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GENERAL CONVERGENCE OF THE METHODS FROM CHEBYSHEV-HALLEY FAMILY

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Abstract. In this paper we study the Chebyshev-Halley family (which contains, as particular cases, the Chebyshev method, Halley method, super-Halley method and the C-method). For Chebyshev and super-Halley methods we give a global theorem of convergence. In the end of the paper we study the basins of attraction of the roots of a polynomial with real coefficients. They are obtained when we apply to that polynomial the methods from the Chebyshev-Halley family methods.

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1. INTRODUCTION

One of the classical problems in numerical analysis is to locate the root of the equation

$$f(x) = 0 \tag{1.1}$$

where $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is an analytic function with simple roots.

Before formulating the problems we investigate here, we recall some basic notions.

Let $g : \mathbb{R} \to \mathbb{R}$ be a rational map, that is, $g(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials without common factors.

We say that ω is a fixed point of g if $g(\omega) = \omega$. A fixed point ω is a periodic point of g if $\exists p \geq 1$ s.t. $g^p(\omega) = \omega$, where $g^p(\omega) = g(g^{p-1}(\omega))$.

The smallest p such that $g^p(\omega) = \omega$ is called the period of ω. A periodic point ω, of period p, of a function g is called repelling if $|(g^p)'(\omega)| > 1$; attractive if $|(g^p)'(\omega)| < 1$; superattractive if $|(g^p)'(\omega)| = 0$ and indifferent if $|(g^p)'(\omega)| = 1$.

The basin of attraction $A(\omega)$ of an attractive fixed point ω, associated with the rational map g, is:

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(1.2) $$A(\omega) = \{z \in \mathbb{C} : g^k(z) \underset{k\to\infty}{\rightarrow} \omega\}.$$

Most well known one-point cubically convergent iteration methods for finding a simple zero of the function f belong to the family of Chebyshev-Halley methods given by the expression:

(1.3)
$$x_{n+1} = M_{f,\theta,c}(x_n) = x_n - \Big(1 + \tfrac{L_f(x_n)}{2(1-\theta L_f(x_n))} + c[L_f(x_n)]^2\Big)u_f(x_n), \text{ for } n > 0,$$

where x_0 is an initial point, $L_f(x) = \frac{f(x)f''(x)}{f'(x)^2}$, $u_f(x) = \frac{f(x)}{f'(x)}$, if $f'(x) \neq 0$, and θ, c are real parameters, both of them should be chosen in a convenient way in every cases. The one parameter family of Chebyshev-Halley methods has been rediscovered by several authors [1], [2], [4], [7].

For $c = 0$ and θ non-negative we obtain a new family of third-order iterative methods which includes, as special cases, the Euler-Chebyshev method ($\theta = 0$), the Halley method ($\theta = \frac{1}{2}$) and the super-Halley method ($\theta = 1$).

In what follows, we assume that $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$, $a < b$, $a, b \in \mathbb{R}$ is a polynomial function.

We notice that when we apply any of these iterative functions of a polynomial, we get a rational application.

2. CONVERGENCE THEOREMS

For the iterative family methods (1.3), we have the following result for which we present a proof.

THEOREM 2.1 (Scaling). [2] *Set $f(x)$ an application, and set $T(x) = \alpha x+\beta$, where $\alpha \neq 0$, an affine application. If $g(x) = \rho(f \circ T(x))$, where ρ is a non-null constant, then $T \circ M_{g,\theta,c} \circ T^{-1}(x) = M_{f,\theta,c}$. That is, $M_{g,\theta,c}$ and $M_{f,\theta,c}$ are conjugate through T.*

Proof. We have

$$\begin{aligned} M_{g,\theta,c} \circ T^{-1}(x) =& M_{g,\theta,c}(T^{-1}(x)) \\ =& T^{-1}(x) - \Big(1 + \tfrac{L_g(T^{-1}(x))}{2(1-\theta L_g(T^{-1}(x)))} + c[L_g(T^{-1}(x))]^2)u_g(T^{-1}(x)\Big). \end{aligned}$$

On the other hand, because $g \circ T^{-1}(x) = f(x)$, we have

$$(g \circ T^{-1})'(x) = \tfrac{1}{\alpha} g'(T^{-1}(x))$$

and

$$(g \circ T^{-1})''(x) = \tfrac{1}{\alpha^2} g''(T^{-1}(x)).$$

It results $g'(T^{-1}(x)) = \alpha \cdot f'(x)$ and $g''(T^{-1}(x)) = \alpha^2 \cdot f''(x)$.

