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UNSTEADY BOUNDARY LAYER FLOW AND HEAT TRANSFER OVER A STRETCHING SHEET: HEAT FLUX CASE[‡]

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Abstract. Unsteady two-dimensional boundary layer flow and heat transfer over a stretching flat plate in a viscous and incompressible fluid of uniform ambient temperature is investigated in this paper. It is assumed that the velocity of the stretching sheet and the heat flux from the surface of the plate vary inverse proportional with time. Two equal and opposite forces are impulsively applied along the plate so that the plate is stretched keeping the origin fixed. Using appropriate similarity variables, the basic partial differential equations are transformed into a set of two ordinary differential equations. These equations are solved numerically for some values of the governing parameters using the Runge-Kutta method of fourth order. Flow and heat transfer characteristics are determined and represented in some tables and figures. It is found that the structure of the boundary layer depends on the ratio of the velocity of the potential flow near the stagnation point to that of the velocity of the stretching surface. In addition, it is shown that the heat transfer from the plate increases when the Prandtl number increases. The present results to include also the steady situation as a special case considered by other authors. Comparison with known results is very good.

MSC 2000. 65H05.
Keywords. Heat transfer, unsteady flow, stretching surface, boundary layer, stagnation point flow, numerical results.

1. INTRODUCTION

The study of unsteady boundary layer flow is important in several physical problems in aero-nautics, missile dynamics, acoustics etc. The work in this area was initiated by Moore [12], Lighthill [5] and Lin [7]. Reviews of unsteady boundary layers were presented by Stuart [17], Riley [15], Telionis [22], [23] and Pop [14]. In recent years certain aspects of the unsteady flows were investigated by Ma and Hui [9] and Ludlow et al. [8] using the classical method of Lie-group. The essence of the Lie-group method is that each of the

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variables in the initial equation is subjected to an infinitesimal transformation and the demand that the equation is invariant under these transformations leads to the determination of the possible symmetries (see Ludlow et al. [8]). The fundamental governing equations of fluid mechanics are the Navier-Stokes equations. This inherently nonlinear set of partial differential equations has no general solution, and only a small number of exact solutions have been found (see Wang [20]). Exact solutions are important for the following reasons; (i) the solutions represent fundamental fluid-dynamic flows. Also, owing to the uniform validity of exact solutions, the basic phenomena described by the Navier-Stokes equations can be more closely studied. (ii) The exact solutions serve as standards for checking the accuracies of the many approximate methods, whether they are numerical, asymptotic, or empirical.

Flow of a viscous fluid over a stretching sheet has an important bearing on several technological processes. In particular in the extrusion of a polymer in a melt-spinning process, the extruded from the die is generally drawn and simultaneously stretched into a sheet which is then solidified through quenching or gradual cooling by direct contact with water. Further, glass blowing, continuous casting of metals and spinning of fibres involve the flow due to a stretching surface, see Lakshmisha et al. [4]. In all these cases, a study of the flow field and heat transfer can be of significant importance since the quality of the final product depends to a large extent on the skin friction coefficient and the surface heat transfer rate. Crane [2] presented a simple closed form exponential solution of the steady two-dimensional flow caused solely by a linearly stretching sheet in an otherwise quiescent incompressible fluid. The simplicity of the geometry and the possibility of obtaining further exact solutions through simple generalizations have generated a lot of interest in extending it to more general situations. Such extensions include consideration of more general stretching velocity, application to non-Newtonian fluids, and inclusion of other physical effects such as suction or blowing, magnetic fields, etc. Unsteady two-dimensional boundary layer flow over a stretching surface has been studied by Na and Pop [13], Wang et al. [21], Elbashbeshy and Badiz [3], Sharidan et al. [16] and Ali and Magyari [1], while Lakshmisha et al. [20], Devi et al. [18] and Takhar et al. [19] have considered the unsteady three-dimensional flow due to the impulsive motion of a stretching surface.

