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CLASSICAL RESULTS VIA MANN–ISHIKAWA ITERATION[‡]

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Abstract. New proofs of existence and uniqueness results for the solution of the Cauchy problem with delay are obtained by use of Mann–Ishikawa iteration.

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1. INTRODUCTION

Consider the following delay differential equation

$$x'(t) = f(t, x(t), x(t-\tau)),\ t \in [t_0, b] \tag{1}$$

with $t_0, b, \tau \in \mathbb{R},\ \tau > 0,\ f \in C([t_0, b] \times \mathbb{R}^2, \mathbb{R})$.

The existence of an approximative solution for equation (1) is given by theorem 1 from [1] (see also [2], [6], [5]). The proof of this theorem is based on the contraction principle. We shall prove it here by applying Mann iteration.

In the last decades, numerous papers were published on the iterative approximation of fixed points of contractive type operators in metric spaces, see for example [7], [8]. The Mann iteration [4] and the Ishikawa iteration [3] are certainly the most studied of these fixed point iteration procedures.

Let X be a real Banach space and $T : X \to X$ a given operator, let $u_0, x_0 \in X$.

The Mann iteration is defined by

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n T u_n. \tag{2}$$

The Ishikawa iteration is defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \end{cases} \tag{3}$$

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where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ and both sequences satisfy

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0,\ \sum_{n=0}^{\infty} \alpha_n = \infty.$$

2. FIXED POINT THEOREMS

We consider the delay differential equation

$$x'(t) = f(t, x(t), x(t-\tau)),\ t \in [t_0, b] \tag{4}$$

with initial condition

$$x(t) = \varphi(t),\ t \in [t_0 - \tau, t_0]. \tag{5}$$

We suppose that the following conditions are fulfilled

(H_1) $t_0, b \in \mathbb{R},\ \tau > 0$;
(H_2) $f \in C([t_0, b] \times \mathbb{R}^2, \mathbb{R})$;
(H_3) $\varphi \in C([t_0 - \tau, b], \mathbb{R})$;
(H_4) there exist $L_f > 0$ such that

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \le L_f(|u_1 - v_1| + |u_2 - v_2|),$$
$$\forall u_i, v_i \in \mathbb{R},\ i = 1, 2,\ t \in [t_0, b];$$

(H_5) $2L_f(b - t_0) < 1$.

By a solution of the problem (4)–(5) we mean the function $x \in C([t_0 - \tau, b], \mathbb{R}) \cap C^1([t_0, b], \mathbb{R})$.

The problem (4)–(5) is equivalent with the integral equation

$$x(t) = \begin{cases} \varphi(t), & t \in [t_0 - \tau, t_0], \\ \varphi(t_0) + \int_{t_0}^{t} f(s, x(s), x(s-\tau))\mathrm{d}s, & t \in [t_0, b]. \end{cases} \tag{6}$$

The operator $T : C([t_0 - \tau, b], \mathbb{R}) \to C([t_0 - \tau, b], \mathbb{R})$, is defined by

$$T(x)(t) = \begin{cases} \varphi(t), & t \in [t_0 - \tau, t_0], \\ \varphi(t_0) + \int_{t_0}^{t} f(s, x(s), x(s-\tau))\mathrm{d}s, & t \in [t_0, b], \end{cases}$$

and the Banach space $C[t_0, b]$ is embedded with Tchebyshev norm

$$d(y, z) = \max_{t_0 \le t \le b} |y(t) - z(t)|\,.$$

Applying contraction principle we have

THEOREM 1. [1] *We suppose conditions* (H_1)-(H_5) *are satisfied. Then the problem* (4)–(5) *has a unique solution in* $C([t_0-\tau, b], \mathbb{R}) \cap C^1([t_0, b], \mathbb{R})$. *Moreover, if* x^* *is the unique solution of the problem* (4)–(5), *then*

$$x^* = \lim_{n\to\infty} T^n(x) \text{ for any } x \in C([t_0 - \tau, b], \mathbb{R}).$$

The following lemma is well-known. For sake of completeness, we shall give a proof here.

LEMMA 2. *Let* $\{a_n\}$ *be a nonnegative sequence satisfying*

$$a_{n+1} \leq (1 - \alpha_n) a_n,$$

where $\{\alpha_n\} \subset (0,1)$, $\sum_{n=0}^{\infty} \alpha_n = \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

Proof. Use $1 - x \leq \mathrm{e}^{-x}$, $\forall x \in (0,1)$ to obtain $\prod_{k=1}^{n}(1-\alpha_k) \leq \mathrm{e}^{-\sum_{k=1}^{n}\alpha_k}$. Actually, one has

$$a_{n+1} \leq \prod_{k=1}^{n}(1-\alpha_k) a_1 \leq a_1 \mathrm{e}^{-\sum_{k=1}^{n}\alpha_k} \to 0.$$

□

3. MAIN RESULT

THEOREM 3. *We suppose that conditions* (H_1)-(H_5) *are satisfied. Then the problem* (4)–(5) *has a unique solution in* $C([t_0-\tau, b], \mathbb{R}) \cap C^1([t_0, b], \mathbb{R})$.

