BEST UNIFORM APPROXIMATION OF SEMI-LIPSCHITZ FUNCTIONS BY EXTENSIONS*

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Abstract. In this paper we consider the problem of best uniform approximation of a real valued semi-Lipschitz function \( F \) defined on an asymmetric metric space \((X, d)\), by the elements of the set \( E_d(F|_Y) \) of all extensions of \( F|_Y \) \((Y \subset X)\), preserving the smallest semi-Lipschitz constant. It is proved that, this problem has always at least a solution, if \((X, d)\) is \((d, d)\)-sequentially compact, or of finite diameter.


Keywords. Semi-Lipschitz functions, uniform approximation, extensions of semi-Lipschitz functions.

1. INTRODUCTION

Let \( X \) be a non-empty set. A function \( d : X \times X \to [0, \infty) \) is called a quasi-metric on \( X \) if the following conditions hold:

1) \( d(x, y) = d(y, x) = 0 \) iff \( x = y \),
2) \( d(x, z) \leq d(x, y) + d(y, z) \), for all \( x, y, z \in X \).

The function \( \overline{d} : X \times X \to [0, \infty) \) defined by \( \overline{d}(x, y) = d(y, x) \), for all \( x, y \in X \) is also a quasi-metric on \( X \), called the conjugate quasi-metric of \( d \).

A pair \((X, d)\) where \( X \) is a non-empty set and \( d \) a quasi-metric on \( X \), is called a quasi-metric space.

If \( d \) can take the value \( +\infty \), then it is called a quasi-distance on \( X \).

Each quasi-metric \( d \) on \( X \) induces a topology \( \tau(d) \) which has as a basis the family of balls (forward open balls)

\[
B^+(x, \varepsilon) := \{ y \in X : d(x, y) < \varepsilon \}, \quad x \in X, \quad \varepsilon > 0.
\]

This topology is called the forward topology of \( X \) (\cite{5}, \cite{9}), and is denoted also by \( \tau_+ \).

Observe that the topology \( \tau_+ \) is a \( T_0 \)-topology. If the condition 1) is replaced by 1') \( d(x, y) = 0 \) iff \( x = y \), then the topology \( \tau_+ \) is a \( T_1 \)-topology (see \cite{14}, \cite{15}).

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Analogously, the quasi-metric \( d \) induces the topology \( \tau(d) \) on \( X \), which has as a basis the family of backward open balls \( (\mathcal{B}^-) \)

\[
B^-(x, \varepsilon) := \{ y \in X : d(y, x) < \varepsilon \}, \quad x \in X, \quad \varepsilon > 0
\]

This topology is called the backward topology of \( X \) \([5], [9]\) and is denoted also by \( \tau^- \).

For more information about quasi-metric spaces and their applications see, for example, the papers \([5], [6], [7], [9], [14]\) and the references quoted therein.

Let \((X, d)\) be a quasi-metric space. A sequence \((x_k)_{k \geq 1} \subseteq X\) is called \( d \)-convergent (forward convergent) to \( x_0 \in X \), respectively \( d \)-convergent (backward convergent) to \( x_0 \in X \) iff

\[
\lim_{k \to \infty} d(x_0, x_k) = 0, \quad \text{respectively} \quad \lim_{k \to 0} d(x_k, x_0) = \lim_{k \to \infty} d(x_0, x_k) = 0.
\]

(see \([5], \text{Definition 2.4}\))

A subset \( K \) of \( X \) is called \( d \)-compact (forward compact) if every open cover of \( K \) with respect to the forward topology \( \tau_+ \) has a finite subcover. We say that a subset \( K \) of \( X \) is \( d \)-sequentially compact (forward-sequentially compact) if every sequence in \( K \) has a \( d \)-convergent (forward convergent) subsequence with limit in \( K \) \([5], \text{Definition 4.1}\).

The \( d \)-compact (backward compact) and \( d \)-sequentially compact (backward-sequentially compact) subset of \( X \) are defined in a similar way.