So we have $L_g(T^{-1}(x)) = L_f(x)$ and $u_g(T^{-1}(x)) = \frac{1}{\alpha} u_f(x)$. By replacing these expressions and by using the definition of $M_{g,\theta,c}(T^{-1}(x))$ we get:

$$T \circ M_{g,\theta,c} \circ T^{-1}(x) = T(M_{g,\theta,c}(T^{-1}(x))) = \alpha M_{g,\theta,c}(T^{-1}(x)) + \beta =$$

$$= \alpha \cdot \Big[T^{-1}(x) - \Big(1 + \frac{L_g(T^{-1}(x))}{2(1-\theta L_g(T^{-1}(x)))} + c[L_g(T^{-1}(x))]^2\Big) u_g(T^{-1}(x))\Big] + \beta$$
$$= x - \Big(1 + \frac{L_f(x)}{2(1-\theta L_f(x))} + c[L_f(x)]^2\Big) u_f(x) = M_{f,\theta,c}(x).$$

□

After a convenient change of coordinate, the Scaling theorem allows us to reduce the study of iterative methods to the study of iterative methods applied to families of simple functions.

We want to give now sufficient conditions for that the sequence $(x_n)_{n\geq 0}$ generated by (1.3) to be convergent, and if $x^* = \lim\limits_{n\to\infty} x_n$, then $f(x^*) = 0$ and, moreover, the convergence order of the sequence that we have considered to be s, $s \geq 2$ natural number.

The following result holds.

Theorem 2.2. [5] *If the function* φ, *the element* $x_0 \in [a, b]$, *and the number* $\delta > 0$ *can be chosen such that the following relations hold:*

a) *the interval* $\Delta = [x_0 - \delta, x_0 + \delta] \subset [a, b]$, $\delta \in \mathbb{R}$;

b) *the function* f *admits derivatives up to the order* s *inclusively on every point of* Δ, *where* $s \in \mathbb{N}$, $s \geq 2$, *and* $\sup\limits_{x\in\Delta} \left|f^{(s)}(x)\right| = M < \infty$;

c) *we have the relation*

$$\left|\sum_{i=0}^{s-1} \frac{1}{i!} f^{(i)}(x)\varphi^i(x)\right| \leq \gamma\, |f(x)|^s$$

for every $x \in \Delta$, *where* $\gamma \in \mathbb{R}$, $\gamma \geq 0$;

d) *the function* φ *verifies the relation* $|\varphi(x)| \leq \beta\, |f(x)|$, *for every* $x \in \Delta$, *where* $\beta \in \mathbb{R}$, $\beta > 0$;

e) *the numbers* λ, β, M *and* δ *verify the relations:*

$$\mu_0 = \lambda\, |f(x_0)| < 1,$$

where $\lambda = \left(\gamma + \frac{M\beta^s}{s!}\right)^{\frac{1}{s-1}}$ *and* $\frac{\beta\mu_0}{\lambda(1-\mu_0)} \leq \delta$,

then the sequence $\{x_n\}_{n\geq 0}$ *generated by* (1.3) *has the following properties:*

i) *is convergent, and if* $x^* = \lim\limits_{n\to\infty} x_n$ *then* $f(x^*) = 0$ *and* $x^* \in \Delta$;

ii) $|x_{n+1} - x_n| \leq \frac{\beta\mu_0^{3^n}}{\lambda}$, *for any* $n = 0, 1, ...$;

iii) $|x^* - x_n| \leq \frac{\beta\mu_0^{3^n}}{\lambda(1-\mu_0^{3^n})}$, $n = 0, 1, 2, ...$

Proof. See [5]. □

Next we are applying Theorem 2.2 for the study of the convergence of the methods from the family of Chebyshev-Halley methods, more exactly we would focus on the Chebyshev method and super-Halley method.

