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THEORETICAL AND NUMERICAL RESULTS ABOUT SOME WEAKLY SINGULAR VOLTERRA-FREDHOLM EQUATIONS

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Abstract. In this paper existence, uniqueness results for the solution of some weakly singular linear Volterra and Volterra-Fredholm integral equations are given. For these equations, a numerical model is proposed and its convergence and rate of convergence are analyzed. Numerical results on some polynomial test functions are given.

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1. INTRODUCTION

The singular integral equations have been studied by many authors. As monographs in this field we quote here W. Pogorzelski [13] (1966), D.V. Ionescu [8] (1972), H.M. Srivastava and R.G. Buschman [17] (1992), R. Estrada and R.P. Kanwall [7] (2000), A. Chakrabarti and G. Vanden Berge [5] (2002). As papers we quote here [1], [6] and [15]. For results in the field of Volterra-Fredholm integral equations by using fixed point theory, we quote [2], [10], [11] and [12].

The aim of this paper is to present existence, uniqueness and numerical results for the solutions of the following weakly singular linear integral equations with linear modification of the argument:

$$y(x) = f(x) + \int_0^x K(x,s)y(\lambda s)\mathrm{d}s, \quad x \in [0,b],\ 0 < \lambda \le 1,$$

where $f \in C[0,b]$, $K(x,s) = \frac{L(x,s)}{|x-s|^\alpha}$ for all $x, s \in [0,b]$, $x \neq s$, $0 < \alpha < 1$ and $L \in C([0,b] \times [0,b])$, respectively

$$y(x) = f(x) + \int_0^x K_1(x,s)y(\lambda s)\mathrm{d}s + \int_0^b K_2(x,u)y(\lambda u)\mathrm{d}u,\ x \in [0,b],\ 0 < \lambda \le 1,$$

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where $f \in C[0,b]$, $K_i(x,s) = \frac{L_i(x,s)}{|x-s|^{\alpha_i}}$, $0 < \alpha_i < 1$, and $L_i \in C([0,b] \times [0,b])$, $i = 1,2$.

At these problems, taking into account the kernel singularity, a particular numerical model [14] is adapted. This model is based on Nystrom collocation method, using Schoenberg variation diminishing (SVD) splines of fourth order. The order of convergence is studied and numerical results are given to test the polynomial exactness.

Section 2 and 3 are respectively devoted to the theoretical results for Volterra and Volterra-Fredholm weakly singular integral equations with linear modification of the argument. Sections 4 and 5 present the numerical model and its convergence analysis; in Section 6 numerical results are given.

2. VOLTERRA INTEGRAL EQUATIONS

Consider the following integral equation:

$$y(x) = f(x) + \int_0^x K(x,s)y(\lambda s)\mathrm{d}s, \quad x \in [0,b],\ 0 < \lambda \leq 1, \tag{2.1$_\lambda$}$$

where $f \in C[0,b]$, $K(x,s) = \frac{L(x,s)}{|x-s|^{\alpha}}$ for all $x,s \in [0,b]$, $x \neq s$, $0 < \alpha < 1$ and $L \in C([0,b] \times [0,b])$.

By using the results given in [8] and [2] for weakly singular Volterra-Fredholm integral equations, we obtain:

Lemma 2.1. *If* $K(x,s) = \frac{L(x,s)}{|x-s|^{\alpha}}$, $0 < \alpha < 1$ *and* $L \in C([0,b] \times [0,b])$, *then the operator* $T : C[0,b] \to C[0,b]$,

$$T(y)(x) := \int_0^x K(x,s)y(s)\mathrm{d}s$$

is well defined $(T(y) \in C[0,b])$.

Lemma 2.2. *If* $K_i(x,s) = \frac{L(x,s)}{|x-s|^{\alpha_i}}$, $0 < \alpha_i < 1$ *and* $L_i \in C([0,b] \times [0,b])$, $i = 1,2$, *then the operator* $T : C[0,b] \to C[0,b]$,

$$T(y)(x) := \int_0^x K_1(x,s)y(s)\mathrm{d}s + \int_0^b K_2(x,u)y(u)\mathrm{d}u$$

is well defined $(T(y) \in C[0,b])$.

We have

Theorem 2.3. *In the conditions mentioned before, the equation* (2.1_λ) *has in* $C[0,b]$ *a unique solution and this solution can be obtained by the successive approximation method, starting from any element of* $C[0,b]$.

