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REMARKS ABOUT AN INEQUALITY USED FOR THE EQUIVALENCE BETWEEN THE CONVERGENCE OF ISHIKAWA AND MANN ITERATIONS

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Abstract. We prove an inequality which is crucial in the proof of the main result from B.E. Rhoades and Ştefan M. Şoltuz, *The Equivalence between the Convergences of Ishikawa and Mann Iterations for an Asymptotically Pseudocontractive Map*, J. Math. Anal. Appl., **283** (2003), 681–688.

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1. INTRODUCTION

Let X be a real Banach space, B a nonempty, convex subset of X, and $T : B \to B$ an operator. Let $u_0, x_0 \in B$. We consider the following iteration, see [3]:

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n T^n u_n. \tag{1}$$

The sequence $(\alpha_n)_n \subset (0, 1)$ satisfies $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$. This iteration is known as the *modified Mann iteration*. We now consider the following iteration, known as the *modified Ishikawa iteration* see [1]:

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T^n y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T^n x_n. \end{aligned} \tag{2}$$

The sequences $(\alpha_n)_n \subset (0, 1)$, $(\beta_n)_n \subset [0, 1)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \ \lim_{n\to\infty} \beta_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty. \tag{3}$$

Let X be an arbitrary real Banach space and $J : X \to 2^{X^*}$ the *normalized duality mapping* given by

$$Jx := \{f \in X^* : \langle x, f\rangle = \|x\|, \|f\| = \|x\|\}, \quad \forall x \in X. \tag{4}$$

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In [5] the following class of maps was introduced

Definition 1. *A map T is said to be asymptotically pseudocontractive if there exists a sequence $(k_n)_n$, $k_n \in [1,\infty), \forall n \in \mathbb{N}$, $\lim_{n\to\infty} k_n = 1$, and there exists $j(x-y) \in J(x-y)$ such that*

$$\langle T^n x - T^n y, j(x-y)\rangle \leq k_n \|x-y\|^2, \quad \forall x,y \in B, \forall n \in \mathbb{N}. \tag{5}$$

The aim of this note is to prove a crucial inequality which was used in [4] in order to prove the main result. We recall the following auxiliary results.

Lemma 2. [2] *Let X be a Banach space and $x, y \in X$. Then*

$$\|x\| \leq \|x + ry\|, \tag{6}$$

for all $r > 0$ if and only if there exists $j(x) \in J(x)$ such that $\langle y, j(x)\rangle \geq 0$.

Lemma 3. [4] *Let B be a nonempty subset of a Banach space X and let $T : B \to B$ be a map. Then the following conditions are equivalent:*

(i) *T is an asymptotically pseudocontractive map,*
(ii) *for $k_n \in [1,\infty), \forall n \in \mathbb{N}$, we have*

$$\|x-y\| \leq \|x-y+r[(k_nI-T^n)x-(k_nI-T^n)y]\|, \quad \forall x,y \in B, \forall r > 0. \tag{7}$$

Definition 4. *Let X be a normed space and B a subset of X, then the map $T : B \to B$ is an uniformly Lipschitzian map if for some $L \geq 1$, we have $\|T^n x - T^n y\| \leq L\|x-y\|$, $\forall x,y \in B, \forall n \in \mathbb{N}$.*

2. MAIN RESULT

We recall the following Theorem.

Theorem 5. [4] *Let B be a closed convex subset of an arbitrary Banach space X, $(x_n)_n$ and $(u_n)_n$ defined by (2) and (1) with $(\alpha_n)_n$ and $(\beta_n)_n$ satisfying (3). Let T be an asymptotically pseudocontractive and uniformly Lipschitzian with $L \geq 1$ self-map of B. Let x^* be a fixed point of T. If $u_0 = x_0 \in B$, then the following two assertions are equivalent:*

(i) *the modified Mann iteration (1) converges strongly to x^*,*
(ii) *the modified Ishikawa iteration (2) converges strongly to x^*.*

In the proof of the above Theorem a less obvious inequality (i.e (8)) was used. Note that, in this case, the triangle inequality is not useful. In order to eliminate any doubts, we shall prove it here.

