ON AN AITKEN TYPE METHOD

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Abstract. In this note we study the convergence of a generalized Aitken type method for approximating the solutions of nonlinear equations in $\mathbb{R}$. We obtain conditions which assure monotone convergence of the generated sequences. We also obtain a posteriori estimations for the errors.

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1. INTRODUCTION

Consider the equation

$$f(x) = 0$$

where $f : [a, b] \rightarrow \mathbb{R}$, $a, b \in \mathbb{R}$, $a < b$.

Consider also the following two equations, both equivalent to (1):

$$x - g_i(x) = 0, 
\quad g_i : [a, b] \rightarrow [a, b], 
\quad i = 1, 2.$$ 

In order to approximate a root $x^*$ of (1), we consider the sequence $(x_n)_{n \geq 0}$, generated by the following relations:

$$x_{n+1} = x_n - \frac{f(x_n)}{[x_n, g_1(x_n); f] - [x_n, g_1(x_n); f][x_n, g_2(x_n); f][g_1(x_n), g_2(x_n); f]}$$

$$x \in [a, b], 
\quad n = 0, 1, 2, ..., $$

where $[x, y; f]$ denotes the first order divided difference of $f$ on $x$ and $y$.

Relation (3) suggests an Aitken type method. If we assume that $f \in C^3[a, b]$ and $f'(x) \neq 0$ for all $x \in [a, b]$, denoting $F = f([a, b])$, it is known that there exists $f^{-1} : F \rightarrow [a, b]$ and $f^{-1} \in C^3([a, b])$. Moreover, the following relation holds

$$(f^{-1}(y))''' = \frac{3(f''(x))^2 - f'(x)f'''(x)}{(f'(x))^5}$$

where $y = f(x)$ and $x \in [a, b]$ (see [1], [2], [4], [7]).
In [3] was studied the convergence of a Steffensen type method, analogous to [5]. In order to obtain a control of the error at each iteration step, we have studied in [3] the cases when the generalized Steffensen type method leads to monotone approximations. Under some reasonable conditions of monotonicity and convexity (concavity) on \( f \), in [3] were obtained monotone sequences which yield bilateral approximations to the solution \( x^* \) of (1). A condition such that the Steffensen type method studied in [6] to lead to monotone approximations is that the sign of the expression

\[
E_f(x) = 3(f''(x))^2 - f'(x)f'''(x)
\]

to be negative: \( E_f(x) < 0, \forall x \in [a, b] \).

In this note we shall show that if \( E_f(x) > 0 \ \forall x \in [a, b] \), one may construct functions \( g_1 \) and \( g_2 \), and on may choose the initial approximations \( x_0 \in [a, b] \) such that, under some reasonable hypotheses on \( f \), the sequence [3] converges monotonically.

In an analogous fashion as in [3], we can easily show that for a given \( x_n \in [a, b] \), then exists \( \xi_n \in]c_n, d_n[, \) where \([c_n, d_n]\) is the smallest interval containing the points \( x_n, g_1(x_n), \) \( g_2(x_n) \), \( x^* \), such that

\[
x^* - x_{n+1} = \frac{E_f(\xi_n)f(x_n)f(g_1(x_n))f(g_2(x_n))}{6[f'(\xi_n)]^5}, \ n = 0, 1, \ldots
\]

2. THE CONVERGENCE OF THE AITKEN METHOD

We consider the following hypotheses on the functions \( f, g_1, g_2 \), and on the initial approximation \( x_0 \in [a, b] \):

i. \( f \in C^5[a, b] \) and \( E_f(x) > 0, \forall x \in [a, b] \) where \( E_f \) is given by [5];
ii. \( f(x_0) < 0 \);
iii. \( f'(x) > 0, \forall x \in [a, b] \);
iv. \( f''(x) \geq 0, x \in [a, b] \);
v. \( g_1 \) and \( g_2 \) are continuous functions, decreasing on \([a, b] \);
vii. equation (1) has a solution \( x^* \in [a, b] \);
viii. \( g_1(x_0) \leq b, g_2(x_0) \leq b \);
viii. equations (1) and (2) are equivalent.

The following result holds:

**Theorem 1.** If functions \( f, g_1, g_2 \) and the initial approximation \( x_0 \in [a, b] \) verify hypotheses i.–viii., then the sequences \( (x_n)_{n \geq 0}, (g_1(x_n)), (g_2(x_n)) \) generated by (3) have the following properties:

j. the sequence \( (x_n)_{n \geq 0} \) is increasing;
jj. the sequences \( (g_1(x_n)), (g_2(x_n))_{n \geq 0} \) are decreasing;
jjj. \( \lim x_n = \lim g_1(x_n) = \lim g_2(x_n) = x^* \);
jv. the following relations hold:

\[
x^* - x_n \leq \min \{g_1(x_n) - x_n, g_2(x_n) - x_n\}, \ n = 0, 1, \ldots
\]
Proof. Let \( x_n \in [a, b] \) be an approximation of the solution \( x^* \) such that \( f(x_n) < 0, g_1(x_n) \leq b, g_2(x_n) \leq b \). From ii. and iii. it follows \( x_n < x^* \).