The aim of this analysis is to study the unsteady flow and heat transfer in the stagnation-point flow on a stretching surface in a viscous and incompressible fluid when the sheet is stretched in its own plane with a velocity proportional to the distance from the stagnation-point and inversely with time and the velocity of the external flow (inviscid fluid) is also proportional with the distance along the plate and inversely with time. The geometry is similar to that proposed by Mahapatra and Gupta [10], [11] for the steady two-dimensional stagnation-point flow towards a stretching sheet. The parabolic partial differential equations governing the flow and heat transfer have been reduced to a system of two ordinary differential equations which are solved using an implicit finite-differential scheme in combination with the shooting method.

2. PROBLEM FORMULATION

We consider the unsteady two-dimensional forced convection flow and heat transfer of a viscous and incompressible fluid near a stagnation point at a surface coinciding with the plain $y = 0$, the flow being confined to $y > 0$. Two equal and opposite forces are applied along the x - axes at the time $t = 0$, so that the surface is stretched keeping the origin fixed as shown in Fig.1. It is assumed that viscous dissipation effects are neglected. Under these assumptions, the system of unsteady boundary layer equations are given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial u_e}{\partial t} + u_e\frac{du_e}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{3}$$

subject to the initial and boundary conditions of the form:

$$t \geq 0: \ u = u_w(t,x), \ v = 0, \ \frac{\partial T}{\partial y} = -\frac{q_w(t)}{k} \text{ for } y = 0 \tag{4}$$

$$u \to u_e(t,x), \ T \to T_\infty \text{ as } y \to \infty \tag{5}$$

where u and v are the velocity components along the $x-$ and $y-$ axes, T is the fluid temperature, $q_w(t)$ is the heat flux at the plate, ν is the kinematic viscosity, α is the thermal diffusivity and k is the thermal conductivity. Following Takhar et al. [19] we assume that $u_w(t,x)$, $u_e(t,x)$ and $q_w(t)$ are given by

$$u_w(t,x) = \frac{c}{1-\alpha t}x, \ u_e(t,x) = \frac{a}{1-\alpha t}x, \ q_w(t) = \frac{q_{w0}}{(1-\alpha t)^{1/2}} \tag{6}$$

where c and a are positive constants. Equations (1)-(3) can be transformed to the corresponding ordinary differential equations using the following similarity variables:

$$\psi = \left(\frac{c\nu}{1-\alpha t}\right)^{1/2} x f(\eta), \tag{7}$$

$$\theta(\eta) = (T - T_\infty)\frac{k}{q_{w0}}\left(\frac{c}{\nu}\right)^{1/2}, \tag{8}$$

$$\eta = \left(\frac{c}{\nu(1-\alpha t)}\right)^{1/2} y \tag{9}$$

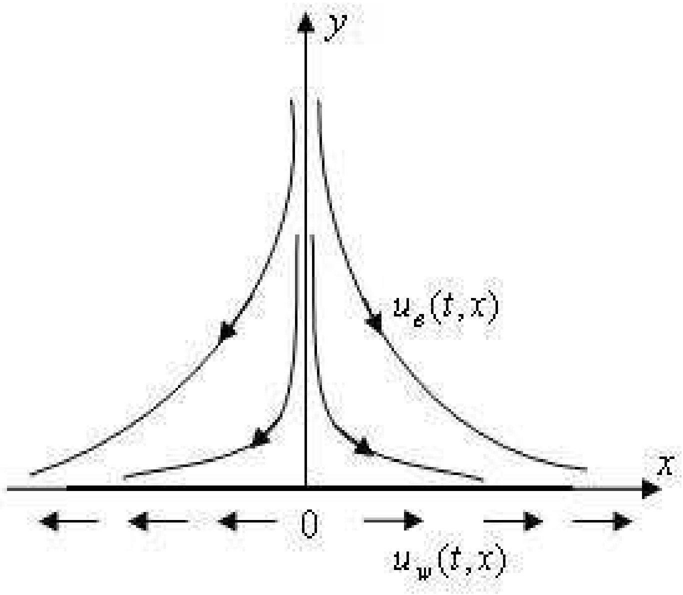

Fig. 1. Physical model and coordinate system.

