THEORETICAL AND NUMERICAL RESULTS ABOUT SOME WEAKLY SINGULAR VOLterra-FREDHOLM EQUATIONS

F. CALIÓ∗, E. MARCHETTI∗ and V. MUREŞAN†

Abstract. In this paper existence, uniqueness results for the solution of some weakly singular linear Volterra and Volterra-Fredholm integral equations are given. For these equations, a numerical model is proposed and its convergence and rate of convergence are analyzed. Numerical results on some polynomial test functions are given.

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1. INTRODUCTION


The aim of this paper is to present existence, uniqueness and numerical results for the solutions of the following weakly singular linear integral equations with linear modification of the argument:

\[ y(x) = f(x) + \int_0^x K(x, s)y(\lambda s)ds, \quad x \in [0, b], \quad 0 < \lambda \leq 1, \]

where \( f \in C[0, b] \), \( K(x, s) = \frac{L(x, s)}{|x-s|^{\alpha}} \) for all \( x, s \in [0, b], \ x \neq s, \ 0 < \alpha < 1 \) and \( L \in C([0, b] \times [0, b]) \), respectively

\[ y(x) = f(x) + \int_0^x K_1(x, s)y(\lambda s)ds + \int_0^b K_2(x, u)y(\lambda u)du, \quad x \in [0, b], \ 0 < \lambda \leq 1, \]

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where \( f \in C[0, b], \ K_i(x, s) = \frac{L_i(x, s)}{|x-s|^{\alpha_i}}, \ 0 < \alpha_i < 1, \) and \( L_i \in C([0, b] \times [0, b]), \ i = 1, 2. \)

At these problems, taking into account the kernel singularity, a particular numerical model \([14]\) is adapted. This model is based on Nystrom collocation method, using Schoenberg variation diminishing (SVD) splines of fourth order. The order of convergence is studied and numerical results are given to test the polynomial exactness.

Section 2 and 3 are respectively devoted to the theoretical results for Volterra and Volterra-Fredholm weakly singular integral equations with linear modification of the argument. Sections 4 and 5 present the numerical model and its convergence analysis; in Section 6 numerical results are given.

### 2. VOLTERRA INTEGRAL EQUATIONS

Consider the following integral equation:

\[
(2.1) \quad y(x) = f(x) + \int_0^x K(x, s)y(\lambda s) ds, \quad x \in [0, b], \ 0 < \lambda \leq 1,
\]

where \( f \in C[0, b], \ K(x, s) = \frac{L(x, s)}{|x-s|^{\alpha}}, \) for all \( x, s \in [0, b], \ x \neq s, \ 0 < \alpha < 1 \) and \( L \in C([0, b] \times [0, b]). \)

By using the results given in \([8]\) and \([2]\) for weakly singular Volterra-Fredholm integral equations, we obtain:

**Lemma 2.1.** If \( K(x, s) = \frac{L(x, s)}{|x-s|^{\alpha}}, \) for all \( x, s \in [0, b], \ x \neq s, \ 0 < \alpha < 1 \) and \( L \in C([0, b] \times [0, b]), \)

then the operator \( T : C[0, b] \rightarrow C[0, b], \)

\[
T(y)(x) := \int_0^x K(x, s)y(s) ds
\]

is well defined \( (T(y) \in C[0, b]). \)

**Lemma 2.2.** If \( K_i(x, s) = \frac{L_i(x, s)}{|x-s|^{\alpha_i}}, \ 0 < \alpha_i < 1 \) and \( L_i \in C([0, b] \times [0, b]), \ i = 1, 2, \) then the operator \( T : C[0, b] \rightarrow C[0, b], \)

\[
T(y)(x) := \int_0^x K_1(x, s)y(s) ds + \int_0^b K_2(x, u)y(u) du
\]

is well defined \( (T(y) \in C[0, b]). \)

We have

**Theorem 2.3.** In the conditions mentioned before, the equation \((2.1)\) has in \( C[0, b] \) a unique solution and this solution can be obtained by the successive approximation method, starting from any element of \( C[0, b]. \)

*Proof.* Because of Lemma 2.1, we have that the operator \( U : C[0, b] \rightarrow C[0, b], \)

\[
U(y)(x) := \int_0^x K(x, s)y(\lambda s) ds,
\]

is well defined.
is well defined. So, we have that \( C[0, b] \) is an invariant set for the operator \( T \), where

\[
T(y)(x) := f(x) + \int_0^x K(x, s)y(\lambda s)ds.
\]