The super-Halley method, called Convex Acceleration of Newton's method, is less known than Chebyshev and Halley methods, and is defined by:

(2.4)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(1 + \frac{\frac{f(x_n)f''(x_n)}{[f'(x_n)]^2}}{2 - 2\frac{f(x_n)f''(x_n)}{[f'(x_n)]^2}}\right) = x_n - \frac{f(x_n)}{f'(x_n)}\left(1 + \frac{L_f(x_n)}{2 - 2L_f(x_n)}\right)$$

Because the methods from the family of Chebyshev-Halley methods have the convergence order 3, we consider $s = 3$ in Theorem 2.2.

For the super-Halley method we obtain the following result.

THEOREM 2.3. *If* $x_0 \in [a,b]$, *the function* f *and the number* $\delta > 0$ *verify the relations:*

a) $\Delta = [x_0 - \delta, x_0 + \delta] \subset [a,b]$, $\delta \in \mathbb{R}$;
b) *the function* f *admits derivatives up to the order* 3 *inclusively at every point of* Δ;
c) $\left|\frac{1}{f'(x)}\right| \leq \beta < 1$ *for every* $x \in \Delta$;
d) $L_f(x) = \frac{f(x)f''(x)}{[f'(x)]^2} \leq \frac{4}{5}$ *for every* $x \in \Delta$;
e) $\sup_{x\in\Delta} |f'''(x)| = M < \infty$;
f) $0 < \sqrt{\frac{27M}{3!}\beta^3 + \gamma} = \sqrt{\frac{9}{2}M\beta^3 + \gamma} = \lambda < 1, \gamma \geq 0$;
g) $\mu_0 = \lambda\,|f(x_0)| < 1$;
h) $\frac{3\beta\mu_0}{\lambda(1-\mu_0)} \leq \delta$,

then the sequence $\{x_n\}_{n\geq 0}$ *generated by* (2.4) *is convergent and, if* $x^* = \lim_{n\to\infty} x_n$, *then the following relations hold:*

i) $f(x^*) = 0$ *and* $x^* \in \Delta$;
ii) $x_n \in \Delta$, $n = 0, 1, 2...$;
iii) $|f(x_n)| \leq \frac{\mu_0^{3^n}}{\lambda}, n = 0, 1, 2, ...$;
iv) $|x^* - x_n| \leq \frac{3\beta\mu_0^{3^n}}{\lambda(1-\mu_0^{3^n})}, n = 0, 1, 2, ...$

Proof. By applying the Taylor expansion we obtain:

$$\begin{aligned}|f''(x)| \leq& |f''(x) - f''(x_0)| + |f''(x_0)|\\ \leq& M\,|x - x_0| + |f''(x_0)| \leq M\delta + |f''(x_0)| \overset{\text{not}}{=} M_2,\end{aligned}$$

for every $x \in \Delta$.

Analogously, we obtain:

$$\begin{aligned}|f'(x)| \leq& |f'(x) - f'(x_0)| + |f'(x_0)| \leq M_2\,|x - x_0| + |f'(x_0)|\\ \leq& M_2\delta + |f'(x_0)| \overset{\text{not}}{=} M_1\\ |f(x)| \leq& |f(x) - f(x_0)| + |f(x_0)| \leq M_1\,|x - x_0| + |f(x_0)|\\ \leq& M_1\delta + |f(x_0)| \overset{\text{not}}{=} M_0,\end{aligned}$$

for every $x \in \Delta$.