Finally, a subset \( Y \) of \((X, d)\) is called \((d, \overline{d})\)-sequentially compact if every sequence \((y_n)_{n \geq 1} \subseteq Y\) has a subsequence \((y_{n_k})_{k \geq 1}\), \(d\)-convergent to some \( u \in Y \) and \( \overline{d}\)-convergent to some \( v \in Y \). By Lemma 3.1 in \([5]\) it follows that we can take \( u = v \) in the definition of \((d, \overline{d})\)-sequentially compactness, if \((X, d)\) is a \(T_1\) quasi-metric space. A subset \( Y \) of \((X, d)\) is called \( d \)-bounded (forward bounded in \([5]\)) if there exist \( x \in X \) and \( r > 0 \), such that \( Y \subseteq B^+(x, r) \). \( Y \) is called \( d \)-totally bounded if for every \( \varepsilon > 0 \), there exists \( n \in \mathbb{N} \), and the forward balls \( B^+(y_1, \varepsilon), B^+(y_2, \varepsilon), \ldots, B_n(y_n, \varepsilon), y_i \in Y \), \( i = 1, \ldots, n \) such that \( Y \subseteq \bigcup_{i=1}^{n} B^+(y_i, \varepsilon) \).

Similar definitions are given for \( \overline{d}\)-boundedness and \( \overline{d}\)-total boundedness of a subset \( Y \) of \((X, d)\).

### 2. THE CONE OF SEMI-LIPSCHITZ FUNCTIONS

**Definition 1.** \([5]\) Let \( Y \) be a non-empty subset of a quasi-metric space \((X, d)\). A function \( f : Y \to \mathbb{R} \) is called \( d \)-semi-Lipschitz if there exists a number \( L \geq 0 \) (named a \( d \)-semi-Lipschitz constant for \( f \)) such that

\[
f(x) - f(y) \leq Ld(x, y),
\]

for all \( x, y \in Y \).

A function \( f : Y \to \mathbb{R} \), is called \( \leq d \)-increasing if \( f(x) \leq f(y) \), whenever \( d(x, y) = 0 \).
Denote by \( \mathbb{R}^Y \leq_d \) the set of all \( \leq_d \)-increasing functions on \( Y \). This set is a cone in the linear space \( \mathbb{R}^Y \) of real valued functions defined on \( Y \), i.e. for each \( f, g \in \mathbb{R}^Y \) and \( \lambda \geq 0 \) it follows that \( f + g \in \mathbb{R}^Y \) and \( \lambda f \in \mathbb{R}^Y \).

For a \( d \)-semi-Lipschitz function \( f \) on \( Y \), put \( \| f \|_d \) the smallest \( d \)-semi-Lipschitz constant of \( f \) (see also \[10, 15\]).

\[
\| f \|_d = \sup \left\{ \frac{(f(x) - f(y)) + \varepsilon}{d(x, y)} : d(x, y) > 0; \ x, y \in Y \right\}
\]

Then \( \| f \|_d \) is the smallest \( d \)-semi-Lipschitz constant of \( f \) (see also \[10, 15\]).

For a fixed element \( \theta \in Y \) denote

\[
d-\text{SLip}_0 Y := \{ f \in \mathbb{R}^Y ; \| f \|_d < \infty \text{ and } f(\theta) = 0 \},
\]

the set of all \( d \)-semi-Lipschitz real valued functions defined on \( Y \) vanishing at the fixed element \( \theta \in Y \).

Observe that if \( (X, d) \) is a \( T_1 \) quasi-metric space, then every real-valued function on \( X \) is \( \leq_d \)-increasing \[14\].

The set \( d-\text{SLip}_0 Y \) is a cone (a subcone of \( \mathbb{R}^Y \leq_d \)) and the functional \( \| \cdot \|_d : d-\text{SLip}_0 Y \to [0, \infty) \) defined by \( \| f \|_d \) is subadditive and positive homogeneous on \( d-\text{SLip}_0 Y \). Moreover \( \| f \|_d = 0 \iff f = 0 \), and consequently \( \| \cdot \|_d \) is a quasi-norm (asymmetric norm) on the cone \( d-\text{SLip}_0 Y \).

In \[13\] some properties of the “normed cone” \( (d-\text{SLip}_0 Y, \| \cdot \|_d) \) are presented. Similar properties in the case of \( d \)-semi-Lipschitz functions on a quasi-metric space with values in a quasi-normed space (space with asymmetric norm) are discussed in \[10, 17\]. For more information concerning other properties of quasi-metric spaces, see also \[7, 13\].