where q_{w0} is the characteristic heat flux and ψ is the stream function which is defined in the usual way as $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$.
Substituting (6) into Eqs. (2) and (3), we obtain the following set of two ordinary differential equations:

$$f''' + f\, f'' + \frac{a^2}{c^2} - f'^{\,2} + \frac{\alpha}{c}\left(\frac{a}{c} - f' - \frac{\eta}{2} f''\right) = 0 \tag{10}$$

$$\frac{1}{Pr}\theta'' + f\,\theta' - \frac{\alpha}{2c}\eta\theta' = 0 \tag{11}$$

subject to the boundary conditions (4) which become

$$f(0) = 0, f'(0) = 1, \theta'(0) = -1 \tag{12}$$

$$f'(\infty) = \frac{a}{c}, \theta(\infty) = 0 \tag{13}$$

where Pr is the Prandtl number and primes denote differentiation with respect to η.
The physical quantities of interest are the skin friction coefficient $C_f = \frac{\tau_w}{\rho u_{ws}^2}$ and the temperature of the wall T_w, where $u_{ws}(x) = cx$, and τ_w is the skin

friction, given by

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} \tag{14}$$

with μ being the dynamic viscosity. Using (6), we get

$$(1-\alpha t)^{-1/2} Re_x^{1/2} C_f = f''(0), \tag{15}$$

$$(T_w - T_\infty)\frac{q_{w0}}{k}\left(\frac{\nu}{c}\right)^{1/2} = \theta(0) \tag{16}$$

Where $Re = (cx)x/\nu$ is the local Reynolds number. It is important to notice that for the steady-state, Eqs. (7) and (8) reduce to

$$f''' + f\, f'' - f'^{\,2} + \frac{a^2}{c^2} = 0 \tag{17}$$

$$\frac{1}{Pr}\theta'' + f\,\theta' = 0 \tag{18}$$

with the boundary conditions

$$f(0) = 0, f'(0) = 1, f'(\infty) = \frac{a}{c} \tag{19}$$

$$\theta'(0) = -1, \theta(\infty) = 0 \tag{20}$$

On the other hand, Eq. (18) subject to the boundary condition (20) for θ is given by

$$\theta(\eta) = \int_0^\infty \exp\Big(-Pr\int_0^\eta f(t)\mathrm{d}t\Big)\mathrm{d}\eta - \int_0^\eta \exp\Big(-Pr\int_0^s f(t)\mathrm{d}t\Big)\mathrm{d}s \tag{21}$$

3. SOLUTION

The systems of ordinary differential equations (10)–(11) subject to the boundary condition(12)–(13) and (17)–(18) subject to the boundary condition (19)–(20) have been solved numerically for some values of the parameters a/c and Pr when $\alpha = -1$ using Runge-Kutta method of fourth order combined with the shooting technique. Some values of $f''(0)$ obtained by solving Eq.(17), which correspond to the steady-state flow case are given in Table 1 for different values of the parameter a/c. The values reported by Mahapatra and Gupta [10] are also included in this table. We can see that there is a very good agreement between our results and those obtained by Mohapatra and Gupta [10].

Figs.(2)–(4) show the velocities profiles f' and f for the case of unsteady flow. The values of the parameters are $a = \{3,\ 2,\ 0.5,\ 0.2,\ 0.1\}$, $c = 1$. The corresponding streamlines for these two solutions are presented in Figs.(6) and (7) for the time step $t = \{0,\ 1,\ 2,\ 3\}$. It is interesting to notice that the solution of Eq. (10) is not unique. Thus, there are two solutions, one (2) and

Table 1. Values of $f''(0)$ for some values of a/c when the flow is steady; ( ) values reported by Mohapatra and Gupta [10].

a/c	0.10	0.20	0.50	2.00
$f''(0)$	-0.9696	-0.9182	-0.6673	2.0175
$f''(0)$	(-0.9694)	(-0.9181)	(-0.6673)	(2.0175)

(3) representing an attached flow and the authors (4) and (5) representing the reversed flow, respectively. This is in agreement with the results obtained by Ma and Hui [9] for the unsteady two-dimensional boundary layer flow near a stagnation point on a fixed flat plate. Therefore, we are confident that the present results are accurate.

