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THE EQUIVALENCE BETWEEN T-STABILITIES OF KRASNOSELSKIJ AND ISHIKAWA ITERATIONS

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Abstract. We prove the equivalence between the T-stabilities of Krasnoselskij and Ishikawa iterations; a consequence is the equivalence with the T-stability of Picard-Banach iteration.

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1. INTRODUCTION

Let X be a normed space and T a selfmap of X. Let x_0 be a point of X, and assume that $x_{n+1} = f(T, x_n)$ is an iteration procedure, involving T, which yields a sequence $\{x_n\}$ of points from X. Suppose $\{x_n\}$ converges to a fixed point x^* of T. Let $\{\xi_n\}$ be an arbitrary sequence in X, and set $\epsilon_n = \|\xi_{n+1} - f(T, \xi_n)\|$ for all $n \in \mathbb{N}$.

DEFINITION 1. [1] *If* $\left((\lim_{n\to\infty} \epsilon_n = 0) \Rightarrow (\lim_{n\to\infty} \xi_n = p)\right)$, *then the iteration procedure* $x_{n+1} = f(T, x_n)$ *is said to be* T*-stable with respect to* T.

REMARK 2. [1] In practice, such a sequence $\{\xi_n\}$ could arise in the following way. Let x_0 be a point in X. Set $x_{n+1} = f(T, x_n)$. Let $\xi_0 = x_0$. Now $x_1 = f(T, x_0)$. Because of rounding or discretization in the function T, a new value ξ_1 approximately equal to x_1 might be obtained instead of the true value of $f(T, x_0)$. Then to approximate x_2, the value $f(T, \xi_1)$ is computed to yield ξ_2, an approximation of $f(T, \xi_1)$. This computation is continued to obtain $\{\xi_n\}$, an approximate sequence of $\{x_n\}$. □

Let X be a normed space, D a nonempty, convex subset of X, and T a selfmap of D, let $p_0 = e_0 \in D$. The Mann iteration (see [4]) is defined by

$$e_{n+1} = (1 - \alpha_n)e_n + \alpha_n T e_n, \tag{1}$$

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where $\{\alpha_n\} \subset (0,1)$. The Ishikawa iteration is defined (see [2]) by

$$a_{n+1} = (1-\alpha_n)a_n + \alpha_n T b_n, \qquad (2)$$
$$b_n = (1-\beta_n)a_n + \beta_n T a_n,$$

where $\{\alpha_n\} \subset (0,1), \{\beta_n\} \subset [0,1)$. The Krasnoselskij iteration (see [3]) is defined by

$$p_{n+1} = (1-\lambda)p_n + \lambda T p_n, \qquad (3)$$

where $\lambda \in (0,1)$. Recently, the equivalence between the T-stabilities of Mann and Ishikawa iterations respectively for modified Mann-Ishikawa iterations was shown in [5]. In [7], it was proven equivalence between the T-stabilities of Krasnoselskij and Mann iterations. Analogously, we shall prove here the equivalence between the T-stabilities of Krasnoselskij and Ishikawa iterations. Note that no additional conditions are imposed on $\{\beta_n\}$. Next, $\{x_n\}, \{v_n\} \subset X$ are arbitrary.

Definition 3. (i) *The Ishikawa iteration* (2), *is said to be T-stable if and only if for all $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ and for every sequence $\{x_n\} \subset X$ we have*

$$\lim_{n\to\infty} \varepsilon_n = 0 \Rightarrow \lim_{n\to\infty} x_n = x^*, \qquad (4)$$

where $y_n = (1-\beta_n)\, x_n + \beta_n T x_n$, $\varepsilon_n := \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T y_n\|$.

(ii) *The Krasnoselskij iteration* (3), *is said to be T-stable if and only if for all $\lambda \in (0,1)$, and for every sequence $\{v_n\} \subset X$ we have*

$$\lim_{n\to\infty} \delta_n = 0 \Rightarrow \lim_{n\to\infty} v_n = x^*, \qquad (5)$$

where $\delta_n := \|v_{n+1} - (1-\lambda)v_n - \lambda T v_n\|$.

2. MAIN RESULTS

Theorem 4. *Let X be a normed space and $T : X \to X$ a map with bounded range, $\{\alpha_n\} \subset (0,1)$ satisfies $\lim_{n\to\infty} \alpha_n = \lambda, \lambda \in (0,1)$, and suppose that*

$$\lim_{n\to\infty} \|Tx_n - Ty_n\| = 0.$$

Then the following are equivalent:

(i) *the Ishikawa iteration is T-stable,*
(i) *the Krasnoselskij iteration is T-stable.*

