}^{q},\mathbb{K}^{q}\right)$ that for $u,v\in\mathbb{K}^{q},\; u=\left(u^{i}\right)_{i=\overline{1,q}},\; v=\left(v^{i}\right)_{i=\overline{1,q}}$ is defined through:

$$B(u,v)=\left(u^{i}v^{i}\right)_{i=\overline{1,q}},$$

the co-ordinates of the vector $B(u,v)$ from $\mathbb{K}^{q}$ will be obtained through the products of the co-ordinates with the same rank from the vectors u and v.

It is easy to verify that (Y_{0},B) is an abelian group. In this respect the null element of this group is $u_{0}=(\underbrace{1,1,...,1}_{q\text{ times}})\in\mathbb{K}^{q}$. Also for $u=\left(u^{i}\right)_{i=\overline{1,q}}$, where $u^{i}\neq 0$ for any $i=\overline{1,q}$, we have $u^{i}\neq 0$, therefore if we chose $u'=\left(\frac{1}{u^{i}}\right)_{i=\overline{1,q}}$ this will be the symmetrical element of u from (Y_{0},B).

Because of the condition $\operatorname{sp}(Y_{0})=Y$, we remark that the only case in which one can apply the theory developed in [6] and [7] is $q=p$. in this case we have $U=\mathbf{I}_{p}$ (the identical mapping from $\mathbb{K}^{p}$).

We now have:

Theorem 10. *If the system of points $x_{0},x_{1},...,x_{n}\in\mathbb{K}^{p}$, where for any $k=\overline{0,n}$; $x_{k}=\left(x_{k}^{s}\right)_{s=\overline{1,p}}\in\mathbb{K}^{p}$ are chosen so that for any $i,j\in\{0,1,...,n\}$; $i\neq j$ and for any $s=\overline{1,p}$ we have $x_{i}^{s}\neq x_{j}^{s}$, then there exists the interpolation polynomial in an abstract sense of the function $f=(f_{1},...,f_{p}):\mathbb{K}^{p}\to\mathbb{K}^{p}$ on the nodes $x_{0},x_{1},...,x_{n}$, this polynomial being:*

$$\mathbf{L}(x_{0},x_{1},...,x_{n};f)(x)=$$

$$=\left(\sum_{k=0}^{n}\frac{\left(x^{s}-x_{0}^{s}\right)...\left(x^{s}-x_{k-1}^{s}\right)\left(x^{s}-x_{k+1}^{s}\right)...\left(x^{s}-x_{n}^{s}\right)}{\left(x_{k}^{s}-x_{0}^{s}\right)...\left(x_{k}^{s}-x_{k-1}^{s}\right)\left(x_{k}^{s}-x_{k+1}^{s}\right)...\left(x_{k}^{s}-x_{n}^{s}\right)}y_{k}^{s}\right)_{s=\overline{1,p}}, \tag{31}$$

where:

$$y_k^s = f_s(x_k) = f_s\left(x_k^1, ..., x_k^p\right);\ k = \overline{0,n};\ s = \overline{1,p}$$

and:

$$[x_0, x_1, ..., x_n; f] = \left(\mathcal{D}^s_{j_1,...,j_n}\right)_{s,j_1,...,j_n=\overline{1,p}} \tag{32}$$

where:

$$\mathcal{D}^s_{j_1,...,j_n} = \begin{cases} \sum\limits_{k=0}^{n} \frac{y_k^s}{(x_k^s - x_0^s)...(x_k^s - x_{k-1}^s)(x_k^s - x_{k+1}^s)...(x_k^s - x_n^s)} & \\ & \text{for } j_1 = j_2 = ... = j_n = s, \\ 0 & \text{differently.} \end{cases} \tag{33}$$

If for any $s = \overline{1,p}$ *and* $k = \overline{0,n}$ *we have* $y_k^s \neq 0$ *for the interpolation polynomial* (31) *and the divided differences* (32)–(33) *we have the conclusions of the Theorems* 1, 5, 6, 8 *and* 9*.*

Proof. From the facts presented in [6] and [7], the remarks in the beginning of the present paper and the previous considerations, we deduce that the existence of the abstract interpolation polynomial and of the divided differences is evident, the requirements referring to the set Y_0, the bilinear mapping $B \in \mathcal{L}_2(\mathbb{K}^p, \mathbb{K}^p)$ and $U = \mathbf{I}_p$ being fulfilled.