The equation (2.1) can be written as a fixed point problem of the form

\[
y = T(y).
\]

Consider \( T : (C[0, b], \| \cdot \|_B) \to (C[0, b], \| \cdot \|_B) \), where \( \| \cdot \|_B \) is a Bielecki norm on \( C[0, b] \) defined by

\[
\|y\|_B = \max_{x \in [0, b]} |y(x)|e^{-\tau x}, \text{ and } \tau > 0.
\]

Denote

\[
L^* = \max_{(x, s) \in [0, b] \times [0, b]} |L(x, s)|.
\]

We have

\[
|T(y)(x) - T(z)(x)| \leq \int_0^x \frac{L^*}{|x - s|^\alpha} |y(\lambda s) - z(\lambda s)|e^{-\tau \lambda s} \cdot e^{\tau \lambda s} ds
\]

\[
\leq L^* \|y - z\|_B \int_0^x \frac{e^{\tau \lambda s}}{(x - s)^\alpha} ds
\]

\[
\leq L^* \|y - z\|_B \left( \frac{\int_0^x ds}{(x - s)^\alpha} \right)^{1/p} \left( \frac{\int_0^x e^{\tau \lambda s} ds}{\int_0^x (x - s)^\alpha} \right)^{1/q},
\]

where \( p > 0, q > 0, \)

\[
\frac{1}{p} + \frac{1}{q} = 1 \text{ and } \alpha p < 1.
\]

So,

\[
|T(y)(x) - T(z)(x)| \leq L^* \left( \frac{b^{1 - \alpha p}}{1 - \alpha p} \right)^{1/p} \frac{e^{\tau x}}{(\tau \lambda q)^{1/q}} \|y - z\|_B
\]

and

\[
|T(y)(x) - T(z)(x)|e^{-\tau x} \leq L^* \left( \frac{b^{1 - \alpha p}}{1 - \alpha p} \right)^{1/p} \frac{1}{(\tau \lambda q)^{1/q}} \|y - z\|_B, \text{ for all } x \in [0, b].
\]

It follows that

\[
\|T(y) - T(z)\|_B \leq L_T \|y - z\|_B, \text{ for all } y, z \in C[0, b],
\]

where

\[
L_T = L^* \left( \frac{b^{1 - \alpha p}}{1 - \alpha p} \right)^{1/p} \frac{1}{(\tau \lambda q)^{1/q}}.
\]

We can choose \( \tau \) large enough such that \( 0 < L_T < 1 \). So, the proof follows from Contraction principle. \( \square \)
3. VOLTERRA-FREDHOLM INTEGRAL EQUATIONS

Consider the following Volterra-Fredholm weakly singular integral equation:

\[ y(x) = f(x) + \int_0^x K_1(x, s)y(\lambda s)ds + \int_0^b K_2(x, u)y(\lambda u)du, \]

\[ x \in [0, b], \ 0 < \lambda \leq 1, \]

where \( f \in C[0, b] \) and \( K_i(x, s) = \frac{L_i(x,s)}{|x-s|^{1/\alpha_i}}, \ 0 < \alpha_i < 1, \ L_i \in C([0, b] \times [0, b]), \ i = 1, 2. \)

We have

**Theorem 3.1.** In the above conditions let \( L_i^* > 0 \) be such that \( |L_i(x, s)| \leq L_i^* \), for all \( x, s \in [0, b], \ i = 1, 2, \) and we suppose that there exist \( p > 0, \ q > 0 \) and \( \tau > 0, \) such that \( \alpha_1 p < 1, \ \alpha_2 p < 1, \ \frac{1}{p} + \frac{1}{q} = 1 \) and \( 0 < \tau L < 1, \) where

\[
L_T = \frac{1}{(\tau Lq)^{1/p}} \left[ L_1^* \left( \frac{b^{1-\alpha_1 p}}{1-\alpha_1 p} \right)^{1/p} + L_2^* \left( 1 + e^{\tau b} \right) \left( \frac{b^{1-\alpha_2 p}}{1-\alpha_2 p} \right)^{1/p} \right].
\]

Then the equation \( (3.1) \) has in \( C[0, b] \) a unique solution \( y^* \) and this solution can be obtained by the successive approximation method starting from any element of \( C[0, b]. \)

**Proof.** Let us consider the operators \( U_i : C[0, b] \to C[0, b], \ i = 1, 2, \) defined by

\[ U_1(y)(x) := \int_0^x K_1(x, s)y(\lambda s)ds \]

and

\[ U_2(y)(x) := \int_0^b K_2(x, u)y(\lambda u)du. \]

By using Lemma 2.2 we obtain that \( U_1 \) and \( U_2 \) are well defined.