Now, let \( (X, d) \) be a quasi-metric space and let \( Y \) be a non-empty subset of \( X \). A real valued function \( f \) defined on \( Y \) is called \( \tau_\downarrow \)-lower semi-continuous (\( \tau_\downarrow \)-l.s.c. in short) (respectively \( \tau_\uparrow \)-upper semi-continuous (\( \tau_\uparrow \)-u.s.c.)) at \( x_0 \in Y \), if for every \( \varepsilon > 0 \) there exists \( r > 0 \) such that for every \( x \in B^+(x_0, r) \) (respectively, for every \( x \in B^-(x_0, r) \)), \( f(x) > f(x_0) - \varepsilon \) (respectively \( f(x) < f(x_0) + \varepsilon \)).

**Proposition 2.** Let \( (X, d) \) be a quasi-metric space, \( \theta \in X \) a fixed element, and \( Y \subset X \) with \( \theta \in Y \). Then every \( f \in d-\text{SLip}_0 Y \) is \( \tau_\downarrow \)-u.s.c. and \( \tau_\uparrow \)-l.s.c., and every \( f \in d-\text{SLip}_0 Y \) is \( \tau_\downarrow \)-u.s.c. and \( \tau_\uparrow \)-l.s.c. on \( Y \).

**Proof.** Let \( f \in d-\text{SLip}_0 Y \) such that \( \| f \|_d = 0 \). Then \( f \equiv 0 \) and \( f \) is \( \tau_\downarrow \)-u.s.c. and \( \tau_\uparrow \)-l.s.c. at every \( y \in Y \).

Now, let \( \| f \|_d > 0 \) and \( y_0 \in Y \). The inequality

\[
f(y) - f(y_0) \leq \| f \|_d d(y, y_0), \ y \in Y
\]

implies

\[
f(y) \leq f(y_0) + \| f \|_d d(y, y_0), \ y \in Y.
\]

So that

\[
f(y) < f(y_0) + \varepsilon,
\]
for every \( \varepsilon > 0 \) and every \( y \in B^-(y_0, \frac{\varepsilon}{\|f\|_{\infty}}) \), showing that \( f \) is \( \tau_-\text{-u.s.c} \) at \( y_0 \in Y \).

Similarly, 
\[
\|f\|_{\infty} - f(y) \leq d(y_0, y), \quad y \in Y,
\]
implies 
\[
f(y) \geq f(y_0) - \|f\|_{\infty} d(y_0, y),
\]
so that 
\[
f(y) > f(y_0) - \varepsilon,
\]
for every \( y \in B^+(y_0, \frac{\varepsilon}{\|f\|_{\infty}}) \), showing that \( f \) is \( \tau_+\text{-l.s.c.} \) in \( y_0 \in Y \).

Similarly one prove that every \( f \in \overline{d}\text{-SLip}_0Y \) is \( \tau_-\text{-u.s.c.} \) and \( \tau_-\text{-l.s.c.} \) on \( Y \).

Observe that if \( f \) is in \( d\text{-SLip}_0Y \), then \( -f \in \overline{d}\text{-SLip}_0Y \), and \( -f \) is \( \tau_+\text{-u.s.c.} \) and \( \tau_-\text{-l.s.c.} \) on \( Y \), i.e. if \( y_0 \in Y \) then

- \( \forall \varepsilon > 0, \exists r > 0 \) such that \( -(f)(y) < -(f)(y) + \varepsilon \), for all \( y \in B^+(y_0, r) \), and respectively
- \( \forall \varepsilon > 0, \exists r > 0 \) such that \( -(f)(y) > -(f)(y_0) - \varepsilon \), for all \( y \in B^-(y_0, r) \).

**Proposition 3.** Let \((X, d)\) be a quasi-metric space, \( \theta \in X \) a fixed element, and \( Y \subseteq X \), with \( \theta \in Y \).