From the definitions of the aforementioned mappings, we deduce that for any $u_1, ..., u_n \in \mathbb{K}^p$ where for any $k = \overline{1,n};\ u_k = (u_k^s)_{s=\overline{1,p}}$ we have:

$$A_n(u_1, ..., u_n) = (u_1^s ... u_n^s)_{s=\overline{1,p}} \tag{34}$$

From this relation it is obvious that the mapping $w_{0,n} : \mathbb{K}^p \to \mathbb{K}^p$ is given through:

$$\begin{aligned} w_{0,n}(x) &= A_{n+1}(x - x_0, x - x_1, ..., x - x_n) = \\ &= ((x^s - x_0^s)(x^s - x_1^s) ... (x^s - x_n^s))_{s=\overline{1,p}} \end{aligned} \tag{35}$$

and for any $k \in \{0, 1, ..., n\}$ we deduce that $w'_{0,n}(x_k) \in \mathcal{L}(\mathbb{K}^p, \mathbb{K}^p)$, and:

$$\begin{aligned} & w'_{0,n}(x_k)h = \\ & = A_{n+1}(x_k - x_0, ..., x_k - x_{k-1}, x_k - x_{k+1}, ..., x_k - x_n, h) = \\ & = \left((x_k^s - x_0^s) ... \left(x_k^s - x_{k-1}^s\right)\left(x_k^s - x_{k+1}^s\right) ... (x_k^s - x_n^s) h^s\right)_{s=\overline{1,p}}. \end{aligned} \tag{36}$$

From the hypotheses of the theorem we deduce the existence of the mapping $\left[w'_{0,n}(x_k)\right]^{-1} : \mathbb{K}^p \to \mathbb{K}^p$ defined through:

$$\begin{aligned} & \left[w'_{0,n}(x_k)\right]^{-1} t = \\ & = \left(\frac{t^s}{(x_k^s - x_0^s)...(x_k^s - x_{k-1}^s)(x_k^s - x_{k+1}^s)...(x_k^s - x_n^s)}\right)_{s=\overline{1,p}} \end{aligned} \tag{37}$$

and
(38)
$$A_{n+1}\left(x-x_0,...,x-x_{k-1},x-x_{k+1},...,x-x_n,\left[w_{0,n}\left(x_k\right)\right]^{-1}f\left(x_k\right)\right)=$$
$$=\left(\frac{\left(x^s-x_0^s\right)...\left(x^s-x_{k-1}^s\right)\left(x^s-x_{k+1}^s\right)...\left(x^s-x_n^s\right)}{\left(x_k^s-x_0^s\right)...\left(x_k^s-x_{k-1}^s\right)\left(x_k^s-x_{k+1}^s\right)...\left(x_k^s-x_n^s\right)}y_k^s\right)_{s=\overline{1,p}},$$

thus the relation (31) is evidently true.

We will evidently also have:

Remark 11. In the case when $n=1$, we obtain for the divided difference the next form:

(39)
$$[x_0,x_1;f]=\begin{bmatrix}\frac{y_1^1-y_0^1}{x_1^1-x_0^1} & 0 & ... & 0\\ 0 & \frac{y_1^2-y_0^2}{x_1^2-x_0^2} & ... & 0\\ ... & ... & ... & ...\\ 0 & 0 & ... & \frac{y_1^p-y_0^p}{x_1^p-x_0^p}\end{bmatrix}$$

where $y_k^i=f_i\left(x_1^k,...,x_p^k\right)$ and $k\in\{0,1\}\,;\ i=\overline{1,p}$.

In the same time it is known that if the functions $f_1,...,f_p:D\to\mathbb{K}$ with $D\subseteq\mathbb{K}^p$ admit continual partial derivatives, then the mapping:

$$f'\left(x\right)=\left(\frac{\partial f_i}{\partial x_j}\left(x\right)\right)_{i,j=\overline{1,p}}\in\mathcal{L}\left(\mathbb{K}^p,\mathbb{K}^p\right)$$

represents the Fréchet derivative of the mapping $f=\left(f_1,...,f_p\right):D\to\mathbb{K}^p$.

In the case of a real function with a real variable, which can be derived in a point $x\in D$, the limit of the divided difference for the nodes tending to x is the Fréchet derivative of the function f on the point x itself. Which is the situation in the case of a function $f=\left(f_1,...,f_p\right):D\to\mathbb{K}^p$ with $D\subseteq\mathbb{K}^p$.

The answer is in the very definition of the existence of the Fréchet derivative in the point $x\in D$. in this case we have:

$$f\left(x+h\right)-f\left(x\right)=f'\left(x\right)h+\omega_f\left(x,h\right),$$

where:

$$\lim_{h\to\theta_p}\frac{\left\|\omega_f(x,h)\right\|_Y}{\|h\|_X}=0.$$

But as from the general theory we have that:

$$f\left(x+h\right)-f\left(x\right)=\left[x,x+h;f\right]h,$$

the previous result will be written as:

$$\lim_{h\to\theta_p}\frac{\|([x,x+h;f]-f'(x))h\|_Y}{\|h\|_X}=0.$$ □

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