We consider the function φ of form:

$$\varphi(x) = -\frac{f(x)}{f'(x)}\left(1 + \frac{L_f(x)}{2-2L_f(x)}\right).$$

Taking into account the above relations, a simple computation leads us to the relation:

$$\begin{aligned}
(2.5)\qquad & \left|f(x) + \tfrac{f'(x)}{1!}\varphi(x) + \tfrac{f''(x)}{2!}\varphi^2(x)\right| = \\
&= \left|f(x) - \tfrac{f'(x)}{1!}\tfrac{f(x)}{f'(x)}\left(1 + \tfrac{L_f(x)}{2-2L_f(x)}\right) + \tfrac{f''(x)}{2!}\left[\tfrac{f(x)}{f'(x)}\left(1 + \tfrac{L_f(x)}{2-2L_f(x)}\right)\right]^2\right| \\
&= \left|\tfrac{f^4(x)[f''(x)]^3}{8[[f'(x)]^3 - f(x)f'(x)f''(x)]^2}\right| \\
&\le |f(x)|^3\left|\tfrac{f(x)[f''(x)]^3}{8[[f'(x)]^3 - f(x)f'(x)f''(x)]^2}\right| \\
&\le |f(x)|^3\left|\frac{f(x)[f''(x)]^3}{8[f'(x)]^6\left[1-\frac{f(x)f''(x)}{[f'(x)]^2}\right]^2}\right| \le |f(x)|^3\,\tfrac{25\beta^6 M_0 M_2^3}{8}
\end{aligned}$$

for every $x \in \Delta$, $\gamma = \frac{25}{8}\beta^6 M_0 M_2^3 \ge 0, \gamma \in \mathbb{R}$.

By condition d) we have:

$$\begin{aligned}
|x_1 - x_0| &= \left|\tfrac{f(x_0)}{f'(x_0)}\right|\left|1 + \frac{\frac{f(x_0)f''(x_0)}{[f'(x_0)]^2}}{2-2\frac{f(x_0)f''(x_0)}{[f'(x_0)]^2}}\right| \le 3\left|\tfrac{f(x_0)}{f'(x_0)}\right| \\
&\le 3\beta\,|f(x_0)| = \tfrac{3\lambda\beta|f(x_0)|}{\lambda} < \tfrac{3\beta\mu_0}{\lambda(1-\mu_0)} \le \delta \Rightarrow x_1 \in \Delta.
\end{aligned}$$

By applying the Taylor expansion of the function f around on x_0 and taking into account the relation (2.5) from above we get:

$$\begin{aligned}
|f(x_1)| &\le \left|f(x_1) - [f(x_0) + f'(x_0)(x_1 - x_0) + \tfrac{1}{2}f''(x_0)(x_1 - x_0)^2]\right| + \\
&\quad + \left|f(x_0) + f'(x_0)(x_1 - x_0) + \tfrac{1}{2}f''(x_0)(x_1 - x_0)^2\right| \le \\
&\le \tfrac{M}{3!}\,|x_1 - x_0|^3 + \gamma\,|f(x_0)|^3 \\
&\le \tfrac{M}{3!}(3\beta\,|f(x_0)|)^3 + \gamma\,|f(x_0)|^3 = \left(\tfrac{27M}{3!}\beta^3 + \gamma\right)|f(x_0)|^3 = \tfrac{\mu_0^3}{\lambda}.
\end{aligned}$$

Because $\left|\frac{1}{f'(x_1)}\right| \le \beta$ we have that

$$|x_2 - x_1| = \left|\tfrac{f(x_1)}{f'(x_1)}\right|\left|1 + \frac{\frac{f(x_1)f''(x_1)}{[f'(x_1)]^2}}{2-2\frac{f(x_1)f''(x_1)}{[f'(x_1)]^2}}\right| \le 3\left|\tfrac{f(x_1)}{f'(x_1)}\right| \le 3\beta\,|f(x_1)| \le \tfrac{3\beta\mu_0^3}{\lambda}.$$

By applying the Taylor expansion of the function f around on x_1 and taking into account the relation (2.5) we get:

$$|f(x_2)| \leq \left|f(x_2) - [f(x_1) + f'(x_1)(x_2 - x_1) + \tfrac{1}{2}f^{(2)}(x_1)(x_2 - x_1)^2]\right| +$$
$$+ \left|f(x_1) + f'(x_1)(x_2 - x_1) + \tfrac{1}{2}f^{(2)}(x_1)(x_2 - x_1)^2\right|$$
$$\leq \tfrac{M}{3!}\,|x_2 - x_1|^3 + \gamma\,|f(x_1)|^3$$
$$\leq \tfrac{M}{3!}(\tfrac{3\beta\mu_0^3}{\lambda})^3 + \tfrac{\gamma\mu_0^{3^2}}{\lambda^3} = (\tfrac{27M}{3!\lambda^3}\beta^3 + \tfrac{\gamma}{\lambda^3})\mu_0^{3^2} = \tfrac{\lambda^2}{\lambda^3}\mu_0^{3^2} = \tfrac{\mu_0^{3^2}}{\lambda}.$$