(a) If \( Y \) is \( \overline{d}\text{-sequentially compact} \), then each \( f \in d\text{-SLip}_0Y \) attains its maximum value on \( Y \);

(b) If \( Y \) is \( d\text{-sequentially compact} \), then each \( f \in d\text{-SLip}_0Y \) attains its minimum value on \( Y \).

**Proof.** (a) Let \( Y \) be \( \overline{d}\text{-sequentially compact} \) and \( M := \sup f(Y) \), where \( M \in \mathbb{R} \cup \{+\infty\} \). Then there exists a sequence \((y_n)_{n \geq 1} \) in \( Y \) such that \( \lim_{n \to \infty} f(y_n) = M \). Because \( Y \) is \( \overline{d}\text{-sequentially compact} \), there exists \( y_0 \in Y \) and a subsequence \((y_{n_k})_{k \geq 1} \) of \((y_n)_{n \geq 1} \) such that \( \lim_{n \to \infty} d(y_{n_k}, y_0) = 0 \). By the \( \tau_-\text{-u.s.c.} \) of \( f \) at \( y_0 \) it follows:

\[
M = \lim_{k \to \infty} f(y_{n_k}) = \lim_{k \to \infty} \sup f(y_{n_k}) \leq f(y_0) = M,
\]
implying \( M < \infty \) and \( f(y_0) = M \).

(b) If \( f \in d\text{-SLip}_0Y \), it follows \( -f \in \overline{d}\text{-SLip}_0Y \), and because \( Y \) is \( d\text{-sequentially compact} \), by (a), it follows that \( -f \) attains its maximum value on \( Y \), i.e. \( f \) attains its minimum value on \( Y \). \( \square \)

**Proposition 4.** Let \((X, d)\) be a quasi-metric space, \( \theta \in X \) a fixed element, and \( Y \subseteq X \) with \( \theta \in Y \).

(a) If \( Y \) is \( \overline{d}\text{-sequentially compact} \), then the functional \( \|f\|_{\infty}^\overline{d} : \text{d-SLip}_0Y \to [0, \infty) \) defined by

\[
\|f\|_{\infty}^\overline{d} = \max\{f(y) : y \in Y\}
\]

is an asymmetric norm on \( d\text{-SLip}_0Y \).
(b) If $Y$ is $d$-sequentially compact, then the functional $\| \cdot \|_d^\infty : d$-$\text{SLip}_0 Y \to [0, \infty)$ defined by

$$\|f\|_d^\infty = \max \{-f(y) : y \in Y\}, f \in d$-$\text{SLip}_0 Y;$$

is an asymmetric norm on $d$-$\text{SLip}_0 Y$.

(c) If $Y$ is $(d, \bar{d})$-sequentially compact, then the functional $\| \cdot \|_\infty : d$-$\text{SLip}_0 Y \to [0, \infty)$ defined by

$$\|f\|_\infty = \|f\|_d^\infty \vee \|f\|_{\bar{d}}^\infty, f \in d$-$\text{SLip}_0 Y$$

is the uniform norm on the cone $d$-$\text{SLip}_0 Y$.

Proof. (a) By Proposition 3 (a), the functional (7) is well defined. For every $f \in d$-$\text{SLip}_0 Y$, we have $\|f\|_\infty \geq f(\theta) = 0$. If $f \in d$-$\text{SLip}_0 Y$ and $\|f\|_\infty > 0$ then there exists $y_0 \in Y$ such that $f(y_0) = \|f\|_\infty > 0$. It follows $f \neq 0$.

- Obviously,

$$\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$$

and

$$\|\lambda f\|_\infty = \lambda \|f\|_\infty$$

for all $f, g \in d$-$\text{SLip}_0 Y$ and $\lambda \geq 0$.

(b) For every $f \in d$-$\text{SLip}_0 Y$ it follows that $-f \in \bar{d}$-$\text{SLip}_0 Y$, and because $Y$ is $d$-sequentially compact, then $-f$ attains its maximum value on $Y$, and

$$\|f\|_\infty = \max \{-f(y) : y \in Y\}$$

is an asymmetric norm on $d$-$\text{SLip}_0 Y$.

(c) By Proposition 3 if $Y$ is $(d, \bar{d})$-sequentially compact, then every $f \in d$-$\text{SLip}_0 Y$, attains its maximum and minimum value on $Y$.