Proof. Consider Mann iteration

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n T u_n,$$

for the operator

$$Tu_n = \begin{cases} \varphi(t), & t \in [t_0 - \tau, t_0], \\ \varphi(t_0) + \int_{t_0}^{t} f(s, u_n(s), u_n(s-\tau))\mathrm{d}s, & t \in [t_0, b]. \end{cases}$$

Denote by $x^* := Tx^*$ the fixed point of T.

For $t \in [t_0 - \tau, t_0]$ we get

$$\|u_{n+1} - x^*\| \leq (1-\alpha_n)\|u_n - x^*\|.$$

Consider Lemma 2 to obtain

$$\lim_{n\to\infty} \|u_n - x^*\| = 0.$$

For $t \in [t_0, b]$ we have

$$\begin{aligned}
\|u_{n+1} - x^*\| = {} & \bigg\|(1-\alpha_n)u_n + \alpha_n(\varphi(t_0) + \int_{t_0}^{t} f(s, u_n(s), u_n(s-\tau))\mathrm{d}s) - \\
& - (1-\alpha_n)x^* + \alpha_n(\varphi(t_0) + \int_{t_0}^{t} f(s, x^*(s), x^*(s-\tau))\mathrm{d}s)\bigg\| \\
\leq {} & (1-\alpha_n)\|u_n - x^*\| + \alpha_n L_f (b - t_0)(|u_n(s) - x^*(s)| + \\
& + |u_n(s-\tau) - x^*(s-\tau)|) \\
\leq {} & (1 - \alpha_n + 2\alpha_n L_f(b-t_0))\|u_n - x^*\| \\
\leq {} & (1 - \alpha_n(1 - 2L_f(b-t_0))\|u_n - x^*\|.
\end{aligned}$$

Assumption (H_5) leads us to

$$1 - \alpha_n(1 - 2L_f(b - t_0) < 1.$$

We take $\alpha_n := \alpha_n(1 - 2L_f(b - t_0))$, $a_n := \|u_n - x^*\|$ and use Lemma 2 to obtain $\lim_{n\to\infty} \|u_n - x^*\| = 0$. □

THEOREM 4. [7] *Let X be a normed space, D a nonempty, convex, closed subset of X and $T : D \to D$ a contraction. If $u_0, x_0 \in D$, then the following are equivalent:*

(i) *the Mann iteration* (2) *converges to x^*;*
(ii) *the Ishikawa iteration* (3) *converges to x^*.*

REMARK 5. Because Mann iteration and Ishikawa iteration are equivalent, it is possible to consider Ishikawa iteration in order to prove Theorem 3. □

Set $\tau = 0$, in (4), to obtain the classical existence and uniqueness result, i.e. Theorem 6, for the Cauchy problem. This problem, see [1], [6], is proved by use of contraction principle.

Consider the following equation:

$$x'(t) = f(t, x(t)),\ t \in [t_0, b], \tag{7}$$

with initial condition

$$x(t_0) = \varphi_0. \tag{8}$$

We suppose that the following conditions are fulfilled

(H_1') $t_0, \varphi_0, b \in \mathbb{R}$;
(H_2') $f \in C([t_0, b] \times \mathbb{R}, \mathbb{R})$;
(H_3') there exist $L_f > 0$ such that

$$|f(t, u_1) - f(t, v_1)| \leq L_f(|u_1 - v_1|),\ \forall u_1, v_1 \in \mathbb{R},\ t \in [t_0, b];$$

(H_4') $L_f(b - t_0) < 1$.

Note that we have supplied here a new proof for the following result using Mann–Ishikawa iteration.

THEOREM 6. *We suppose conditions* (H_1')-(H_4') *are satisfied. Then the problem* (7)–(8) *has a unique solution in $C([t_0, b], \mathbb{R})$. Moreover, if x^* is the unique solution of the problem* (7)–(8)*, then*

$$x^* = \lim_{n\to\infty} T^n(x) \text{ for any } x \in C([t_0, b], \mathbb{R}).$$

REFERENCES

[1] COMAN, GH., PAVEL, G., RUS, I. and RUS, I. A., *Introduction in the theory of operatorial equation*, Ed. Dacia, Cluj-Napoca, 1976 (in Romanian).
[2] HARTMAN, P., *Ordinary differential equations*, John Wiley & Sons, Inc., New York, London, Sydney, 1964.
[3] ISHIKAWA, S., *Fixed points by a new iteration method*, Proc. Amer. Math. Soc., **44**, pp. 147–150, 1974.
[4] MANN, W. R., *Mean value in iteration*, Proc. Amer. Math. Soc., **4**, pp. 506–510, 1953.

[5] Otrocol, D., *Data dependence for the solution of a Lotka-Volterra system with two delays,* Mathematica, Tome **48** (71), 1, pp. 61–68, 2006.
[6] Rus, I. A., *Principles and applications of the fixed point theory,* Ed. Dacia, Cluj Napoca, 1979 (in Romanian).
[7] Şoltuz, Ş. M., *The equivalence of Picard, Mann and Ishikawa iteration dealing with quasi-contractive operators*, Math. Comm. **10**, pp. 81–88, 2005.
[8] Şoltuz, Ş. M., *An equivalence between the convergence of Ishikawa, Mann and Picard iterations*, Math. Comm. **8**, pp. 15–22, 2003.

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