Analogously, one can prove the following:

$$|f(x_n)| \leq \tfrac{\mu_0^{3^n}}{\lambda},\ n = 0, 1, 2, ..., \tag{2.6}$$

$$|x_{n+1} - x_n| = \left|\tfrac{f(x_n)}{f'(x_n)}\right| \left|1 + \frac{\frac{f(x_n)f''(x_n)}{[f'(x_n)]^2}}{2-2\frac{f(x_n)f''(x_n)}{[f'(x_n)]^2}}\right| \leq \tfrac{3\beta\mu_0^{3^n}}{\lambda},\ n = 0, 1, ..., \tag{2.7}$$

(2.8)

$$|x_{n+1}-x_0| \leq \sum_{i=0}^{k} |x_{i+1} - x_i| \leq \sum_{i=0}^{k} \tfrac{3\beta\mu_0^{3^i}}{\lambda} \leq \tfrac{3\beta\mu_0}{\lambda}(1 + \mu_0^{3-1} + \mu_0^{3^2-1} + ... + \mu_0^{3^k-1})$$
$$< \tfrac{3\beta\mu_0}{\lambda(1-\mu_0)} \leq \delta \Rightarrow x_{n+1} \in \Delta,\ n = 0, 1, 2, ...$$

By using relation (2.7) we deduce that

$$|x_{n+p} - x_n| \leq \sum_{i=n}^{n+p-1} |x_{i+1} - x_i| \leq \sum_{i=n}^{n+p-1} \tfrac{3\beta\mu_0^{3^i}}{\lambda} \tag{2.9}$$
$$< \tfrac{3\beta\mu_0^{3^n}}{\lambda}(1 + \mu_0^{3^{n+1}-3^n} + ... + \mu_0^{3^{n+p-1}-3^n})$$
$$< \tfrac{3\beta\mu_0^{3^n}}{\lambda(1-\mu_0^{3^n})},\ p \in \mathbb{N}, n = 0, 1, 2, ...$$

Because $\mu_0 < 1$ it results that the sequence $\{x_n\}_{n\geq 0}$ is fundamental, so according to the Cauchy theorem, it is convergent.

If $x^* = \lim\limits_{n\to\infty} x_n$, for $p \to \infty$ from inequality (2.9) we deduce:

$$|x^* - x_n| \leq \tfrac{3\beta\mu_0^{3^n}}{\lambda(1-\mu_0^{3^n})},\ n = 0, 1, 2, ... \tag{2.10}$$

We show now that x^* is a root of the equation $f(x) = 0$.

From the continuity of the function f and from iii) for $n \to \infty$ it results:

$$0 \leq |f(x^*)| \leq \lim_{n\to\infty} \frac{\mu_0^{3^n}}{\lambda} = 0 \Leftrightarrow f(x^*) = 0.$$

From inequality (2.10) for $n = 0$ we obtain:

$$|x^* - x_0| \leq \tfrac{3\beta\mu_0^{3^0}}{\lambda(1-\mu_0^{3^0})} \leq \delta \Leftrightarrow x^* \in \Delta. \qquad \square$$

We apply Theorem 2.2 to the study of the convergence of the Chebyshev method. As we know, the Chebyshev method can be applied to any nonlinear equation from $\mathbb{R}$. It is the most studied third-order method in the literature. The method is known by the name parable-tangent method or the super-Newton method because of its geometrical interpretations.

We obtain the following result.