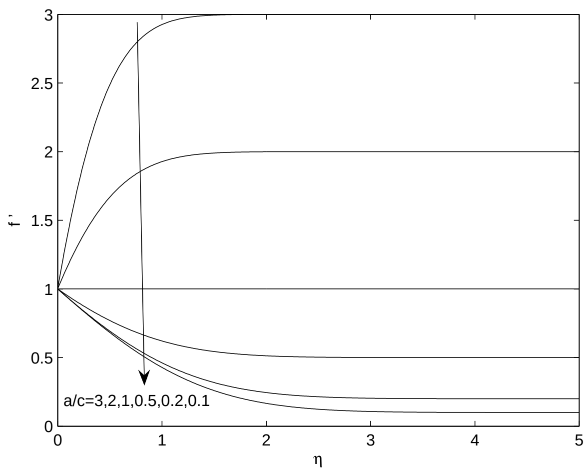

Fig. 2. The first solution of f' for some valuers of a/c.

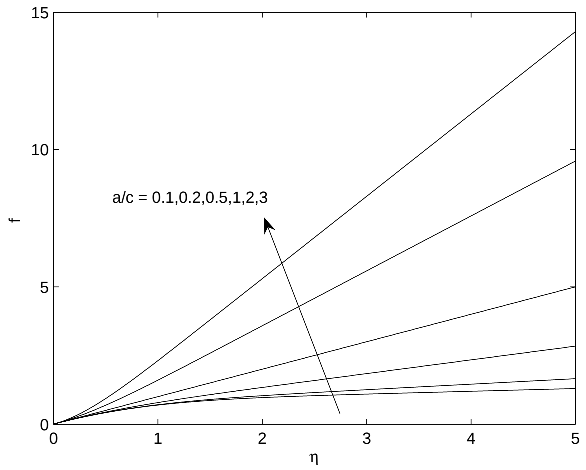

Fig. 3. The first solution of f for some valuers of a/c.

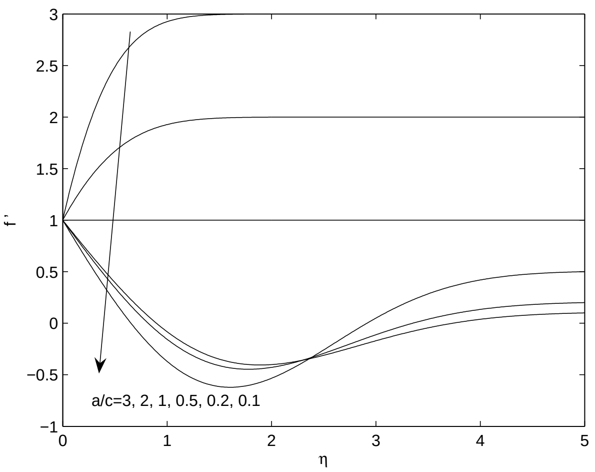

Fig. 4. The second solution of f' for some valuers of a/c.

Some values of the wall temperature $\theta(0) = \theta_w$ described by Eq.(18) for different values of the Prandtl number (Pr) are given in Table 2. Also, the variation of θ_w with Pr in this case, is shown in Fig. 8 when $a = 0$, and $c = 1$. It is seen that θ_w decreases with the increasing of Pr, which is in agreement with the results reported by Lin and Chen [6]. Further, Figs. 9 and 10 show the temperature profiles $\theta(\eta)$ given by Eq. (11) for $a/c = 2$ respectively, $a/c = 0.1$ and for different values of Pr. It is evident from these figures that an increase in Pr results in a decrease in the thermal boundary layer thickness.

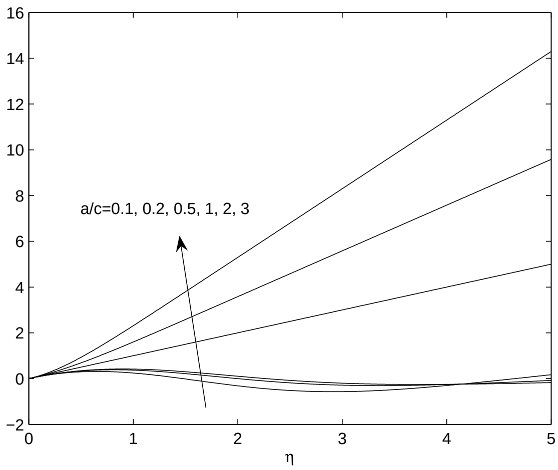

Fig. 5. The second solution of f for some valuers of a/c.

