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BEST UNIFORM APPROXIMATION OF SEMI-LIPSCHITZ FUNCTIONS BY EXTENSIONS*

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Abstract. In this paper we consider the problem of best uniform approximation of a real valued semi-Lipschitz function F defined on an asymmetric metric space (X,d), by the elements of the set $\mathcal{E}_d(F|_Y)$ of all extensions of $F|_Y$ $(Y\subset X)$, preserving the smallest semi-Lipschitz constant. It is proved that, this problem has always at least a solution, if (X,d) is $(d,\overline{d})$-sequentially compact, or of finite diameter.

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1. INTRODUCTION

Let X be a non-empty set. A function $d: X\times X\to[0,\infty)$ is called a *quasi-metric* on X [14] if the following conditions hold:

1) $d(x,y)=d(y,x)=0 \quad \text{iff} \quad x=y$,
2) $d(x,z)\leq d(x,y)+d(y,z)$, for all $x,y,z\in X$.

The function $\overline{d}: X\times X\to[0,\infty)$ defined by $\overline{d}(x,y)=d(y,x)$, for all $x,y\in X$ is also a quasi-metric on X, called the *conjugate* quasi-metric of d.

A pair (X,d) where X is a non-empty set and d a quasi-metric on X, is called a quasi-metric space.

If d can take the value $+\infty$, then it is called a *quasi-distance* on X.

Each quasi-metric d on X induces a topology $\tau(d)$ which has as a basis the family of balls (*forward open balls* [5])

$$B^+(x,\varepsilon):=\{y\in X: d(x,y)<\varepsilon\},\ x\in X,\ \varepsilon>0. \tag{1}$$

This topology is called the *forward topology of* X ([5], [9]), and is denoted also by τ_+.

Observe that the topology τ_+ is a T_0-topology. If the condition 1) is replaced by 1′) $d(x,y)=0$ iff $x=y$, then the topology τ_+ is a T_1-topology (see [14], [15]).

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Analogously, the quasi-metric $\overline{d}$ induces the topology $\tau(\overline{d})$ on X, which has as a basis the family of *backward open balls* ([5])

$$B^-(x,\varepsilon) := \{y \in X : d(y,x) < \varepsilon\},\ x \in X,\ \varepsilon > 0 \tag{2}$$

This topology is called the *backward topology* of X ([5], [9]) and is denoted also by τ_-.

For more information about quasi-metric spaces and their applications see, for example, the papers [5], [6], [7], [9], [14] and the references quoted therein.

Let (X,d) be a quasi-metric space. A sequence $(x_k)_{k\geq 1} \subset X$ is called *d-convergent (forward convergent)* to $x_0 \in X$, respectively $\overline{d}$*-convergent* (*backward convergent*) to $x_0 \in X$ iff

$$\lim_{k\to\infty} d(x_0,x_k) = 0,\ \text{respectively}\ \lim_{k\to 0} d(x_k,x_0) = \lim_{k\to\infty} \overline{d}(x_0,x_k) = 0. \tag{3}$$

(see [5], Definition 2.4)

A subset K of X is called *d-compact* (*forward compact*) if every open cover of K with respect to the forward topology τ_+ has a finite subcover. We say that a subset K of X is *d-sequentially compact* (*forward-sequentially compact*) if every sequence in K has a d-convergent (forward convergent) subsequence with limit in K ([5], Definition 4.1).

The $\overline{d}$-compact (backward compact) and $\overline{d}$-sequentially compact (backward -sequentially compact) subset of X - are defined in a similar way.