Theorem 2.4. *If $x_0 \in [a,b]$, the function f and the number $\delta > 0$ verify the relations:*

a) $\Delta = [x_0 - \delta, x_0 + \delta] \subset [a,b]$, $\delta \in \mathbb{R}$;
b) *the function f admits derivatives up to the order 3 inclusively, at every point of Δ;*
c) $\left|\frac{1}{f'(x)}\right| \leq \beta < 1$ *for every $x \in \Delta$;*
d) $-2 \leq L_f(x) = \frac{f(x)f''(x)}{[f'(x)]^2} \leq 2$ *for every $x \in \Delta$;*
e) $\sup\limits_{x\in\Delta} |f'''(x)| = M < \infty$;
f) $0 < \sqrt{\frac{8M}{3!}\beta^3 + \gamma} = \sqrt{\frac{4}{3}M\beta^3 + \gamma} = \lambda <, \ \gamma \geq 0$;
g) $\mu_0 = \lambda\,|f(x_0)| < 1$;
h) $\frac{2\beta_0\mu_0}{\lambda(1-\mu_0)} \leq \delta$,

then the sequence $\{x_n\}_{n\geq 0}$ generated by (1.3) for $\theta = 0$ is convergent and if $x^* = \lim\limits_{n\to\infty} x_n$ the following relations hold:

i) $f(x^*) = 0$ *and* $x^* \in \Delta$;
ii) $x_n \in \Delta$, $n = 0, 1, 2...$;
iii) $|f(x_n)| \leq \frac{\mu_0^{3^n}}{\lambda}, n = 0, 1, 2, ...$;
iv) $|x^* - x_n| \leq \frac{2\beta_0\mu_0^{3^n}}{\lambda(1-\mu_0^{3^n})}, n = 0, 1, 2, ...$

Proof. By applying the Taylor expansion we obtain:

$$\begin{aligned}|f''(x)| \leq& |f''(x) - f''(x_0)| + |f''(x_0)| \leq M\,|x - x_0| + |f''(x_0)|\\ \leq& M\delta + |f''(x_0)| \overset{\text{not}}{=} M_2,\end{aligned}$$

for every $x \in \Delta$.

We will consider the function φ of the form:

$$\varphi(x) = -\frac{f(x)\left(1+\frac{1}{2}L_f(x)\right)}{f'(x)}.$$

Taking into account the above notations, a simple computation takes us to the relation:

$$\left|f(x)+\tfrac{f'(x)}{1!}\varphi(x)+\tfrac{f''(x)}{2!}\varphi^2(x)\right| = \tag{2.11}$$

$$=\left|f(x)-\tfrac{f'(x)}{1!}\tfrac{f(x)\left(1+\frac{1}{2}L_f(x)\right)}{f'(x)}+\tfrac{f''(x)}{2!}\left[\tfrac{f(x)\left(1+\frac{1}{2}L_f(x)\right)}{f'(x)}\right]^2\right|$$

$$=\left|\tfrac{f^3(x)[f''(x)]^2}{2[f'(x)]^4}+\tfrac{f^4(x)[f''(x)]^3}{8[f'(x)]^6}\right|$$

$$\le \tfrac{|f(x)|^3\left|[f''(x)]^2\right|}{|2[f'(x)]^4|}\left|1+\tfrac{f(x)f''(x)}{4[f'(x)]^2}\right|$$

$$\le |f(x)|^3\,\tfrac{3\beta^4M_2^2}{4}$$

for every $x\in\Delta$, $\gamma=\frac{3\beta^4M_2^2}{4}\ge 0,\gamma\in\mathbb{R}$.

By condition d) we have:

$$|x_1-x_0|=\left|\tfrac{f(x_0)}{f'(x_0)}\right|\left|1+\frac{\frac{f(x_0)f''(x_0)}{[f'(x_0)]^2}}{2}\right|\le 2\left|\tfrac{f(x_0)}{f'(x_0)}\right|$$

$$\le 2\beta\,|f(x_0)|\le\tfrac{2\lambda\beta|f(x_0)|}{\lambda}<\tfrac{2\beta\mu_0}{\lambda(1-\mu_0)}\le\delta\Rightarrow x_1\in\Delta.$$