- We have

$$\|f\|_\infty = \max \{|f(y)| : y \in Y\} =$$

$$= (\max \{f(y) : y \in Y\}) \vee (\max \{-f(y) : y \in Y\})$$

$$= \|f\|_d^\infty \vee \|f\|_{\bar{d}}^\infty.$$

\[\square\]

3. BEST UNIFORM APPROXIMATION BY EXTENSIONS

In the following the quasi-metric space $(X, d)$ is supposed $(d, \bar{d})$-sequentially compact. Let $\theta \in X$ be a fixed element, and $Y \subseteq X$ with $\theta \in Y$. Consider also the normed cones $(d$-$\text{SLip}_0 Y, \|\cdot\|_d)$ and $(\bar{d}$-$\text{SLip}_0 X, \|\cdot\|_{\bar{d}})$, where $\|\cdot\|_d$ is the asymmetric norm defined as in (5), where $d$ is replaced by $\bar{d}$.

An extension results for semi-Lipschitz functions, analogous to Mc Shane’s Extension Theorem [8] for real-valued Lipschitz functions defined on a subset of a metric space was proved in [10] (see also [12]).
Proposition 5. \[10\] For every \( f \in d\text{-SLip}_0Y \) there exists at least one function \( F \in d\text{-SLip}_0X \), such that
\[
F|_Y = f \quad \text{and} \quad \|F\|_d = \|f\|_d.
\]
A function \( F \) with the properties included in Proposition 5 is called an extension, preserving the asymmetric norm of \( f \) (or an extension preserving the smallest semi-Lipschitz constant of \( f \)).

Denote the set of all extensions of \( f \) preserving asymmetric norm, by
\[
\mathcal{E}_d(f) = \{ F \in d\text{-SLip}_0X : F|_Y = f \quad \text{and} \quad \|F\|_d = \|f\|_d \}.
\]
The set \( \mathcal{E}_d(f) \) is convex in \( d\text{-SLip}_0X \), the functions
\[
F_d(f)(x) = \inf \{ f(y) + \|f\|_d d(x,y) : y \in Y \}, \quad x \in X,
\]
and
\[
G_d(f)(x) = \sup \{ f(y) - \|f\|_d d(y,x) : y \in Y \}, \quad x \in X,
\]
are extremal elements of \( \mathcal{E}_d(f) \), and
\[
G_d(f)(x) \leq F(x) \leq F_d(f)(x),
\]
for all \( F \in \mathcal{E}_d(f) \) (see \[10\], \[11\]).

Now let \( \mathbb{R}^X \) be the linear space of all real valued functions defined on \((X,d)\). One considers the quasi-distance (\[15\], p.67)
\[
D_d : \mathbb{R}^X \times \mathbb{R}^X \to [0,\infty)
\]
defined by
\[
D_d(f,g) = \sup \{ (f(x) - g(x)) \vee 0 : x \in X \}.
\]
Obviously, \( d\text{-SLip}_0X \subset \mathbb{R}^X_{d} \subset \mathbb{R}^X \), and the quasi-distance \( D_d \) may be restricted to \( d\text{-SLip}_0X \).

The quasi-distance \( D_d \) generates the topology \( \tau(D_d) \), named the topology of quasi-uniform convergence. In \[15\] (Corollary 4, p.67), it is proved that the unit ball \( U_0 \) of \( d\text{-SLip}_0X \) is compact with respect to the topology of quasi-uniform convergence \( \tau(D_d) \), (and \( \tau(D_d) \) too, where \( D_d(f,g) = D_d(g,f) \), \( f,g \in d\text{-SLip}_0X \)).

We have

Proposition 6. For every \( f \in d\text{-SLip}_0Y \), the set \( \mathcal{E}_d(f) \) is compact with respect to the topology \( \tau(D_d) \), (and \( \tau(D_d) \), too).