Finally, a subset Y of (X,d) is called $(d,\overline{d})$*-sequentially compact* if every sequence $(y_n)_{n\geq 1}$ in Y has a subsequence $(y_{n_k})_{k\geq 1}$, d-convergent to some $u \in Y$ and $\overline{d}$-convergent to some $v \in Y$. By Lemma 3.1 in [5] if follows that we can take $u = v$ in the definition of $(d,\overline{d})$-sequentially compactness, if (X,d) is a T_1 quasi-metric space. A subset Y of (X,d) is called *d-bounded* (*forward bounded* in [5]) if there exist $x \in X$ and $r > 0$, such that $Y \subset B^+(x,r)$. Y is called *d-totally bounded* if for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$, and the forward balls $B^+(y_1,\varepsilon), B^+(y_2,\varepsilon), ..., B_n(y_n,\varepsilon),\ y_i \in Y, i = \overline{1,n}$ such that $Y \subset \bigcup_{i=1}^{n} B^+(y_i,\varepsilon)$.

Similar definitions are given for $\overline{d}$-boundedness and $\overline{d}$-total boundedness of a subset Y of (X,d).

2. THE CONE OF SEMI-LIPSCHITZ FUNCTIONS

Definition 1. [15] *Let Y be a non-empty subset of a quasi-metric space (X,d). A function $f : Y \to \mathbb{R}$ is called d-semi-Lipschitz if there exists a number $L \geq 0$ (named a d-semi-Lipschitz constant for f) such that*

$$f(x) - f(y) \leq Ld(x,y), \tag{4}$$

for all $x,y \in Y$.

A function $f : Y \to \mathbb{R}$, is called $\leq_d$-increasing if $f(x) \leq f(y)$, whenever $d(x,y) = 0$.

Denote by $\mathbb{R}^Y_{\leq d}$ the set of all $\leq_d$-increasing functions on Y. This set is a cone in the linear space $\mathbb{R}^Y$ of real valued functions defined on Y, i.e. for each $f, g \in \mathbb{R}^Y_{\leq d}$ and $\lambda \geq 0$ it follows that $f + g \in \mathbb{R}^Y_{\leq d}$ and $\lambda f \in \mathbb{R}^Y_{\leq d}$.

For a d-semi-Lipschitz function f on Y, put [14]:

$$\|f|_d = \sup\left\{\frac{(f(x)-f(y))\vee 0}{d(x,y)} : d(x,y) > 0;\ x, y \in Y\right\} \tag{5}$$

Then $\|f|_d$ is the smallest d-semi-Lipschitz constant of f (see also [10], [15]).

For a fixed element $\theta \in Y$ denote

$$d\text{-SLip}_0 Y := \{f \in \mathbb{R}^Y_{\leq d} : \|f|_d < \infty \text{ and } f(\theta) = 0\}, \tag{6}$$

the set of all d-semi-Lipschitz real valued functions defined on Y vanishing at the fixed element $\theta \in Y$.

Observe that if (X, d) is a T_1 quasi-metric space, then every real-valued function on X is $\leq_d$-increasing [14].

The set d-SLip$_0Y$ is a cone (a subcone of $\mathbb{R}^Y_{\leq d}$) and the functional $\|\cdot|_d : d$-SLip$_0Y \to [0, \infty)$ defined by (5) is subadditive and positive homogeneous on d-SLip$_0Y$. Moreover $\|f|_d = 0$ iff $f = 0$, and consequently $\|\cdot|_d$ is a quasi-norm (asymmetric norm) on the cone d-SLip$_0Y$.

In [15] some properties of the "normed cone" (d-SLip$_0Y$, $\|\cdot|_d$) are presented. Similar properties in the case of $d-$semi-Lipschitz functions on a quasi-metric space with values in a quasi-normed space (space with asymmetric norm) are discussed in [16], [17]. For more information concerning other properties of quasi-metric spaces, see also [7], [13].

Now, let (X, d) be a quasi-metric space and let Y be a non-empty subset of X. A real valued function f defined on Y is called τ_+-lower semi-continuous (τ_+-*l.s.c* in short) (respectively τ_--upper semi-continuous ($\tau_- - u.s.c.$)) at $x_0 \in Y$, if for every $\varepsilon > 0$ there exists $r > 0$ such that for every $x \in B^+(x_0, r)$ (respectively, for every $x \in B^-(x_0, r)$), $f(x) > f(x_0) - \varepsilon$ (respectively $f(x) < f(x_0) + \varepsilon$).