By v. we have \( g_1(x_n) > x^*, g_2(x_n) > x^* \). Taking into account the fact that \( f(x_n) < 0, f(g_1(x_n)) > 0 \) and using hypothesis iii. and iv., from (8) it follows that \( x_{n+1} > x_n \). Since \( f(g_2(x_n)) > 0 \), by iii., i. and (8) we get \( x_{n+1} < x^* \).

By v. and \( x_n < x_{n+1} \) it follows \( g_1(x_n) > g_1(x_{n+1}) \) and \( g_2(x_n) > g_2(x_{n+1}) \).

By v. it also follows that \( g_1(x_{n+1}) > x^* \) and \( g_2(x_{n+1}) > x^* \). These relations imply j. and jj.

In order to prove jjj., let \( \ell = \lim n x_n \). For \( n \to \infty \), from (8) we obtain \( f(\ell) = 0 \) and hence \( \ell = x^* \). Obviously, from the continuity of \( g_1 \) and \( g_2 \) it follows that \( \lim g_1(x_n) = g_1(x^*) = x^* \) and \( \lim g_2(x_n) = g_2(x^*) = x^* \).

Relations jv. are obvious, and allow us to evaluate the a posteriori error at each iteration step.

\[ \square \]

3. CONSTRUCTION OF FUNCTIONS \( g_1 \) AND \( g_2 \)

The basic conditions on \( g_1 \) and \( g_2 \) on v. and vii.

We shall consider a function \( g : [a, b] \to \mathbb{R} \) given by

\[ (7) \quad g(x) = x - \lambda f(x) \]

where \( \lambda \in \mathbb{R} \).

If \( g'(x) \leq 0 \), then \( g \) is decreasing. This attracts

\[ 1 - \lambda f'(x) < 0 \]

i.e.,

\[ (8) \quad \lambda > \frac{1}{f'(x)} \]

But since \( f'(x) > 0 \) and \( f''(x) \geq 0 \), we obtain

\[ f'(a) \leq f'(x) \leq f'(b) \]

or

\[ (9) \quad \frac{1}{f'(a)} \geq \frac{1}{f'(x)} \geq \frac{1}{f'(b)}. \]

By (8), it is obvious that if \( \lambda > \frac{1}{f'(a)} \), then (8) is verified for every \( x \in [a, b] \), from which \( g \) is decreasing.

In order to obtain \( g(x_0) \leq b \), we need that \( x_0 - \lambda(x_0) \leq b \), which leads to

\[ \lambda \leq \frac{x_0 - b}{f(x_0)} \]

which in turn holds if \( x_0 \) is sufficiently close to \( x^* \).

Let \( \lambda_i, i = 1, 2, \lambda_1 \neq \lambda_2 \) be two numbers such that

\[ \frac{1}{f'(a)} \leq \lambda_i \leq \frac{x_0 - b}{f(x_0)} \]

then functions \( g_i, i = 1, 2 \) given by

\[ g_i(x) = x - \lambda_i f(x) \]

verify hypotheses v. and vii.
4. NUMERICAL EXAMPLE

Consider the equation
\[ f(x) = x - 2 \cos x = 0, \]
with \( x \in \left[ \frac{\pi}{6}, \frac{\pi}{2} \right] \).

Obviously, \( f'(x) > 0 \), for all \( x \in \left[ \frac{\pi}{6}, \frac{\pi}{2} \right] \), and \( f''(x) > 0 \), \( x \in \left[ \frac{\pi}{6}, \frac{\pi}{2} \right] \). If in (7) we take \( \lambda_1 = 0.5, \lambda_2 = 0.6 \) and \( x_0 = \frac{\pi}{6} \), then \( g_1 \) and \( g_2 \) verify v. and vii.

We have:
\[
\begin{align*}
g_1(x) &= \cos x + \frac{x^2}{2}, \\
g_2(x) &= 6 \cos x + \frac{x^2}{5}.
\end{align*}
\]

Obviously, \( g_1 \left( \frac{\pi}{6} \right) < \frac{\pi}{2} \) and \( g_2 \left( \frac{\pi}{6} \right) < \frac{\pi}{2} \). Function \( E_f \) is given by
\[
E_f(x) = 4 + 8 \cos^2 x + 2 \sin x > 0, \forall x \in \left[ \frac{\pi}{6}, \frac{\pi}{2} \right].
\]

In the table below we have obtained the following results.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
<th>( g_1(x_n) )</th>
<th>( g_2(x_n) )</th>
<th>( f(x_n) )</th>
</tr>
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<td>0.5235987755982988</td>
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<td>(-1.208452031970579 \cdot 10^{-15})</td>
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<tr>
<td>2</td>
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<td>1.030632925047758</td>
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<td>(-5.830220460.33369 \cdot 10^{-3})</td>
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<td>3</td>
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<td>1.029866529322259</td>
<td>1.029866529322259</td>
<td>0</td>
</tr>
</tbody>
</table>

The numerical results confirm that inequalities
\[ 0 \leq x^* - x_4 < 10^{-15} \]
hold.

REFERENCES


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