By applying the Taylor expansion to the function f at x_0 and taking into account relation (2.11) we get:

$$|f(x_1)|\le\left|f(x_1)-[f(x_0)+f'(x_0)(x_1-x_0)+\tfrac{1}{2}f^{(2)}(x_0)(x_1-x_0)^2]\right|+$$

$$+\left|f(x_0)+f'(x_0)(x_1-x_0)+\tfrac{1}{2}f^{(2)}(x_0)(x_1-x_0)^2\right|$$

$$\le\tfrac{M}{3!}\,|x_1-x_0|^3+\gamma\,|f(x_0)|^3$$

$$\le\tfrac{M}{3!}(2\beta\,|f(x_0)|)^3+\gamma\,|f(x_0)|^3=\left(\tfrac{8M}{3!}\beta^3+\gamma\right)|f(x_0)|^3=\tfrac{\mu_0^3}{\lambda}.$$

By condition c) we have that

$$|x_2-x_1|=\left|\frac{f(x_1)}{f'(x_1)}\right|\left|1+\frac{\frac{f(x_1)f''(x_1)}{[f'(x_1)]^2}}{2}\right|\le 2\left|\frac{f(x_1)}{f'(x_1)}\right|\le 2\beta\,|f(x_1)|\le\frac{2\beta\mu_0^3}{\lambda}.$$

By applying the Taylor expansion to the function f at x_1 and taking into account relation (2.11) we get:

$$|f(x_2)|\le\left|f(x_2)-[f(x_1)+f'(x_1)(x_2-x_1)+\tfrac{1}{2}f^{(2)}(x_1)(x_2-x_1)^2]\right|+$$

$$+\left|f(x_1)+f'(x_1)(x_2-x_1)+\tfrac{1}{2}f^{(2)}(x_1)(x_2-x_1)^2\right|$$

$$\le\tfrac{M}{3!}\,|x_2-x_1|^3+\gamma\,|f(x_1)|^3$$

$$\le\tfrac{M}{3!}\left(\tfrac{3\beta\mu_0^3}{\lambda}\right)^3+\tfrac{\gamma\mu_0^{3^2}}{\lambda^3}=\left(\tfrac{27M}{3!\lambda^3}\beta^3+\tfrac{\gamma}{\lambda^3}\right)\mu_0^{3^2}=\tfrac{\lambda^2}{\lambda^3}\mu_0^{3^2}=\tfrac{\mu_0^{3^2}}{\lambda}.$$

Analogously, one can prove the following inequalities:

$$|f(x_n)| \leq \frac{\mu_0^{3^n}}{\lambda}, n = 0, 1, 2, ..., \tag{2.12}$$

$$|x_{n+1} - x_n| = \left|\frac{f(x_n)}{f'(x_n)}\right| \left|1 + \frac{1}{2}\frac{f(x_n)f''(x_n)}{[f'(x_n)]^2}\right| \leq \frac{2\beta\mu_0^{3^n}}{\lambda},\ n = 0, 1, ... \tag{2.13}$$

By using the inequality (2.13) we obtain the inequalities:

$$\begin{aligned} |x_{n+1} - x_0| &\leq \sum_{i=0}^{k} |x_{i+1} - x_i| \leq \sum_{i=0}^{k} \frac{2\beta\mu_0^{3^i}}{\lambda} \\ &\leq \frac{2\beta\mu_0}{\lambda}(1 + \mu_0^{3-1} + \mu_0^{3^2-1} + ... + \mu_0^{3^k-1}) \\ &< \frac{2\beta\mu_0}{\lambda(1-\mu_0)} \leq \delta \Rightarrow x_{n+1} \in \Delta, n = 0, 1, 2, ... \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} |x_{n+p} - x_n| &\leq \sum_{i=n}^{n+p-1} |x_{i+1} - x_i| \leq \sum_{i=n}^{n+p-1} \frac{2\beta\mu_0^{3^i}}{\lambda} \\ &< \frac{2\beta\mu_0^{3^n}}{\lambda}(1 + \mu_0^{3^{n+1}-3^n} + ... + \mu_0^{3^{n+p-1}-3^n}) \\ &< \frac{2\beta\mu_0^{3^n}}{\lambda(1 - \mu_0^{3^n})},\ p \in \mathbb{N},\ n = 0, 1, 2, ... \end{aligned} \tag{2.15}$$

Because $\mu_0 < 1$ it results that the sequence $\{x_n\}_{n\geq 0}$ is fundamental, so according to the theorem of Cauchy, it is convergent.