Proposition 2. *Let (X, d) be a quasi-metric space, $\theta \in X$ a fixed element, and $Y \subseteq X$ with $\theta \in Y$. Then every $f \in d$-SLip$_0Y$ is τ_--u.s.c and τ_+-l.s.c.,and every $f \in \bar{d}$-SLip$_0Y$ is τ_+-u.s.c. and τ_--l.s.c. on Y.*

Proof. Let $f \in d$-SLip$_0Y$ such that $\|f|_d = 0$. Then $f \equiv 0$ and f is τ_--*u.s.c.* and τ_+-*l.s.c* at every $y \in Y$.

Now, let $\|f|_d > 0$ and $y_0 \in Y$. The inequality

$$f(y) - f(y_0) \leq \|f|_d\, d(y, y_0),\ y \in Y$$

implies

$$f(y) \leq f(y_0) + \|f|_d\, d(y, y_0),\ y \in Y.$$

So that

$$f(y) < f(y_0) + \varepsilon,$$

for every $\varepsilon > 0$ and every $y \in B^-\left(y_0, \frac{\varepsilon}{\|f|_d}\right)$, showing that f is τ_--*u.s.c* at $y_0 \in Y$.

Similarly,

$$f(y_0) - f(y) \leq \|f|_d \cdot d(y_0, y),\ y \in Y,$$

implies

$$f(y) \geq f(y_0) - \|f|_d\, d(y_0, y),$$

so that

$$f(y) > f(y_0) - \varepsilon,$$

for every $y \in B^+\left(Y_0, \frac{\varepsilon}{\|f|_d}\right)$, showing that f is τ_+-*l.s.c.* in $y_0 \in Y$.

Similarly one prove that every $f \in \overline{d}$-$\mathrm{SLip}_0 Y$ is τ_+-*u.s.c.* and τ_--*l.s.c.* on Y. □

Observe that if f is in d-$\mathrm{SLip}_0 Y$, then $-f \in \overline{d}$-$\mathrm{SLip}_0 Y$, and $-f$ is τ_+-*u.s.c*, and τ_--*l.s.c.* on Y, i.e. if $y_0 \in Y$ then

- $\forall \varepsilon > 0, \exists r > 0$ such that $(-f)(y) < (-f)(y_0) + \varepsilon$, *for all* $y \in B^+(y_0, r)$, and respectively
- $\forall \varepsilon > 0, \exists r > 0$ *such that* $(-f)(y) > (-f)(y_0) - \varepsilon$, *for all* $y \in B^-(y_0, r)$.

PROPOSITION 3. *Let (X, d) be a quasi-metric space, $\theta \in X$ a fixed element, and $Y \subset X$, with $\theta \in Y$.*

(a) *If Y is $\overline{d}$-sequentially compact, then each $f \in d$-$\mathrm{SLip}_0 Y$ attains its maximum value on Y;*

(b) *If Y is d- sequentially compact, then each $f \in d$-$\mathrm{SLip}_0 Y$ attains its minimum value on Y.*

Proof. (a) Let Y be $\overline{d}$-sequentially compact and $M := \sup f(Y)$, where $M \in \mathbb{R} \cup \{+\infty\}$. Then there exists a sequence $(y_n)_{n\geq 1}$ in Y such that $\lim\limits_{n\to\infty} f(y_n) = M$. Because Y is $\overline{d}$-sequentially compact, there exists $y_0 \in Y$ and a subsequence $(y_{n_k})_{k\geq 1}$ of $(y_n)_{n\geq 1}$ such that $\lim\limits_{n\to\infty} d(y_{n,k}, y_0) = 0$. By the τ_--*u.s.c.* of f at y_0 it follows:

$$M = \lim_{k\to\infty} f(y_{n_k}) = \limsup_k f(y_{n_k}) \leq f(y_0) = M,$$

implying $M < \infty$ and $f(y_0) = M$.