Proposition 6. *Let all assumptions of Theorem* 5 *to be satisfied, then the following inequality (used for proving Theorem* 5) *is true,*

(8)

$$||(1+\alpha_n^2)\,(x_{n+1}-u_{n+1})+\alpha_n\,((\alpha_n k_n I-T^n)x_{n+1}-(\alpha_n k_n I-T^n)u_{n+1})\,||$$
$$+\,[1-(1+k_n)\,\alpha_n]\,\alpha_n\,\,\|x_n-u_n\|-(1+k_n)\,\alpha_n^3\,\,\|x_n-u_n-T^n y_n+T^n u_n\|$$
$$-\,\alpha_n\,\,\|T^n x_{n+1}-T^n u_{n+1}-T^n y_n+T^n u_n\|$$
$$\geq(1+\alpha_n^2)\,\|x_{n+1}-u_{n+1}\|+[1-(1+k_n)\,\alpha_n]\,\alpha_n\,\,\|x_n-u_n\|-(1+k_n)\,\alpha_n^3$$
$$\cdot\,\|x_n-u_n-T^n y_n+T^n u_n\|-\alpha_n\,\,\|T^n x_{n+1}-T^n u_{n+1}-T^n y_n+T^n u_n\|\,.$$

Proof. We have

(9)

$$x_n=x_{n+1}+\alpha_n x_n-\alpha_n T^n y_n=(1+\alpha_n^2)\,x_{n+1}+\alpha_n\,(\alpha_n\,k_n I-T^n)\,x_{n+1}$$
$$-\,(1+k_n)\,\alpha_n^2\,x_{n+1}+\alpha_n x_n+\alpha_n\,(T^n x_{n+1}-T^n y_n)$$
$$=(1+\alpha_n^2)\,x_{n+1}+\alpha_n\,(\alpha_n\,k_n I-T^n)\,x_{n+1}$$
$$-\,(1+k_n)\,\alpha_n^2\,[\,x_n+\alpha_n\,(T^n y_n-x_n)]+\alpha_n x_n+\alpha_n\,(T^n x_{n+1}-T^n y_n)$$
$$=(1+\alpha_n^2)\,x_{n+1}+\alpha_n\,(\alpha_n\,k_n I-T^n)\,x_{n+1}-(1+k_n)\,\alpha_n^2\,x_n$$
$$+\,(1+k_n)\,\alpha_n^3\,\,(x_n-T^n y_n)+\alpha_n x_n+\alpha_n\,(T^n x_{n+1}-T^n y_n)$$
$$=(1+\alpha_n^2)\,x_{n+1}+\alpha_n\,(\alpha_n\,k_n I-T^n)\,x_{n+1}+[1-(1+k_n)\,\alpha_n]\,\alpha_n\,x_n$$
$$+\,(1+k_n)\,\alpha_n^3\,\,(x_n-T^n y_n)+\alpha_n\,(T^n x_{n+1}-T^n y_n).$$

Also

(10)

$$u_n=u_{n+1}+\alpha_n u_n-\alpha_n T^n u_n=(1+\alpha_n^2)\,u_{n+1}+\alpha_n\,(\alpha_n\,k_n I-T^n)\,u_{n+1}$$
$$-\,(1+k_n)\alpha_n^2\,u_{n+1}+\alpha_n u_n+\alpha_n(T^n u_{n+1}-T^n u_n)$$
$$=(1+\alpha_n^2)\,u_{n+1}+\alpha_n\,(\alpha_n\,k_n I-T^n)\,u_{n+1}$$
$$-\,(1+k_n)\alpha_n^2\,[u_n+\alpha_n(T^n u_n-u_n)]+\alpha_n u_n+\alpha_n\,(T^n u_{n+1}-T^n u_n)$$
$$=(1+\alpha_n^2)\,u_{n+1}+\alpha_n\,(\alpha_n\,k_n I-T^n)\,u_{n+1}+(1+k_n)\,\alpha_n^3\,(u_n-T^n u_n)$$
$$+\,[1-(1+k_n)\,\alpha_n]\,\alpha_n\,u_n+\alpha_n\,(T^n u_{n+1}-T^n u_n).$$