Proof. Because of Lemma 2.1, we have that the operator $U : C[0,b] \to C[0,b]$,

$$U(y)(x) := \int_0^x K(x,s)y(\lambda s)\mathrm{d}s,$$

is well defined. So, we have that $C[0,b]$ is an invariant set for the operator T, where

$$T(y)(x) := f(x) + \int_0^x K(x,s)y(\lambda s)\mathrm{d}s.$$

The equation (2.1_λ) can be written as a fixed point problem of the form $y = T(y)$.

Consider $T : (C[0,b], \|\cdot\|_B) \to (C[0,b], \|\cdot\|_B)$, where $\|\cdot\|_B$ is a Bielecki norm on $C[0,b]$ defined by

$$\|y\|_B = \max_{x\in[0,b]} |y(x)|\mathrm{e}^{-\tau x}, \text{ and } \tau > 0.$$

Denote

$$L^* = \max_{(x,s)\in[0,b]\times[0,b]} |L(x,s)|.$$

We have

$$\begin{aligned}
|T(y)(x) - T(z)(x)| &\le \int_0^x \frac{L^*}{|x-s|^\alpha} |y(\lambda s) - z(\lambda s)|\mathrm{e}^{-\tau\lambda s} \cdot \mathrm{e}^{\tau\lambda s}\mathrm{d}s \\
&\le L^*\|y-z\|_B \int_0^x \frac{\mathrm{e}^{\tau\lambda s}}{(x-s)^\alpha}\mathrm{d}s \\
&\le L^*\|y-z\|_B \Big(\int_0^x \frac{\mathrm{d}s}{(x-s)^{\alpha p}}\Big)^{1/p} \Big(\int_0^x \mathrm{e}^{\tau\lambda s q}\mathrm{d}s\Big)^{\frac{1}{q}},
\end{aligned}$$

where $p > 0$, $q > 0$,

$$\tfrac{1}{p} + \tfrac{1}{q} = 1 \quad \text{and} \quad \alpha p < 1.$$

So,

$$|T(y)(x) - T(z)(x)| \le L^*\Big(\frac{b^{1-\alpha p}}{1-\alpha p}\Big)^{1/p} \frac{\mathrm{e}^{\tau x}}{(\tau\lambda q)^{1/q}} \|y-z\|_B$$

and

$$|T(y)(x) - T(z)(x)|\mathrm{e}^{-\tau x} \le L^*\Big(\frac{b^{1-\alpha p}}{1-\alpha p}\Big)^{1/p} \frac{1}{(\tau\lambda q)^{1/q}} \|y-z\|_B, \text{ for all } x \in [0,b].$$

It follows that

$$\|T(y) - T(z)\|_B \le L_T\|y-z\|_B, \text{ for all } y,z \in C[0,b],$$

where

$$L_T = L^*\Big(\frac{b^{1-\alpha p}}{1-\alpha p}\Big)^{1/p} \frac{1}{(\tau\lambda q)^{1/q}}.$$

We can choose τ large enough such that $0 < L_T < 1$. So, the proof follows from Contraction principle. $\square$

3. VOLTERRA-FREDHOLM INTEGRAL EQUATIONS

Consider the following Volterra-Fredholm weakly singular integral equation:

$$y(x) = f(x) + \int_0^x K_1(x,s)y(\lambda s)\mathrm{d}s + \int_0^b K_2(x,u)y(\lambda u)\mathrm{d}u, \tag{3.1$_\lambda$}$$

$$x \in [0,b],\ 0 < \lambda \le 1,$$

where $f \in C[0,b]$ and $K_i(x,s) = \frac{L_i(x,s)}{|x-s|^{\alpha_i}}$, $0 < \alpha_i < 1$, $L_i \in C([0,b]\times[0,b])$, $i = 1,2$.