If $x^* = \lim\limits_{n\to\infty} x_n$ from the inequality (2.15), for $p \to \infty$, we deduce

$$|x^* - x_n| \leq \frac{2\beta\mu_0^{3^n}}{\lambda(1-\mu_0^{3^n})}, n = 0, 1, 2, ... \tag{2.16}$$

We show now that x^* is a root of the equation $f(x) = 0$.

From the continuity of the function f and from iii) for $n \to \infty$ it results

$$0 \leq |f(x^*)| \leq \lim_{n\to\infty} \frac{\mu_0^{3^n}}{\lambda} = 0 \Leftrightarrow f(x^*) = 0.$$

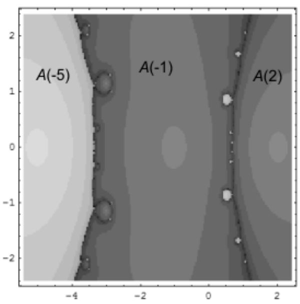
For $n = 0$ from inequality (2.16) we obtain:

$$|x^* - x_0| \leq \frac{2\beta\mu_0^{3^0}}{\lambda(1 - \mu_0^{3^0})} \leq \delta \Leftrightarrow x^* \in \Delta. \qquad \square$$

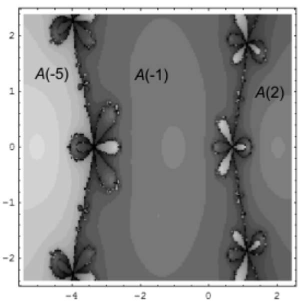
3. NUMERICAL EXAMPLE

We apply the iterative methods that we have considered in the previous sections to approximate the real roots of a polynomial with real coefficients.

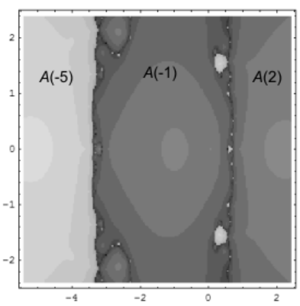
Example 3.5. Next we will apply the iterative methods from above to get the real roots of the polynomial $p(x) = x^3 + 4x^2 - 7x - 10$. It is clear that the roots of the polynomial p are -5, -1 and 2. We take a rectangle $D = [-5.4, 2.4] \times [-2.4, 2.4]$, which contains these three roots and we apply



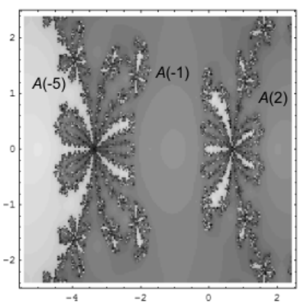
(a) Halley method



(b) Chebyshev method



(c) super-Halley method



(d) C method for C=2

these iterative methods starting from each $x_0 \in D$. In practice we will take a grid of 1024×1024 points in D and we will use these points as initial points $x_0 \in D$. The numerical methods starting from a point in D can converge to some of the roots or, eventually, diverge. We will use a tolerance $\epsilon = 10^{-8}$ and compute maximum 10 iterations. In the next figures are presented the graphical images of the iterative methods from the Chebyshev-Halley methods, in the above described region. If the fractal that appears becomes more complicated, then there seems that the method requires more conditions on the initial point. We assign a gray color with different nuances, light colors or dark ones in correspondence with the number of the iterations useful to find the roots. With the precision that we have taken to each point $x_0 \in D$ according to the root at which the iterative methods starting from x_0 converge, and we mark the point as black if the methods does not converge. We marked with black the points $x_0 \in D$ for which the iterative methods, starting from the initial point $x_0 \in D$ they are not approaching to any root, with the tolerance $\epsilon = 10^{-8}$ in maximum 10 iterations. The region $A(-1)$ constitutes the basin of attraction of the root -1, the region $A(2)$ constitutes the basin of the attraction of the root 2, and $A(-5)$ constitutes the basin of attraction of the root -5. The graphics that are shown here were generated with Mathematica 4.0.

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