(b) If $f \in d$-$\mathrm{SLip}_0 Y$, it follows $-f \in \overline{d}$-$\mathrm{SLip}_0 Y$, and because Y is d-sequentially compact, by (a), it follows that $-f$ attains its maximum value on Y, i.e. f attains its minimum value on Y. □

PROPOSITION 4. *Let (X, d) be a quasi-metric space, $\theta \in X$ a fixed element, and $Y \subseteq X$ with $\theta \in Y$.*

(a) *If Y is $\overline{d}$-sequentially compact, then the functional $\|\cdot|_\infty^{\overline{d}} : d-\mathrm{SLip}_0 Y \to [0, \infty)$ defined by*

$$\|f|_\infty^{\overline{d}} = \max\{f(y) : y \in Y\} \tag{7}$$

is an asymmetric norm on d-$\mathrm{SLip}_0 Y$.

(b) *If Y is d-sequentially compact, then the functional* $\|\cdot|^{d}_{\infty} : d\text{-SLip}_0 Y \to [0,\infty)$ *defined by*

$$\|f|^{d}_{\infty} = \max\{-f(y) : y \in Y\}, f \in d\text{-SLip}_0 Y, \tag{8}$$

is an asymmetric norm on d-$\text{SLip}_0 Y$*;*

(c) *If Y is $(d,\bar{d})$-sequentially compact, then the functional* $\|\cdot|_{\infty} : d\text{-SLip}_0 Y \to [0,\infty)$ *defined by*

$$\|f|_{\infty} = \|f|^{d}_{\infty} \vee \|f|^{\bar{d}}_{\infty}, f \in d\text{-SLip}_0 Y \tag{9}$$

is the uniform norm on the cone d-$\text{SLip}_0 Y$*.*

Proof. (a) By Proposition 3 (a), the functional (7) is well defined. For every $f \in d\text{-SLip}_0 Y$, we have $\|f|^{\bar{d}}_{\infty} \geq f(\theta) = 0$. If $f \in d\text{-SLip}_0 Y$ and $\|f|^{d}_{\infty} > 0$ then there exists $y_0 \in Y$ such that $f(y_0) = \|f|^{\bar{d}}_{\infty} > 0$. It follows $f \neq 0$.

- Obviously,

$$\|f+g|^{\bar{d}}_{\infty} \leq \|f|^{\bar{d}}_{\infty} + \|g|^{\bar{d}}_{\infty}$$

and

$$\|\lambda f|^{\bar{d}}_{\infty} = \lambda \|f|^{\bar{d}}_{\infty}$$

for all $f, g \in d\text{-SLip}_0 Y$ and $\lambda \geq 0$.

(b) For every $f \in d\text{-SLip}_0 Y$ it follows that $-f \in \bar{d}\text{-SLip}_0 Y$, and because Y is d-sequentially compact, then $-f$ attains its maximum value on Y, and

$$\|f|^{d}_{\infty} = \max\{-f(y) : y \in Y\}$$

is an asymmetric norm on d-$\text{SLip}_0 Y$.

(c) By Proposition 3, if Y is $(d,\bar{d})$-sequentially compact, then every $f \in d$-$\text{SLip}_0 Y$, attains its maximum and minimum value on Y.

- We have

$$\begin{aligned} \|f\|_{\infty} &= \max\{|f(y)| : y \in Y\} = \\ &= (\max\{f(y) : y \in Y\}) \vee (\max\{-f(y) : y \in Y\}) \\ &= \|f|^{d}_{\infty} \vee \|f|^{\bar{d}}_{\infty}. \end{aligned}$$

□

3. BEST UNIFORM APPROXIMATION BY EXTENSIONS

In the following the quasi-metric space (X, d) is supposed $(d,\bar{d})$-sequentially compact. Let $\theta \in X$ be a fixed element, and $Y \subseteq X$ with $\theta \in Y$. Consider also the normed cones $(d\text{-SLip}_0 Y, \|\cdot|_d)$ and $(\bar{d}\text{-SLip}_0 X, \|\cdot|_{\bar{d}})$, where $\|\cdot|_{\bar{d}}$ is the asymmetric norm defined as in (5), where d is replaced by $\bar{d}$.