Proof. Because \( F_d(f) \) defined in \[12\] and \( G_d(f) \) defined in \[13\] are in \( \mathcal{E}_d(f) \), and they satisfy the inequalities \[14\], it follows
\[
D_d(F,F_d(f)) = 0, \quad \text{and} \quad D_d(F,G_d(f)) = D_d(G_d(f),F) = 0
\]
for every \( F \in \mathcal{E}_d(f) \). It follows that \( \mathcal{E}_d(f) \) is \( D_d \)-totally bounded (and \( D_d \)-totally bounded too).
Let \((F_n)_{n \geq 1}\) be a sequence in \(E_d(f)\). Because \(F_n(x) \leq F_d(f)(x)\), for all \(x \in X\), it follows that \(D_d(F_n, F_d(f)) = 0\), \(n = 1, 2, \ldots\), i.e. \((F_n)_{n \geq 1}\) is \(D_d\)-convergent to \(F_d(f)\). It follows that \(E_d(f)\) is \(D_d\)-sequentially compact. By Proposition 4.6 in [5], because \(E_d(f)\) is totally \(D_d\)-bounded and \(D_d\)-sequentially compact it follows that the set \(E_d(f)\) is \(D_d\)-compact (i.e. compact with respect to the topology \(\tau(D_d)\)).

Because \(E_d(f)(x) \leq F(x)\), for all \(x \in X\) and every \(F \in E_d(f)\), it follows that \(D_d(G_d(f), F) = \overline{D}_d(F, G_d(f)) = 0\). Consequently, \(E_d(f)\) is \(\overline{D}_d\)-compact too. (i.e. with respect to the topology \(\tau(\overline{D}_d)\)).

Obviously, for every \(F \in d\)-SLip_0 X, \(F_{|Y} \in d\)-SLip_0 Y and the set \(E_d(F_{|Y})\) is a \((D_d, \overline{D}_d)\)-compact subset of \(d\)-SLip_0 X, by Proposition 6.

Now, we consider the following optimization problem:

For \(F \in d\)-SLip_0 X, find \(G_0 \in E_{d_{|Y}}(F_{|Y})\) such that

\[
D_d(F, G_0) = \inf \{D_d(F, G) : G \in E_d(F_{|Y})\}. \tag{16}
\]

This problem (of best approximation) has always at least one solution, because \(E_d(F_{|Y})\) is \(D_d\)-compact. Analogously, the problem of existence of an element \(\overline{G}_0 \in E_{d_{|Y}}(F_{|Y})\) such that

\[
\overline{D}_d(F, \overline{G}_0) = \inf \{\overline{D}_d(F, G) : G \in E_d(F_{|Y})\}; \tag{17}
\]

is also assured, because \(E_d(F_{|Y})\) is \(\overline{D}_d\)-compact too.

Now, because \((X, d)\) is supposed \((d, \overline{d})\)-sequentially compact, every \(F \in d\)-SLip_0 X is bounded, and the uniform norm

\[
\|F\|_{\infty} = \max \{|F(x) : x \in X\} \lor \max \{-F(x) : x \in X\} \tag{18}
\]

is well defined, by Proposition 4 (c).

Moreover, for every \(G \in E_d(F_{|Y})\), we have

\[
\|F - G\|_{\infty} = D_d(F, G) \lor \overline{D}_d(F, G). \tag{19}
\]

Now, we consider the following problem of uniform best approximation:

For \(F \in d\)-SLip_0 X, find \(G_0 \in E_d(F_{|Y})\), such that

\[
\|F - G_0\|_{\infty} = \inf \{\|F - G\|_{\infty} : G \in E_d(F_{|Y})\}; \tag{20}
\]

**Proposition 7.** Let \((X, d)\) be a \((d, \overline{d})\)-sequentially compact quasi-metric space, \(\theta \in X\) a fixed element, and \(Y \subset X\) with \(\theta \in Y\). Then for every \(F \in d\)-SLip_0 X, there exists at least one element \(G_0 \in E_d(F_{|Y})\), such that

\[
\|F - G_0\|_{\infty} = \inf \{\|F - G\|_{\infty} : G \in E_d(F_{|Y})\}. \tag{21}
\]

**Proof.** For every \(G \in E_d(F_{|Y})\), using the equality (18), one obtains

\[
\inf \{\|F - G\|_{\infty} : G \in E_d(F_{|Y})\} = \inf \{D_d(F, G) \lor D_d(G, F) : G \in E_d(F_{|Y})\}.
\]
Because $E_d(F|_Y)$ is $(D_d, \overline{D}_d)$-compact, the conclusion of Proposition follows. 

\[ \square \]

Any solution $G_0 \in E_d(F|_Y)$ of problem (20) is called an element of best uniform approximation of $F$ by elements of $E_d(F|_Y)$.