The equation \( (3.1) \) is equivalent to the following fixed point problem:

\[ y = T(y), \] where \( T : C[0, b] \to C[0, b], \) is given by

\[ T(y)(x) := f(x) + U_1(y)(x) + U_2(y)(x), \ x \in [0, b], \ 0 < \lambda \leq 1, \]

and \( T \) is well defined.

Consider \( T : (C[0, b], \| \cdot \|_B) \to (C[0, b], \| \cdot \|_B), \) where \( \| \cdot \|_B \) is a Bielecki norm on \( C[0, b] \) defined by

\[ \|y\|_B := \max_{x \in [0, b]} |y(x)| e^{-\tau x} \] and \( \tau > 0. \)

We have

\[ |T(y)(x) - T(z)(x)| \leq |U_1(y)(x) - U_1(z)(x)| + |U_2(y)(x) - U_2(z)(x)|. \]

But from results given in Section 2, the following inequality holds:

\[ |U_1(y)(x) - U_2(y)(x)| \leq L_1^* \left( \frac{b^{1-\alpha_1 p}}{1-\alpha_1 p} \right)^{1/p} e^{\tau b} \|y - z\|_B. \]
We estimate
\[
|U_2(y)(x) - U_2(z)(x)| \leq \int_0^b \frac{L_2^*}{|x-u|^\alpha} |y(\lambda u) - z(\lambda u)| e^{-\tau \lambda u} e^{\tau u} du \leq \\
\leq L_2^* \|y - z\|_B \left( \int_0^x \frac{e^{\tau \lambda u}}{|x-u|^\alpha} du + \int_x^b \frac{e^{\tau \lambda u}}{|x-u|^\alpha} du \right)^{1/p} \left( \int_x^b e^{\tau \lambda u} du \right)^{1/q} \\
\leq L_2^* \|y - z\|_B \left[ \left( \frac{b^{1-a_2p}}{1-a_2p} \right)^{1/p} e^{\tau x} + \left( \frac{e^{\tau x}}{\lambda q} \right)^{1/q} \right] \left( \frac{b^{1-a_2p}}{1-a_2p} \right)^{1/p} \left( \frac{e^{\tau \lambda u}}{\lambda q} \right)^{1/q} \\
\leq L_2^* \|y - z\|_B \left[ \left( \frac{b^{1-a_2p}}{1-a_2p} \right)^{1/p} e^{\tau x} + \left( \frac{b^{1-a_2p}}{1-a_2p} \right)^{1/p} \left( \frac{e^{\tau \lambda u}}{\lambda q} \right)^{1/q} \right] \\
\leq L_2^* \|y - z\|_B \left[ \left( \frac{b^{1-a_2p}}{1-a_2p} \right)^{1/p} e^{\tau x} + \left( \frac{b^{1-a_2p}}{1-a_2p} \right)^{1/p} \left( \frac{e^{\tau \lambda u}}{\lambda q} \right)^{1/q} \right] \\
\leq L_2^* \|y - z\|_B \left[ \left( \frac{b^{1-a_2p}}{1-a_2p} \right)^{1/p} e^{\tau x} + \left( \frac{b^{1-a_2p}}{1-a_2p} \right)^{1/p} \left( \frac{e^{\tau \lambda u}}{\lambda q} \right)^{1/q} \right] \\
\leq L_2^* \|y - z\|_B e^{\tau x} \left( \frac{b^{1-a_2p}}{1-a_2p} \right)^{1/p} \left( \frac{1 + e^{\tau \lambda u}}{\lambda q} \right)^{1/q}.
\]

It follows that
\[
\|T(y)(x) - T(z)(x)\|e^{-\tau x} \leq \\
\leq L_T^* \left[ \left( \frac{b^{1-a_1p}}{1-a_1p} \right)^{1/p} + L_2^*(1 + e^{\tau b}) \right] \left( \frac{1}{\lambda q} \right)^{1/q} \|y - z\|_B.
\]