Proof. We prove that (i) $\Rightarrow$ (ii). If $\lim_{n\to\infty} \delta_n = 0$, then $\{v_n\}$ is bounded. Set

$$M_1 := \max\left\{\sup_{x\in X}\{\|T(x)\|\}, \|v_0\|, \|u_0\|\right\}.$$

Observe that $\|v_1\| \leq \delta_0+(1-\lambda)\|v_0\|+\lambda\|Tv_0\| \leq \delta_0+M_1$. Set $M := M_1+1/\lambda$. Suppose that $\|v_n\| \leq M$ to prove that $\|v_{n+1}\| \leq M$. Remark that

$$\begin{aligned}\|v_{n+1}\| &\leq \delta_n+(1-\lambda)\delta_{n-1}+...+(1-\lambda)^n\delta_0+M_1\\ &\leq 1+(1-\lambda)+...+(1-\lambda)^n+M_1\\ &\leq \tfrac{1}{1-(1-\lambda)}+M_1=M.\end{aligned}$$

Suppose $\lim\limits_{n\to\infty}\delta_n=0$, to note that

$$\begin{aligned}\varepsilon_n &= \|v_{n+1}-(1-\alpha_n)v_n-\alpha_n Ty_n\| =\\ &= \|v_{n+1}-(1-\alpha_n)v_n-\alpha_n Ty_n+\lambda v_n-\lambda v_n+\lambda Tv_n-\lambda Tv_n\|\\ &= \|v_{n+1}-(1-\lambda)v_n-\lambda Tv_n+\alpha_n v_n-\alpha_n Ty_n-\lambda v_n+\lambda Tv_n\|\\ &\leq \|v_{n+1}-(1-\lambda)v_n-\lambda Tv_n\|+|\lambda-\alpha_n|\,\|v_n\|+\|-\alpha_n Ty_n+\lambda Tv_n\|\\ &\leq \delta_n+M|\lambda-\alpha_n|+\|-\alpha_n Ty_n+\alpha_n Tv_n-\alpha_n Tv_n+\lambda Tv_n\|\\ &\leq \delta_n+M|\lambda-\alpha_n|+\alpha_n\|Tv_n-Ty_n\|+|\lambda-\alpha_n|\,\|Tv_n\|\\ &\leq \delta_n+M|\lambda-\alpha_n|+\alpha_n\|Tv_n-Ty_n\|+|\lambda-\alpha_n|\,M\to 0 \text{ as } n\to\infty.\end{aligned}$$

Condition (i) assures that if $\lim\limits_{n\to\infty}\varepsilon_n=0$, then $\lim\limits_{n\to\infty}v_n=x^*$. Thus, for a $\{v_n\}$ satisfying

$$\lim_{n\to\infty}\delta_n=\lim_{n\to\infty}\|v_{n+1}-(1-\lambda)v_n-\lambda Tv_n\|=0,$$

we have shown that $\lim\limits_{n\to\infty}v_n=x^*$.

Conversely, we prove (ii) $\Rightarrow$ (i) . First, we prove that $\{x_n\}$ is bounded. Since $\lim\limits_{n\to\infty}\alpha_n=\lambda$, for $\gamma\in(0,1)$ given, there exists $n_0\in N$, such that $1-\alpha_n\leq\gamma, \forall n\geq n_0$. Set $M_1 := \max\{\sup_{x\in X}\|Tx\|,\|u_0\|\}$ and $M := n_0+1+\frac{\gamma}{1-\gamma}+M_1$ to obtain

$$\begin{aligned}\|x_{n+1}\| \leq&[\varepsilon_n+(1-\alpha_1)\varepsilon_{n-1}+(1-\alpha_1)(1-\alpha_2)\varepsilon_{n-2}+...\\ &+(1-\alpha_1)(1-\alpha_2)...(1-\alpha_{n_0})\varepsilon_{n-n_0}]\\ &+(1-\alpha_1)(1-\alpha_2)...(1-\alpha_{n_0})(1-\alpha_{n_0+1})\varepsilon_{n-n_0-1}+...\\ &+(1-\alpha_1)(1-\alpha_2)...(1-\alpha_n)\varepsilon_0+M_1\\ \leq&(n_0+1)+(1-\alpha_{n_0+1})+(1-\alpha_{n_0+1})(1-\alpha_{n_0+2})...\\ &+(1-\alpha_{n_0+1})...(1-\alpha_n)\varepsilon_0+M_1\\ \leq& n_0+1+\gamma+\gamma^2+...+\gamma^{n-n_0}+M_1<M.\end{aligned}$$