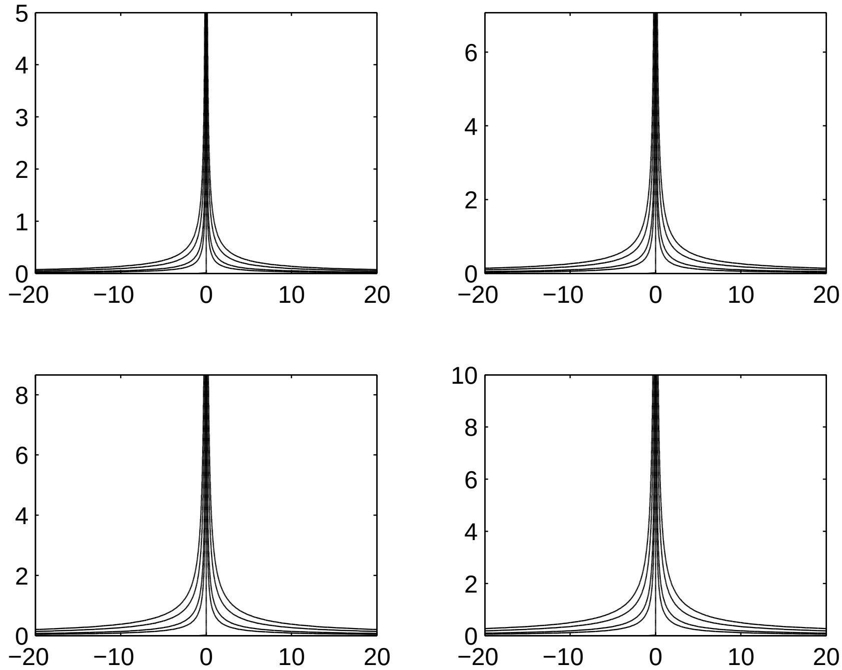

Fig. 6. The streamlines for: $t = 0, 1, 2, 3$ corresponding to the first solution.

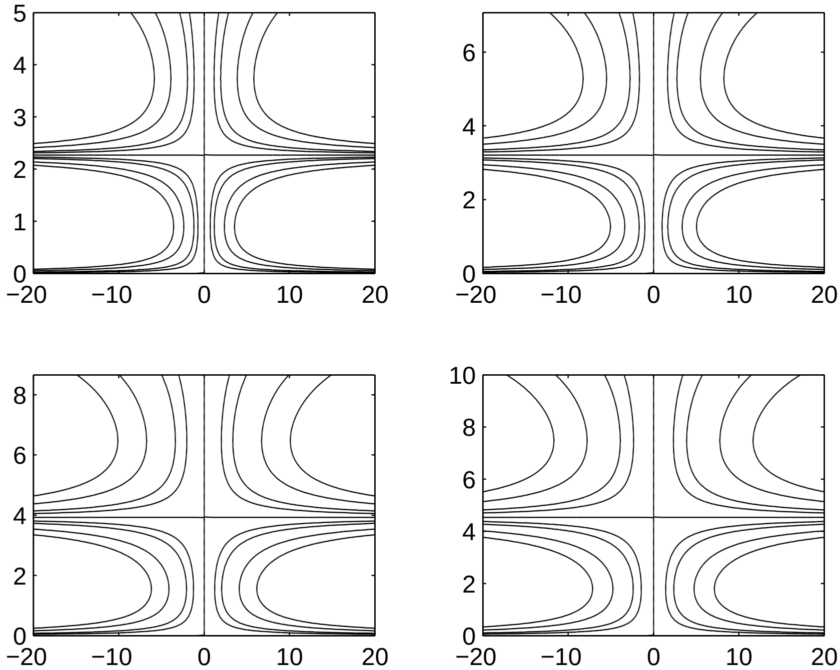

Fig. 7. The streamlines for: $t = 0, 1, 2, 3$ corresponding to the second solution.