So,
\[
\|T(y) - T(z)\|_B \leq L_T \|y - z\|_B, \text{ for all } y, z \in C[0,b],
\]
where
\[
L_T = \left( \frac{1}{\lambda q} \right)^{1/q} \left[ \left( \frac{b^{1-a_1p}}{1-a_1p} \right)^{1/p} + L_2^*(1 + e^{\tau b}) \right]^{1/p}.
\]

So, the proof follows from Contraction principle. \(\square\)

**Remark 3.1.** For \(\lambda = 1\) in (2.1) and (3.1) we have the equations considered by Sz. András in [1] and [2]. \(\square\)

**Example 3.2.** Consider (3.1) in which \(K_i(x,s) := -\frac{1}{|x-s|^\alpha}, i = 1, 2,\) and \(b := 1, \lambda := \frac{1}{2}\). \(\square\)

We have

**Theorem 3.2.** If there exist \(0 < p < 2\) and \(\tau > 0\) such that
\[
\left( \frac{2}{\tau} \right)^{\frac{p-1}{p}} \left( \frac{e-1}{p} \right)^{\frac{p-1}{p}} \left( \frac{2}{2-p} \right)^{\frac{1}{p}} (2 + e^\tau) < 1,
\]
then the equation
\[ y(x) = f(x) - \int_0^x \frac{1}{(x-s)^{1/2}} y \left( \frac{s}{2} \right) \, ds - \int_0^1 \frac{1}{(x-s)^{1/2}} y \left( \frac{s}{2} \right) \, ds, \quad x \in [0, 1], \] (3.2)
where \( f \in C[0, b] \), has in \( C[0, 1] \) a unique solution.

Remark 3.3. By choosing \( p = \frac{11}{10} \) and \( \tau = \frac{1}{10} \), the condition in Theorem 3.2 is satisfied. \( \square \)

4. NUMERICAL MODEL

In this section we present a numerical model suitable to (3.1) based on a global collocation method using approximating splines, in particular the so-called Schoenberg variation-diminishing (SVD) splines [16].

In the following we recall the necessary background on SVD splines.

4.1. The SVD splines. Let \( J := [a, b] \) be a partition of the interval \( J := [\alpha, \beta] \) with \( H_m := \max_{0 \leq j \leq m} (x_{j+1} - x_j) \), \( H_m \to 0 \) as \( m \to \infty \), and let \( \{d_j: j = 0, \ldots, m + 1\} \) be a vector of positive integers where \( d_0 = d_{m+1} = p \) (\( p > 1 \)) and \( d_j \leq p - 1, \, j = 1, \ldots, m \).

We set \( n + p := \sum_{j=0}^{m+1} d_j \) and define \( \Pi_n = \{t_i : i = 1, 2, \ldots, n + p\} \) as the nondecreasing sequence obtained from \( X_m \) by repeating \( x_j \) exactly \( d_j \) times, \( j = 0, \ldots, m + 1 \).

\( \Pi_n \) is assumed as mesh of the set of normalized B-splines \( B_{i,p} \) of order \( p \) defined by the following recurrence relation:

\begin{align}
B_{i,p}(x) &= \frac{x-t_i}{t_{i+p-1} - t_i} B_{i,p-1}(x) + \frac{t_{i+p} - x}{t_{i+p} - t_{i+1}} B_{i+1,p-1}(x) \\
B_{i,1}(x) &= \begin{cases} 
1, & t_i \leq x < t_{i+1} \\
0, & \text{otherwise}
\end{cases}
\end{align}

(4.1) \hspace{1cm} (4.2)

Let \( \xi_i = \frac{t_{i+1} + \cdots + t_{i+p-1}}{p-1} \) (\( i = 1, 2, \ldots, n \)) be a set of nodes, the so-called Schoenberg points, belonging to \([t_i, t_{i+p}]\) for \( i = 1, 2, \ldots, n \).

For all \( g \in C(J) \) we define the following spline operator:

\[ W_n g := \sum_{i=1}^{n} g(\xi_i) B_{i,p}(x), \quad \xi_i \in J \, (i = 1, 2, \ldots, n). \] (4.3)

According to [9] \( W_n \) is a SVD spline operator. In [9] it is shown to be a projector operator.
5. NUMERICAL SOLUTION OF THE PROBLEM

Let us consider the function:

\[ y_n(x) := \sum_{i=1}^{n} \alpha_i B_{i,p}(x), \]

where \( \alpha_i \) \((i = 1, 2, \ldots, n)\) are chosen to satisfy the so-called generalized Nyström collocation system. Precisely, we introduce \( y_n(\lambda x) \) instead of \( y(\lambda x) \) in (3.1.3) obtaining:

\begin{equation}
(5.1) \quad y_n(x) = f(x) + \int_0^{\lambda x} K_1(x, s)y_n(\lambda s)ds + \int_0^{\lambda x} K_2(x, u)y_n(\lambda u)du,
\end{equation}

\( x \in [0, b], \ \lambda \in (0, 1] \).