Suppose $\lim_{n\to\infty} \varepsilon_n = 0$. Observe that

$$\begin{aligned}
\delta_n &= \|x_{n+1} - (1-\lambda)x_n - \lambda T x_n\| \\
&= \|x_{n+1} - x_n + \lambda x_n - \lambda T x_n + \alpha_n x_n - \alpha_n x_n - \alpha_n T y_n + \alpha_n T y_n\| \\
&\le \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T y_n\| + \|\lambda x_n - \lambda T x_n - \alpha_n x_n + \alpha_n T y_n\| \\
&= \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T y_n\| + \\
&\quad + \|\lambda x_n - \lambda T x_n - \alpha_n x_n + \alpha_n T y_n + \alpha_n T x_n - \alpha_n T x_n\| \\
&\le \varepsilon_n + |\lambda - \alpha_n| \|x_n\| + |\lambda - \alpha_n| \|T x_n\| + \alpha_n \|T x_n - T y_n\| \\
&\le \varepsilon_n + |\lambda - \alpha_n| M + |\lambda - \alpha_n| M + \|T x_n - T y_n\| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Condition (ii) assures that if $\lim_{n\to\infty} \delta_n = 0$, then $\lim_{n\to\infty} x_n = x^*$. Thus, for a $\{x_n\}$ satisfying

$$\lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T y_n\| = 0,$$

we have shown that $\lim_{n\to\infty} x_n = x^*$. □

Remark 5. Let X be a normed space and $T : X \to X$ a map with bounded range and $\{\alpha_n\} \subset (0,1)$ satisfies $\lim_{n\to\infty} \alpha_n = \lambda,\ \lambda \in (0,1)$. If Ishikawa iteration is not T-stable, then the Krasnoselskij iteration is not T-stable, and conversely. So Mann iteration is not T-stable. Actually, by use of Theorem 4 one can easily obtain the non T-stability of the other iteration, provided that the previous one is not stable. □

The following result takes in consideration the case in which no condition on $\{\alpha_n\}$ nor $\{\beta_n\}$ are imposed.

Theorem 6. *Let X be a normed space and $T : X \to X$ a map, and $\{\alpha_n\} \subset (0,1)$. If*

$$\lim_{n\to\infty} \|x_n - T y_n\| = 0,\ \lim_{n\to\infty} \|v_n - T y_n\| = 0 \text{ and } \lim_{n\to\infty} \|x_n - T x_n\| = 0,$$

then the following are equivalent:

(i) *the Ishikawa iteration is T-stable,*
(ii) *the Krasnoselskij iteration is T-stable.*

Proof. We prove that (i) $\Rightarrow$ (ii). Suppose $\lim_{n\to\infty} \delta_n = 0$, to note that,

$$\begin{aligned}
\varepsilon_n &= \|v_{n+1} - (1-\alpha_n)v_n - \alpha_n T y_n\| = \\
&= \|v_{n+1} - (1-\alpha_n) v_n - \alpha_n T y_n + \lambda v_n - \lambda v_n + \lambda T v_n - \lambda T v_n\| \\
&= \|v_{n+1} - (1-\lambda) v_n - \lambda T v_n + \alpha_n v_n - \alpha_n T y_n - \lambda v_n + \lambda T v_n\| \\
&\le \delta_n + \alpha_n \|v_n - T y_n\| + \lambda \|v_n - T v_n\| \\
&\le \delta_n + \alpha_n \|v_n - T y_n\| + \lambda \|v_n - T v_n\| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Condition (i) assures that if $\lim_{n\to\infty} \varepsilon_n = 0$, then $\lim_{n\to\infty} v_n = x^*$. Thus, for a $\{v_n\}$ satisfying

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \|v_{n+1} - (1-\lambda)v_n - \lambda T v_n\| = 0,$$

we have shown that $\lim_{n\to\infty} v_n = x^*$.

Conversely, we prove (ii) $\Rightarrow$ (i) . Suppose $\lim_{n\to\infty} \varepsilon_n = 0$. Observe that

$$\begin{aligned}
\delta_n &= \|x_{n+1} - (1-\lambda)x_n - \lambda T x_n\| \\
&= \|x_{n+1} - x_n + \lambda x_n - \lambda T x_n + \alpha_n x_n - \alpha_n x_n - \alpha_n T y_n + \alpha_n T y_n\| \\
&\le \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T y_n\| + \|\lambda x_n - \lambda T x_n - \alpha_n x_n + \alpha_n T y_n\| \\
&= \varepsilon_n + \alpha_n \|x_n - T y_n\| + \lambda \|x_n - T x_n\| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Condition (ii) assures that if $\lim_{n\to\infty} \delta_n = 0$, then $\lim_{n\to\infty} x_n = x^*$. Thus, for a $\{x_n\}$ satisfying

$$\lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T y_n\| = 0,$$

we have shown that $\lim_{n\to\infty} x_n = x^*$. □

REMARK 7. Let X be a normed space and $T : X \to X$ a map, $\{\alpha_n\} \subset (0,1)$ and $\lim_{n\to\infty} \|v_n - T v_n\| = 0$, $\lim_{n\to\infty} \|u_n - T u_n\| = 0$. If Mann iteration is not T-stable, then the Krasnoselskij iteration is not T-stable, and conversely. □

3. FURTHER RESULTS

Let $q_0 \in X$ be fixed, and $q_{n+1} = T q_n$ be the Picard-Banach iteration.