From (9) and (10) we get

(11)

$$x_n-u_n=$$
$$=(1+\alpha_n^2)(x_{n+1}-u_{n+1})+\alpha_n\,\,((\alpha_n\,k_n I-T^n)x_{n+1}-(\alpha_n\,k_n I-T^n)u_{n+1})$$
$$+\,[1-(1+k_n)\,\alpha_n]\,\alpha_n\,(x_n-u_n)+(1+k_n)\,\alpha_n^3\,(x_n-u_n-T^n y_n+T^n u_n)$$
$$+\,\alpha_n\,(T^n x_{n+1}-T^n u_{n+1}-T^n y_n+T^n u_n).$$

The norm of the sum of the first two terms on the right hand side of (11) is equal to

(12)

$$(1+\alpha_n^2)\ \left\|(x_{n+1}-u_{n+1})+\tfrac{\alpha_n}{1+\alpha_n^2}\ \left((\alpha_n\ k_nI-T^n)x_{n+1}-(\alpha_n\ k_nI-T^n)u_{n+1}\right)\right\|.$$

Using (6) with

$$x:=(x_{n+1}-u_{n+1}),\qquad (13)$$
$$y:=\tfrac{\alpha_n}{1+\alpha_n^2}\ \left((\alpha_n\ k_nI-T^n)x_{n+1}-(\alpha_n\ k_nI-T^n)u_{n+1}\right),$$

we obtain

(14)

$$\left\|(1+\alpha_n^2)(x_{n+1}-u_{n+1})+\alpha_n\ \left((\alpha_n\ k_nI-T^n)x_{n+1}-(\alpha_n\ k_nI-T^n)u_{n+1}\right)\right\|$$
$$\geq(1+\alpha_n^2)\left\|x_{n+1}-u_{n+1}\right\|$$

From (11) it follows that

(15)

$$\begin{aligned}&\|x_n-u_n\|\\ &\geq\left\|(1+\alpha_n^2)\ (x_{n+1}-u_{n+1})+\alpha_n\left((\alpha_nk_nI-T^n)x_{n+1}-(\alpha_nk_nI-T^n)u_{n+1}\right)\right\|\\ &\quad+[1-(1+k_n)\ \alpha_n]\ \alpha_n\ \|x_n-u_n\|-(1+k_n)\ \alpha_n^3\ \|x_n-u_n-T^ny_n+T^nu_n\|\\ &\quad-\alpha_n\ \|T^nx_{n+1}-T^nu_{n+1}-T^ny_n+T^nu_n\|\\ &\geq^{(?)}(1+\alpha_n^2)\|x_{n+1}-u_{n+1}\|+[1-(1+k_n)\ \alpha_n]\ \alpha_n\ \|x_n-u_n\|-(1+k_n)\ \alpha_n^3\\ &\quad\cdot\|x_n-u_n-T^ny_n+T^nu_n\|-\alpha_n\ \|T^nx_{n+1}-T^nu_{n+1}-T^ny_n+T^nu_n\|.\end{aligned}$$

To prove the last inequality we need the following notations:

$$\begin{aligned}a&=\left(1+\alpha_n^2\right)(x_{n+1}-u_{n+1}),\\ a'&=\left(1+\alpha_n^2\right)(x_{n+1}-u_{n+1})\\ &\qquad+\alpha_n\left((\alpha_nk_nI-T^n)\,x_{n+1}-(\alpha_nk_nI-T^n)\,u_{n+1}\right),\\ b&=(1-(1+k_n)\,\alpha_n)\,\alpha_n\,\|x_n-u_n\|,\\ c&=(1+k_n)\,\alpha_n^3\,(x_n-u_n-T^ny_n+T^nu_n),\\ d&=\alpha_n\,(T^nx_{n+1}-T^nu_{n+1}-T^ny_n+T^nu_n).\end{aligned}\qquad(16)$$