Using (19), one obtains:

If $F$ is such that

\[ F(x) \geq F_d(F|_Y)(x), \quad x \in X, \]

then $G_0 = F_d(F|_Y)$ is the unique solution of (20), where $F_d(F|_Y)$ is defined as in (12):

If $F$ is such that

\[ F(x) \leq G_d(F|_Y)(x), \quad x \in X, \]

then $G_0 = G_d(F|_Y)$ is the unique solution of (20), where $G_d(F|_Y)$ is defined as in (13);

Finally, if $F \in E_d(F|_Y)$ i.e. $\|F\|_d = \|F|_Y\|_d$, then $G_0 = F$.

In the following we consider another situation where a uniform best approximation problem by extensions may be posed and solved.

This is the case when the quasi-metric space $(X, d)$ is of finite diameter, i.e. such that $\sup\{d(x, y) : x, y \in X\} = \text{diam}X < \infty$.

For $\theta \in (X, d)$ denote $cl_{\tau(d)}\{\theta\} = \{x \in X : d(\theta, x) = 0\}$ and $cl_{\tau(\overline{d})}\{\theta\} = \{x \in X : d(x, \theta) = 0\}$ (see [15], p.68). Let also $cl\{\theta\} = cl_{\tau(d)}\{\theta\} \cup cl_{\tau(\overline{d})}\{\theta\}$.

The following proposition holds:

**Proposition 8.** Let $(X, d)$ be a quasi-metric space of finite diameter, and $\theta \in X$ a fixed element. Then every $f \in d\text{-Slip}_0X$ is bounded on $X \setminus cl\{\theta\}$.

**Proof.** Let $f$ be in $d\text{-Slip}_0X$. By definition, we have $f(\theta) = 0$, and for $x \in cl_{\tau(\overline{d})}\{\theta\} = \{x \in X : d(x, \theta) = 0\}$ -it follows $f(x) \leq 0$, because $d(x, \theta) = 0$ implies $f(x) \leq f(\theta) = 0$.

Analogously, for $x \in cl_{\tau(d)}\{\theta\} = \{x \in X : d(\theta, x) = 0\}$ it follows $0 = f(\theta) \leq f(x)$.

For every $x \in X \setminus cl_{\tau(\overline{d})}\{\theta\}$, we have

\[ f(x) - f(\theta) \leq \|f\|_d d(x, \theta) \leq \|f\|_d \text{diam}X, \]

and consequently $f(x) \leq \|f\|_d \text{diam}X < \infty$.

It follows, $f(x) \leq \|f\|_d \text{diam}X < \infty$ for all $x \in X \setminus cl_{\tau(\overline{d})}\{\theta\}$.

For every $x \in X \setminus cl_{\tau(d)}\{\theta\}$ it follows

\[ f(\theta) - f(x) \leq \|f\|_d d(\theta, x) \leq \|f\|_d \text{diam}X. \]

Then $f(x) \geq -\|f\|_d \text{diam}X > -\infty$, for all $x \in X \setminus cl_{\tau(d)}\{\theta\}$. Consequently

\[ -\|f\|_d \text{diam}X \leq f(x) \leq \|f\|_d \text{diam}X, \quad x \in X \setminus cl\{\theta\}. \]

\[ \square \]
Now, let \((X, d)\) be a quasi-metric space of finite diameter, \(\theta \in X\) a fixed element, and \(Y \subset X\) with \(\theta \in Y\). Then, for every \(F \in d - \text{SLip}_0 X\), it follows \(F|_Y \in d - \text{SLip}_0 Y\), and the set

\[
\mathcal{E}_d(F|_Y) = \{G \in d - \text{SLip}_0 X : G|_Y = F|_Y, \|G\|_d = \|F|_Y\|_d\}
\]

is non empty.