DEFINITION 8. *The Picard iteration is said to be T-stable if and only if for every sequence $\{q_n\} \subset X$ given, we have*

$$\lim_{n\to\infty} \Delta_n = 0 \Rightarrow \lim_{n\to\infty} q_n = x^*,$$

where $\Delta_n := \|q_{n+1} - T q_n\|$.

In [6], the equivalence between the T-stabilities of Picard-Banach iteration and Mann iteration is given, i.e.

THEOREM 9. *Let X be a normed space and $T : X \to X$ a map. If*

$$\lim_{n\to\infty} \|q_n - T q_n\| = 0 \textit{ and } \lim_{n\to\infty} \|x_n - T y_n\| = 0,$$

then the following are equivalent:

(i) *for all $\{\alpha_n\} \subset (0,1)$, the Ishikawa iteration is T-stable,*
(ii) *the Picard iteration is T-stable.*

Proof. We prove that (i) $\Rightarrow$ (ii) . Suppose $\lim_{n\to\infty} \Delta_n = 0$, to note that

$$\begin{aligned}\varepsilon_n &= \|v_{n+1} - (1-\alpha_n)v_n - \alpha_n T y_n\| = \\ &= \|v_{n+1} - (1-\alpha_n)\, v_n - \alpha_n T y_n + \lambda v_n - \lambda v_n + \lambda T v_n - \lambda T v_n\| \\ &= \|v_{n+1} - (1-\lambda)\, v_n - \lambda T v_n + \alpha_n v_n - \alpha_n T y_n - \lambda v_n + \lambda T v_n\| \\ &\le \delta_n + \alpha_n \|v_n - T y_n\| + \lambda \|v_n - T v_n\| \to 0 \text{ as } n \to \infty.\end{aligned}$$

Condition (i) assures that if $\lim_{n\to\infty} \varepsilon_n = 0$, then $\lim_{n\to\infty} v_n = x^*$. Thus, for a $\{v_n\}$ satisfying

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \|v_{n+1} - (1-\lambda)v_n - \lambda T v_n\| = 0,$$

we have shown that $\lim_{n\to\infty} v_n = x^*$.

In order to prove (ii) $\Rightarrow$ (i), suppose $\lim_{n\to\infty} \varepsilon_n = 0$ and note that

$$\begin{aligned}\Delta_n &= \|p_{n+1} - T p_n\| \\ &= \|p_{n+1} - (1-\alpha_n)\, p_n - \alpha_n T y_n + (1-\alpha_n)\, p_n + \alpha_n T y_n - T p_n\| \\ &\le \|p_{n+1} - (1-\alpha_n)\, p_n - \alpha_n T y_n\| + \|p_n - T p_n\| + \alpha_n \|p_n - T y_n\| \\ &= \varepsilon_n + \|p_n - T p_n\| + \alpha_n \|p_n - T y_n\| .\end{aligned}$$

□

REMARK 10. Let X be a normed space and $T : X \to X$ a map, $\{\alpha_n\} \subset (0,1)$ and $\lim_{n\to\infty} \|q_n - T q_n\| = 0$, $\lim_{n\to\infty} \|v_n - T v_n\| = 0$, $\lim_{n\to\infty} \|u_n - T u_n\| = 0$. If the Ishikawa or Krasnoselskij iteration is not T-stable, then the Picard-Banach iteration is not T-stable, and conversely. □

REFERENCES

[1] HARDER, A.M. and HICKS, T., *Stability results for fixed point iteration procedures*, Math. Japonica, **33**, pp. 693–706, 1988.

[2] ISHIKAWA, S., *Fixed points by a new iteration method*, Proc. Amer. Math. Soc., **44**, pp. 147–150, 1974.

[3] KRASNOSELSKIJ, M.A., *Two remarks on the method of succesive approximations*, Uspehi Mat. Nauk., **10**, pp. 123–127, 1955.

[4] MANN, W.R., *Mean value methods in iteration*, Proc. Amer. Math. Soc., **4**, pp. 506–510, 1953.

[5] RHOADES, B.E. and ŞOLTUZ, Ş.M., *The equivalence between the T-stabilities of Mann and Ishikawa iterations*, J. Math. Anal. Appl., **318**, pp. 472–475, 2006.

[6] ŞOLTUZ, Ş.M., *The equivalence between the T-stabilities of Picard-Banach and Mann-Ishikawa iterations*, Appl. Math. E-Notes, **8**, pp. 109–114, 2008.

[7] ŞOLTUZ, Ş.M., *The equivalence between T-stabilities of the Krasnoselskij and the Mann iterations*, Fixed Point Theory and Applications, 2007, Article ID 60732.

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