We shall prove

$$\begin{aligned}\|a'+b+c+d\|+\|c\|+\|d\|&\geq\|a'+b+c+d\|+\|c+d\|\geq^{(?)}\\ &\geq^{(?)}\|a'\|+\|b\|\geq\|a\|+\|b\|.\end{aligned}\qquad(17)$$

The last inequality is true because we know from Lemma 2 that $\|a'\| \geq \|a\|$. By use of

$$x_{n+1} = x_n - \alpha_n x_n + \alpha_n T^n y_n, \qquad (18)$$
$$u_{n+1} = u_n - \alpha_n u_n + \alpha_n T^n u_n,$$

we obtain:

$$\begin{aligned}
&\|a'\| = \\
&= \left\|\left(1+\alpha_n^2\right)(x_{n+1}-u_{n+1}) + \alpha_n\left((\alpha_n k_n I - T^n)\,x_{n+1} - (\alpha_n k_n I - T^n)\,u_{n+1}\right)\right\| \\
&= \left\|\left(1+\alpha_n^2\right)(x_{n+1}-u_{n+1}) + k_n\alpha_n^2\,(x_{n+1}-u_{n+1}) - \alpha_n\,(T^n x_{n+1} - T^n u_{n+1})\right\| \\
&= ||\,(1+k_n)\,\alpha_n^2\,(-\alpha_n x_n + \alpha_n T^n y_n + \alpha_n u_n - \alpha_n T^n u_n) \\
&\quad + (1+k_n)\,\alpha_n^2\,(x_n - u_n) + (x_{n+1}-u_{n+1}) - \alpha_n\,(T^n x_{n+1} - T^n u_{n+1})\,|| \\
&= ||\,(1+k_n)\,\alpha_n^3\,(-x_n + T^n y_n + u_n - T^n u_n) - \alpha_n\,(T^n x_{n+1} - T^n u_{n+1}) \\
&\quad + (1+k_n)\,\alpha_n^2\,(x_n-u_n) + (x_n-u_n) + \alpha_n\,(-x_n + T^n y_n + u_n - T^n u_n)\,|| \\
&= ||\,(1+k_n)\,\alpha_n^3\,(-x_n + T^n y_n + u_n - T^n u_n) \\
&\quad - \alpha_n\,(T^n x_{n+1} - T^n u_{n+1} - T^n y_n + T^n u_n) \\
&\quad + \left(1 - \alpha_n + (1+k_n)\,\alpha_n^2\right)(x_n - u_n)\,|| \\
&\leq ||-\left((1+k_n)\,\alpha_n^3\,(x_n - T^n y_n - u_n + T^n u_n)\right. \\
&\quad \left. +\alpha_n\,(T^n x_{n+1} - T^n u_{n+1} - T^n y_n + T^n u_n)\right)|| \\
&\quad + \left(1 - \alpha_n + (1+k_n)\,\alpha_n^2\right)\|x_n - u_n\| \\
&= ||\,(1+k_n)\,\alpha_n^3\,(x_n - T^n y_n - u_n + T^n u_n) \\
&\quad + \alpha_n\,(T^n x_{n+1} - T^n u_{n+1} - T^n y_n + T^n u_n)\,|| \\
&\quad - (1 - (1+k_n)\,\alpha_n)\,\alpha_n\,\|x_n - u_n\| + \|x_n - u_n\| \\
&= \|c+d\| - \|b\| + \|x_n - u_n\| = \|c+d\| - \|b\| + \|a' + b + c + d\|. \qquad \square
\end{aligned}$$

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