This set is also \((D_d, D_d^-)\)-compact and the following proposition holds:

**Proposition 9.** Let \((X, d)\) be a quasi-metric space of finite diameter, \(\theta \in X\) a fixed element, and \(Y \subset X\) with \(\theta \in Y\). Then for every \(F \in d - \text{SLip}_0 X\), there exists at least one element \(G_0 \in \mathcal{E}_d(F|_Y)\) such that

\[
\|F - G_0\|_{X \setminus \text{cl}\{\theta\}} = \inf \{\|F - G\|_{X \setminus \text{cl}\{\theta\}} : G \in \mathcal{E}_d(F|_Y)\}.
\]

The proof is immediate.

**Example 10.** Let \(X = [-10, 10]\) and the quasi-metric \(d : X \times X \to [0, \infty)\) defined by

\[
d(x, y) = \begin{cases} 
  y - x & \text{if } x \leq y, \\
  2(x - y) & \text{if } x > y.
\end{cases}
\]

Consider \(\theta = 0\) and \(Y = \{-1, 0, 1\}\). Then the function \(f : Y \to \mathbb{R}\)

\[
f(y) = \begin{cases} 
  -1, & y = -1, \\
  0, & y = 0, \\
  3, & y = 1,
\end{cases}
\]

is in \(d - \text{SLip}_0 Y\) and \(\|f\|_d = 3\).

The functions

\[
F_d(f)(x) = \inf_{y \in Y} \{f(y) + 3d(x, y)\}
\]

\[
= \begin{cases} 
  -4 - 3x, & x \in [-10, -1], \\
  6x + 5, & x \in (-1, -\frac{5}{9}], \\
  -3x, & x \in (-\frac{5}{9}, 0], \\
  6x, & x \in (0, \frac{2}{3}], \\
  6 - 3x, & x \in \left(\frac{2}{3}, 1\right], \\
  6x - 3, & x \in (1, 10].
\end{cases}
\]
\[ G_d(f)(x) = \sup_{y \in Y} \{ f(y) - 3d(y, x) \} = \]
\[
\begin{cases}
6x + 5, & x \in [-10, -1], \\
-3x + 4, & x \in \left(-1, \frac{-4}{9}\right], \\
6x, & x \in \left(\frac{-4}{9}, 0\right], \\
-3x, & x \in \left(0, \frac{7}{3}\right], \\
6x - 3, & x \in \left(\frac{7}{3}, 1\right], \\
-3x - 6, & x \in (1, 10].
\end{cases}
\]

verifies the conditions:
\[ F_d(f) | Y = G_d(f) | Y = f, \]
\[ \| F_d(f) \|_d = \| G_d(f) \|_d = \| f \|_d = 3, \]

and
\[ F_d(f)(x) \geq H(x) \geq G_d(f)(x), \ x \in [-10, 10], \]
where \( H \in \mathcal{E}_d(f) \) is an arbitrary extension of \( f \).

Obviously, \((X, d)\) is \((d, d)\)-sequentially compact and \( \mathcal{E}_d(f) \) is compact in the uniform topology.

Let \( F \in d - \text{SLip}_0X \) such that \( F|_Y = f \).

Then
\[ \mathcal{E}_d(F|_Y) = \mathcal{E}_d(f). \]

If
\[ F(x) \geq F_d(f)(x), \ \forall x \in [-10, 10] \]
then
\[ \| F - F_d(f) \|_\infty = \inf \{ \| F - H \|_\infty : H \in \mathcal{E}_d(F|_Y) \} \]

For example, let \( F \) be the function
\[ F(x) = \begin{cases} 
F_d(f)(x), & x \in [-1, 1], \\
-4x - 5, & x \in [-10, -1) \\
7x - 4, & x \in (1, 10).
\end{cases} \]

Then
\[ \| F - F_d(f) \|_\infty = \max_{x \in [-10, 1]} \{-x - 1\} \vee \max_{x \in [1, 10]} \{x - 1\} = 9 \]

Similarly, if \( F(x) \leq G_d(f)(x), \ \forall x \in [-10, 10] \)
then
\[ \| F - G_d(f) \|_\infty = \inf \{ \| F - H \|_\infty : H \in \mathcal{E}_d(F|_Y) \}. \]
REFERENCES


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