We can rewrite (5.1) as:

\begin{equation}
(5.2) \quad y_n(x) = f(x) + \frac{1}{\lambda} \int_0^{\lambda x} K_1(x, \frac{s}{\lambda})y_n(s)ds + \frac{1}{\lambda} \int_0^{\lambda x} K_2(x, \frac{u}{\lambda})y_n(u)du,
\end{equation}

\( x \in [0, b], \ \lambda \in (0, 1] \).

Let \( J := [0, b] \), we choose in \( J \) a set of collocation points \( \tau_k \) \((k = 1, 2, \ldots, n)\), decoupled from the set of the \( \xi_i \) \((i = 1, 2, \ldots, n)\). Consequently from (5.2) we obtain the following collocation system:

\begin{equation}
(5.3) \quad \sum_{i=1}^{n} \alpha_i B_{i,p}(\tau_k) - \frac{1}{\lambda} \sum_{i=1}^{n} \alpha_i \left[ \int_0^{\lambda \tau_k} K_1(\tau_k, \frac{s}{\lambda})B_{i,p}(s)ds + \int_0^{\lambda \tau_k} K_2(\tau_k, \frac{u}{\lambda})B_{i,p}(u)du \right] = f(\tau_k), \quad \tau_k \in J (k = 1, 2, \ldots, n), \ \lambda \in (0, 1].
\end{equation}

The evaluation of the singular integrals

\[ I(K_1, B_{i,p}) = \int_0^{\lambda \tau_k} K_1(\tau_k, \frac{s}{\lambda})B_{i,p}(s)ds \]

and

\[ I(K_2, B_{i,p}) = \int_0^{\lambda \tau_k} K_2(\tau_k, \frac{u}{\lambda})B_{i,p}(u)du \]

is carried out by a recurrence formula analogous to (4.1).

The basis integrals

\[ I(K_1, B_{1,p}) = \int_0^{\lambda \tau_k} K_1(\tau_k, \frac{s}{\lambda})s^{p-1}B_{1,p}(\tau_k)ds \]

and

\[ I(K_2, B_{1,p}) = \int_0^{\lambda \tau_k} K_2(\tau_k, \frac{u}{\lambda})u^{p-1}B_{1,p}(u)du \]

are evaluated by a closed analytical formula.
We assume \( y_n(x) \) as approximated solution of (3.1). Now the problem is to analyze the convergence of \( y_n(x) \) to \( y(x) \).

5.1. **Convergence analysis.** In order to carry out the error analysis for the proposed method we can rewrite (3.1) as

\[
y(x) = f(x) + \int_0^b \tilde{K}(x, s)y(\lambda s)ds, \quad x \in [0, b], \; \lambda \in (0, 1]
\]

and (5.1) as

\[
y_n(x) = f(x) + \int_0^b \tilde{K}(x, s)y_n(\lambda s)ds, \quad x \in [0, b], \; \lambda \in (0, 1],
\]

where \( \tilde{K}(x, s) = \tilde{K}_1(x, s) + K_2(x, s) \)

and

\[
\tilde{K}_1(x, s) = \begin{cases} K_1(x, s) & \text{if } 0 \leq s \leq x \\ 0 & \text{if } s > x \\ \end{cases}
\]

**Theorem 5.1.** Let \( W_n \) as in (4.3) and let

\[
W_n \tilde{K}g = \frac{1}{\lambda} \int_0^b \tilde{K}(x, s)W_n g(s)ds, \quad x \in [0, b], \; \lambda \in (0, 1].
\]

Then

\[
\|y - y_n\| \leq \|I - W_n \tilde{K}\|^{-1}\|I - W_n y\|.
\]

**Proof.** We can transform (5.4) and (5.5) in the operator form

\[
(I - \tilde{K})y = f
\]

and

\[
(I - \tilde{K})y_n = f
\]

where

\[
y = Iy, \quad \tilde{K}y = \frac{1}{\lambda} \int_0^b \tilde{K}(x, s)y(s)ds, \quad \tilde{K}y_n = \frac{1}{\lambda} \int_0^b \tilde{K}(x, s)y_n(s)ds,
\]

\[
x \in [0, b], \; \lambda \in (0, 1].
\]