θ_w	
Pr	$Numerical$
0.05	4.1277
0.5	1.7717
0.72	1.4770
1	1.2533
1.5	1.0233

4. CONCLUSION

The unsteady two-dimensional stagnation-point flow and heat transfer of an incompressible fluid over a stretching flat plate in its own plane has been numerically analyzed in detailed. The case of variable heat flux from the wall is considered. Following Takhar [19] similarity variables where used to reduce the governing partial differential equations to ordinary differential equations. Solving numerically these equations, we have been able to determine the velocity, temperature profile, skin friction and temperature at the wall. For the case of steady-state flow, we have compared our results with those of Mahapatra [10]. The agreement between the results is very good. Effects of a/c and Pr

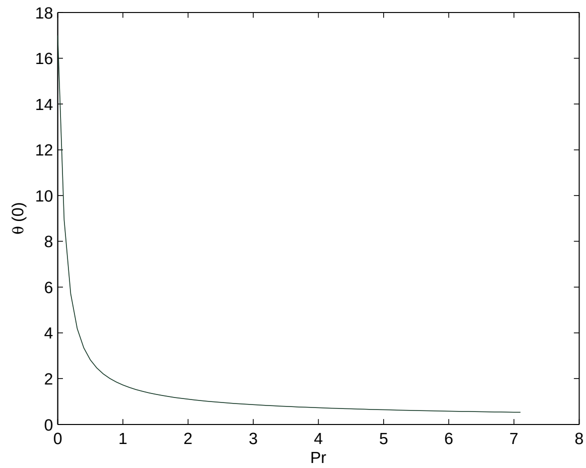

Fig. 8. The solution of $\theta(0)$ in respect with Prandtl number for the particular equation.

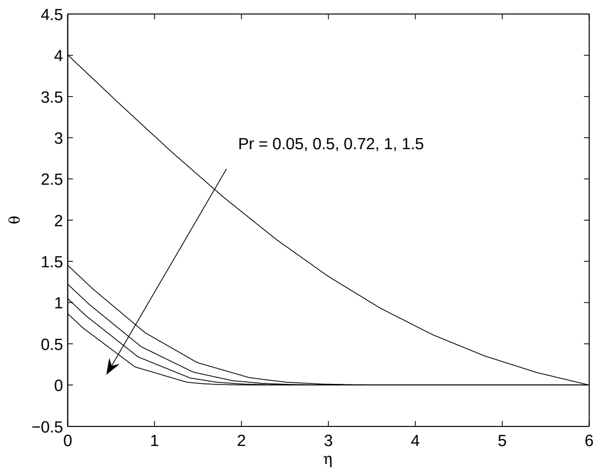

Fig. 9. Temperature profiles of $\theta(\eta)$ for several values of Pr and $a/c = 2$ in respect with η.

on the flow and heat transfer characteristic have been examined and discussed in detail. It is shown that for small values of a/c the solution of the ordinary differential equation is not unique. One solution represents an attached flow and the other one a reversed flow. It is worth mentioning that solutions of the problem for more values of the governing parameters have been determined. However, in order to save space we have present results here only for some values of these parameters.

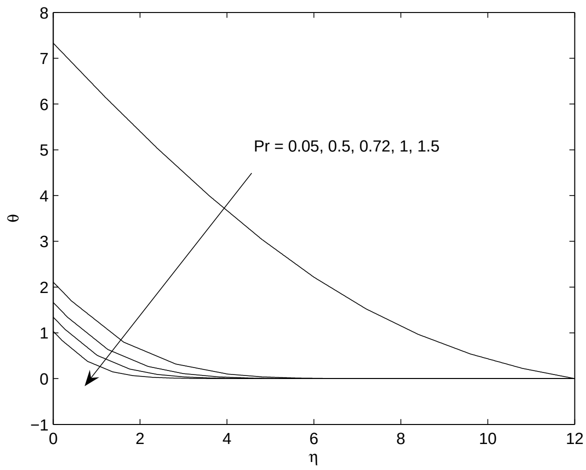

Fig. 10. Temperature profiles of $\theta(\eta)$ for several values of Pr and $a/c = 0.1$ in respect with η.

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