An extension results for semi-Lipschitz functions, analogous to Mc Shane's Extension Theorem [8] for real-valued Lipschitz functions defined on a subset of a metric space was proved in [10] (see also [12]).

Proposition 5. [10] *For every* $f \in d\text{-SLip}_0 Y$ *there exists at least one function* $F \in d\text{-SLip}_0 X$, *such that*

$$F|_Y = f \text{ and } \|F|_d = \|f|_d \,. \tag{10}$$

A function F with the properties included in Proposition 5, is called an *extension*, preserving the asymmetric norm of f (or an extension preserving the smallest semi-Lipschitz constant of f).

Denote the set of all extensions of f preserving asymmetric norm, by

$$\mathcal{E}_d(f) = \{F \in d\text{-SLip}_0 X : F|_Y = f \text{ and } \|F|_d = \|f|_d\} \tag{11}$$

The set $\mathcal{E}_d(f)$ is convex in $d\text{-SLip}_0 X$, the functions

$$F_d(f)(x) = \inf \{f(y) + \|f|_d \, d(x,y) : y \in Y\}, \; x \in X, \tag{12}$$

and

$$G_d(f)(x) = \sup \{f(y) - \|f|_d \cdot d(y,x) : y \in Y\}, \; x \in X, \tag{13}$$

are extremal elements of $\mathcal{E}_d(f)$, and

$$G_d(f)(x) \le F(x) \le F_d(f)(x), \tag{14}$$

for all $F \in \mathcal{E}_d(f)$ (see [10], [11]).

Now let $\mathbb{R}^X$ be the linear space of all real valued functions defined on (X,d). One considers the quasi-distance ([15], p.67)

$$D_d : \mathbb{R}^X \times \mathbb{R}^X \to [0,\infty)$$

defined by

$$D_d(f,g) = \sup \{(f(x) - g(x)) \vee 0 : x \in X\}. \tag{15}$$

Obviously, $d\text{-SLip}_0 X \subset \mathbb{R}^X_{\le d} \subset \mathbb{R}^X$, and the quasi-distance D_d may be restricted to $d\text{-SLip}_0 X$.

The quasi-distance D_d generates the topology $\tau(D_d)$, named the topology of quasi-uniform convergence. In [15] (Corollary 4, p.67), it is proved that the unit ball U_0 of $d\text{-SLip}_0 X$ is compact with respect to the topology of quasi-uniform convergence $\tau(D_d)$, (and $\tau(\overline{D}_d)$ too, where $\overline{D}_d(f,g) = D_d(g,f), \; f,g \in d\text{-SLip}_0 X$).

We have

Proposition 6. *For every* $f \in d\text{-SLip}_0 Y$, *the set* $\mathcal{E}_d(f)$ *is compact with respect to the topology* $\tau(D_d)$, *(and* $\tau(\overline{D}_d)$, *too).*

Proof. Because $F_d(f)$ defined in (12) and $G_d(f)$ defined in (13) are in $\mathcal{E}_d(f)$, and they satisfy the inequalities (14), it follows

$$D_d(F, F_d(f)) = 0, \text{ and } \overline{D}_d(F, G_d(f) = D_d(G_d(f), F) = 0$$

for every $F \in \mathcal{E}_d(f)$. It follows that $\mathcal{E}_d(f)$ is D_d-totally bounded (and $\overline{D}_d$-totally bounded too).