Applying the operator (4.3) to (5.7) we obtain

\[
W_n(I - \tilde{K})y = W_n f,
\]

that is equivalent to

\[
(I - W_n \tilde{K})y = W_n f + (I - W_n)y.
\]

Analogously for (5.8) we obtain

\[
(I - W_n \tilde{K})y_n = W_n f + (I - W_n)y_n.
\]
From (5.10) and (5.11) and taking account that $W_n$ is a projector operator, it follows
\begin{equation}
(I - W_n\tilde{K})(y - y_n) = (I - W_n)y.
\end{equation}

Then (5.6) holds. \hfill \Box

Corollary 5.2. Let $W_n$ and $\tilde{K}$ be the operators as in (5.12). Then for $n$ sufficiently large, say $n \geq N$, the operator $(I - W_n\tilde{K})^{-1}$ from $C(I)$ to $C(I)$ exists. Moreover it is uniformly bounded, i.e.:
\[ \sup_{n \geq N} \| I - W_n\tilde{K} \|^{-1} \leq M < \infty. \]

Proof. From Theorem 1 and Corollary 2 in [14] it follows that $\| \tilde{K} - W_n\tilde{K} \| \rightarrow 0$ as $n \rightarrow \infty$. Consequently, by the proof of Theorem 1 in [16], the Corollary 5.2 is proved. \hfill \Box

Remark 5.1. From (5.12) and from the Corollary 5.2 it follows that $\| y - y_n \| \rightarrow 0$ exactly with the same rate of convergence as $\| y - W_ny \|$ does. \hfill \Box

6. NUMERICAL RESULTS

In what follows we present some numerical results for some Volterra-Fredholm integral equation, by using the numerical method presented above. In particular, the exactness of the method for polynomial functions till third degree is tested. In all examples the hypotheses of existence and uniqueness of the solution are guaranteed.

We consider the following equation:
\[ y(x) = f(x) + \int_0^x K_1(x,s)y(\lambda s)ds + \int_0^x K_2(x,u)y(\lambda u)du, \quad x \in [0,b], \lambda \in (0,1], \]
where $y$ is the unknown function and $K_1, K_2, f$ are given functions.

<table>
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<tr>
<th>$f(x)$</th>
<th>$y(x)$</th>
<th>$\lambda = 1$</th>
<th>$\lambda = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 + 2\sqrt{b - x + 2\sqrt{x}}$</td>
<td>$x$</td>
<td>$1.43 \cdot 10^{-16}$</td>
<td>$2.39 \cdot 10^{-15}$</td>
</tr>
<tr>
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<td>$x^2$</td>
<td>$4.45 \cdot 10^{-16}$</td>
<td>$7.78 \cdot 10^{-16}$</td>
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<td>$x^3$</td>
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<td>$5.50 \cdot 10^{-17}$</td>
</tr>
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<td>$x^4$</td>
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</tr>
<tr>
<td>$x^4 + 2\lambda^4/9(\sqrt{b - x(b^4 + 8xb^3/7)} + 48x^{2b^2}/35 + 64x^3(b + 2x)/35) + 256x^{7/2}/35$</td>
<td>$x^4$</td>
<td>$1.88 \cdot 10^{-5}$</td>
<td>$1.29 \cdot 10^{-5}$</td>
</tr>
</tbody>
</table>

In all cases the interval $[0, b]$ has been divided by $m = 11$ equispaced simple nodes $x_j = (0.1)j b$, $(j = 0, 1, \ldots, 10)$, except for $x_0$ and $x_{10}$ of multiplicity $p = 4$. The corresponding vector $t$ has $n + p = 17$ components.
The unknown function is approximated in \( n = 13 \) nodes belonging to \([0, b]\).

In Table 1 we show the results obtained with the choice \( K_1(x, t) = K_2(x, t) = -|t-x|^{-1/2}, \ b = 1, \ \lambda = 1 \) and \( \lambda = 0.5 \). For \( \lambda = 0.5 \) we have the equation (3.2), considered in Example 3.2. For brevity we indicate the mean of the absolute values of the errors evaluated in the interval. Our computer programs are written in MATLAB 7.3.

REFERENCES


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