Let $(F_n)_{n\geq 1}$ be a sequence in $\mathcal{E}_d(f)$. Because $F_n(x) \leq F_d(f)(x)$, for all $x \in X$, it follows that $D_d(F_n, F_d(f)) = 0,\ n = 1, 2, ...$, i.e. $(F_n)_{n\geq 1}$ is D_d-convergent to $F_d(f)$. It follows that $\mathcal{E}_d(f)$ is D_d-sequentially compact. By Proposition 4.6 in [5], because $\mathcal{E}_d(f)$ is totally D_d-bounded an D_d-sequentially compact it follows that the set $\mathcal{E}_d(f)$ is D_d-compact (i.e. compact with respect to the topology $\tau(D_d)$).

Because $G_d(f)(x) \leq F(x)$, for all $x \in X$ and every $F \in \mathcal{E}_d(f)$, it follows that $D_d(G_d(f), F) = \overline{D}_d(F, G_d(f)) = 0$. Consequently, $\mathcal{E}_d(f)$ is $\overline{D}_d$-compact too. (i.e. with respect to the topology $\tau(\overline{D}_d)$). □

Obviously, for every $F \in d\text{-SLip}_0X$, $F|_Y \in d\text{-SLip}_0Y$ and the set $\mathcal{E}_d(F|_Y)$ is a $(D_d, \overline{D}_d)$-compact subset of $d\text{-SLip}_0X$, by Proposition 6.

Now, we consider the following optimization problem:

For $F \in d\text{-SLip}_0X$, find $G_0 \in \mathcal{E}_d\ (F|_Y)$ such that

$$D_d(F, G_0) = \inf\{D_d(F, G) : G \in \mathcal{E}_d(F|_Y)\}. \tag{16}$$

This problem (of best approximation) has always at least one solution, because $\mathcal{E}_d(F|_Y)$ is D_d-compact. Analogously, the problem of existence of an element $\overline{G}_0 \in \mathcal{E}_d(F|_Y)$ such that

$$\overline{D}_d(F, \overline{G}_0) = \inf\{\overline{D}_d(F, G) : G \in \mathcal{E}_d(F|_Y)\}, \tag{17}$$

is also assured, because $\mathcal{E}_d(F|_Y)$ is $\overline{D}_d$-compact too.

Now, because (X, d) is supposed $(d, \overline{d})$-sequentially compact, every $F \in d$-SLip_0X is bounded, and the uniform norm

$$\|F\|_\infty = \max\{F(x) : x \in X\} \vee \max\{-F(x) : x \in X\} \tag{18}$$

is well defined, by Proposition 4, (c).

Moreover, for every $G \in \mathcal{E}_d(F|_Y)$, we have

$$\|F - G\|_\infty = D_d(F, G) \vee \overline{D}_d(F, G). \tag{19}$$

Now, we consider the following *problem of uniform best approximation*:

For $F \in d\text{-SLip}_0X$, find $G_0 \in \mathcal{E}_d(F|_Y)$, such that

$$\|F - G_0\|_\infty = \inf\{\|F - G\|_\infty : G \in \mathcal{E}_d(F|_Y)\}. \tag{20}$$

Proposition 7. *Let (X, d) be a $(d, \overline{d})$-sequentially compact quasi-metric space, $\theta \in X$ a fixed element, and $Y \subset X$ with $\theta \in Y$. Then for every $F \in d$-SLip_0X, there exists at least one element $G_0 \in \mathcal{E}_d(F|_Y)$, such that*

$$\|F - G_0\|_\infty = \inf\{\|F - G\|_\infty : G \in \mathcal{E}_d(F|_Y)\}.$$

Proof. For every $G \in \mathcal{E}_d(F|_Y)$, using the equality (18), one obtains

$$\inf\{\|F - G\|_\infty : G \in \mathcal{E}_d(F|_Y)\} =$$
$$= \inf\{D_d(F, G) \vee D_d(G, F) : G \in \mathcal{E}_d(F|_Y)\}$$

Because $\mathcal{E}_d(F|_Y)$ is $(D_d, \overline{D}_d)$-compact, the conclusion of Proposition follows. □

Any solution $G_0 \in \mathcal{E}_d(F|_Y)$ of problem (20) is called an element of best uniform approximation of F by elements of $\mathcal{E}_d(F|_Y)$.

Using (19), one obtains:

If F is such that

$$F(x) \geq F_d(F|_Y)(x), x \in X,$$

then $G_0 = F_d(F|_Y)$ is the unique solution of (20), where $F_d(F|_Y)$ is defined as in (12);

If F is such that

$$F(x) \leq G_d(F|_Y)(x),\ x \in X,$$

then $G_0 = G_d(F|_Y)$ is the unique solution of (20), where $G_d(F|_Y)$ is defined as in (13);

Finally, if $F \in \mathcal{E}_d(F|_Y)$ i.e. $\|F|_d = \|F\ |_Y|_d$, then $G_0 = F$.

In the following we consider another situation where a uniform best approximation problem by extensions may be posed and solved.

This is the case when the quasi-metric space (X, d) is of finite diameter, i.e. such that $sup\{d(x, y) : x, y \in X\} = \text{diam}X < \infty$.

For $\theta \in (X, d)$ denote $cl_{\tau(d)}\{\theta\} = \{x \in X : d(\theta, x) = 0\}$ and $cl_{\tau(\bar{d})}\{\theta\} = \{x \in X : d(x, \theta) = 0\}$ (see [15], p.68). Let also $cl\{\theta\} = cl_{\tau(d)}\{\theta\} \cup cl_{\tau(\bar{d})}\{\theta\}$.

The following proposition holds:

Proposition 8. *Let (X, d) be a quasi-metric space of finite diameter, and $\theta \in X$ a fixed element. Then every $f \in d$-SLip_0X is bounded on $X \setminus cl\{\theta\}$.*

Proof. Let f be in d-SLip_0X. By definition, we have $f(\theta) = 0$, and for $x \in cl_{\tau(\bar{d})}\{\theta\} = \{x \in X : d(x, \theta) = 0\}$ -it follows $f(x) \leq 0$, because $d(x, \theta) = 0$ implies $f(x) \leq f(\theta) = 0$.

Analogously, for $x \in cl_{\tau(d)}\{\theta\} = \{x \in X : d(\theta, x) = 0\}$ it follows $0 = f(\theta) \leq f(x)$.

For every $x \in X \setminus cl_{\tau(\bar{d})}\{\theta\}$, we have

$$f(x) - f\,(\theta) \leq \|f|_d\, d(x, \theta) \leq \|f|_d\, \text{diam}X,$$

and consequently $f(x) \leq \|f|_d\, \text{diam}X < \infty$.

It follows, $f(x) \leq \|f|_d\, \text{diam}X < \infty$ for all $x \in X \setminus cl_{\tau(\bar{d})}\{\theta\}$.

For every $x \in X \setminus cl_{\tau(d)}\{\theta\}$ it follows

$$f(\theta) - f(x) \leq \|f|_d\, d(\theta, x) \leq \|f|_d\, \text{diam}X.$$

Then $f(x) \geq -\|f|_d\, \text{diam}X > -\infty$, for all $x \in X \setminus cl_{\tau(d)}\{\theta\}$. Consequently $-\|f|_d\, \text{diam}X \leq f(x) \leq \|f|_d\, \text{diam}X$, $x \in X \setminus cl\{\theta\}$. □

Now, let (X,d) be a quasi-metric space of finite diameter, $\theta \in X$ a fixed element, and $Y \subset X$ with $\theta \in Y$. Then, for every $F \in d - \mathrm{SLip}_0 X$, it follows $F|_Y \in d\text{-}\mathrm{SLip}_0 Y$, and the set

$$\mathcal{E}_d(F|_Y) = \{G \in d\text{-}\mathrm{SLip}_0 X \colon G|_Y = F|_Y \,, \|G|_d = \|F|_Y|_d\}$$

is non empty.

This set is also $(D_d, \overline{D}_d)$-compact and the following proposition holds:

PROPOSITION 9. *Let (X,d) be a quasi-metric space of finite diameter, $\theta \in X$ a fixed element, and $Y \subset X$ with $\theta \in Y$. Then for every $F \in d\text{-}\mathrm{SLip}_0 X$, there exists at least one element $G_0 \in \mathcal{E}_d(F|_Y)$ such that*

$$\left\|(F - G_0)|_{X \setminus cl\{\theta\}}\right\|_\infty = \inf\{\left\|(F - G)|_{X \setminus cl\{\theta\}}\right\|_\infty : G \in \mathcal{E}_d(F|_Y)\}.$$

The proof is immediate.

EXAMPLE 10. Let $X = [-10, 10]$ and the quasi-metric $d : X \times X \to [0, \infty)$ defined by

$$d(x,y) = \begin{cases} y - x \text{ if } x \leq y, \\ 2(x - y) \text{ if } x > y. \end{cases}$$

Consider $\theta = 0$ and $Y = \{-1, 0, 1\}$. Then the function $f : Y \to \mathbb{R}$

$$f(y) = \begin{cases} -1, & y = -1, \\ 0, & y = 0, \\ 3, & y = 1, \end{cases}$$

is in $d - \mathrm{SLip}_0 Y$ and $\|f|_d = 3$.

The functions

$$\begin{aligned} F_d(f)(x) &= \inf_{y \in Y}\{f(y) + 3d(x,y)\} \\ &= \begin{cases} -4 - 3x, & x \in [-10, -1], \\ 6x + 5, & x \in \left(-1, \frac{-5}{9}\right], \\ -3x, & x \in \left(\frac{-5}{9}, 0\right], \\ 6x, & x \in \left(0, \frac{2}{3}\right], \\ 6 - 3x, & x \in \left(\frac{2}{3}, 1\right], \\ 6x - 3, & x \in (1, 10]. \end{cases} \end{aligned}$$

and, respectively

$$G_d(f)(x) = \sup_{y\in Y}\{f(y) - 3d(y,x)\} =$$

$$= \begin{cases} 6x+5, & x\in[-10,-1], \\ -3x+4, & x\in\left(-1,\frac{-4}{9}\right], \\ 6x, & x\in\left(\frac{-4}{9},0\right], \\ -3x, & x\in\left(0,\frac{1}{3}\right], \\ 6x-3, & x\in\left(\frac{1}{3},1\right], \\ -3x-6, & x\in(1,10]. \end{cases}$$

verifies the conditions:

$$F_d(f)\,|_Y = G_d(f)\,|_Y = f,$$
$$\|F_d(f)|_d = \|G_d(f)|_d = \|f|_d = 3,$$

and

$$F_d(f)(x) \geq H(x) \geq G_d(f)(x),\ x\in[-10,10],$$

where $H \in \mathcal{E}_d(f)$ is an arbitrary extension of f.

Obviously, (X,d) is $(d,\overline{d})$-sequentially compact and $\mathcal{E}_d(f)$ is compact in the uniform topology.

Let $F \in d - \mathrm{SLip}_0 X$ such that $F|_Y = f$.

Then

$$\mathcal{E}_d(F|_Y) = \mathcal{E}_d(f).$$

If

$$F(x) \geq F_d(f)(x),\ \forall x\in[-10,10]$$

then

$$\|F - F_d(f)\|_\infty = \inf\{\|F-H\|_\infty : H\in\mathcal{E}_d(F|_Y)\}$$

For example, let F be the function

$$F(x) = \begin{cases} F_d(f)(x), & x\in[-1,1], \\ -4x-5, & x\in[-10,-1) \\ 7x-4, & x\in(1,10). \end{cases}$$

Then

$$\|F - F_d(f)\|_\infty = \max_{x\in[-10,1]}\{-x-1\} \vee \max_{x\in[1,10]}\{x-1\} =$$
$$= 9$$

Similarly, if $F(x) \leq G_d(f)(x)$, $\forall x\in[-10,10]$

then

$$\|F - G_d(f)\|_\infty = \inf\{\|F-H\|_\infty : H\in\mathcal{E}_d(F|_Y)\}.$$

□

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