We have

Theorem 3.1. *In the above conditions let $L_i^* > 0$ be such that $|L_i(x,s)| \le L_i^*$, for all $x,s \in [0,b]$, $i = 1,2$, and we suppose that there exist $p > 0$, $q > 0$ and $\tau > 0$, such that $\alpha_1 p < 1$, $\alpha_2 p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $0 < L_T < 1$, where*

$$L_T = \frac{1}{(\tau\lambda q)^{1/q}} \left[L_1^* \left(\frac{b^{1-\alpha_1 p}}{1-\alpha_1 p}\right)^{1/p} + L_2^*(1+e^{\tau b}) \left(\frac{b^{1-\alpha_2 p}}{1-\alpha_2 p}\right)^{1/p} \right].$$

Then the equation (3.1_λ) has in $C[0,b]$ a unique solution y^* and this solution can be obtained by the successive approximation method starting from any element of $C[0,b]$.

Proof. Let us consider the operators $U_i : C[0,b] \to C[0,b]$, $i = 1,2$, defined by

$$U_1(y)(x) := \int_0^x K_1(x,s)y(\lambda s)\mathrm{d}s$$

and

$$U_2(y)(x) := \int_0^b K_2(x,u)y(\lambda u)\mathrm{d}u.$$

By using Lemma 2.2 we obtain that U_1 and U_2 are well defined.

The equation (3.1_λ) is equivalent to the following fixed point problem: $y = T(y)$, where $T : C[0,b] \to C[0,b]$, is given by

$$T(y)(x) := f(x) + U_1(y)(x) + U_2(y)(x), \quad x \in [0,b],\ 0 < \lambda \le 1,$$

and T is well defined.

Consider $T : (C[0,b], \|\cdot\|_B) \to (C[0,b], \|\cdot\|_B)$, where $\|\cdot\|_B$ is a Bielecki norm on $C[0,b]$ defined by

$$\|y\|_B := \max_{x\in[0,b]} |y(x)|\mathrm{e}^{-\tau x} \quad \text{and} \quad \tau > 0.$$

We have

$$|T(y)(x) - T(z)(x)| \le |U_1(y)(x) - U_1(z)(x)| + |U_2(y)(x) - U_2(z)(x)|.$$

But from results given in Section 2, the following inequality holds:

$$|U_1(y)(x) - U_2(y)(x)| \le L_1^* \left(\frac{b^{1-\alpha_1 p}}{1-\alpha_1 p}\right)^{1/p} \frac{\mathrm{e}^{\tau x}}{(\tau\lambda q)^{1/q}} \|y - z\|_B.$$

We estimate

$$|U_2(y)(x) - U_2(z)(x)| \leq \int_0^b \tfrac{L_2^*}{|x-u|^{\alpha_2}} |y(\lambda u) - z(\lambda u)| \mathrm{e}^{-\tau\lambda u} \mathrm{e}^{\tau\lambda u} \mathrm{d}u \leq$$

$$\leq L_2^* \|y-z\|_B \left(\int_0^x \tfrac{\mathrm{e}^{\tau\lambda u}}{|x-u|^{\alpha_2}} \mathrm{d}u + \int_x^b \tfrac{\mathrm{e}^{\tau\lambda u}}{|x-u|^{\alpha_2}} \mathrm{d}u \right)$$

$$\leq L_2^* \|y-z\|_B \left[\left(\tfrac{b^{1-\alpha_2 p}}{1-\alpha_2 p} \right)^{1/p} \tfrac{\mathrm{e}^{\tau x}}{(\tau\lambda q)^{1/q}} + \left(\int_x^b \tfrac{\mathrm{d}u}{(u-x)^{\alpha_2 p}} \right)^{1/p} \left(\int_x^b \mathrm{e}^{\tau\lambda u q} \mathrm{d}u \right)^{1/q} \right]$$

$$\leq L_2^* \|y-z\|_B \left[\left(\tfrac{b^{1-\alpha_2 p}}{1-\alpha_2 p} \right)^{1/p} \tfrac{\mathrm{e}^{\tau x}}{(\tau\lambda q)^{1/q}} + \left(\tfrac{(u-x)^{1-\alpha_2 p}}{1-\alpha_2 p} \Big|_x^b \right)^{1/p} \left(\tfrac{\mathrm{e}^{\tau\lambda u q}}{\tau\lambda q} \Big|_x^b \right)^{1/q} \right]$$

$$\leq L_2^* \|y-z\|_B \left[\left(\tfrac{b^{1-\alpha_2 p}}{1-\alpha_2 p} \right)^{1/p} \tfrac{\mathrm{e}^{\tau x}}{(\tau\lambda q)^{1/q}} + \left(\tfrac{b^{1-\alpha_2 p}}{1-\alpha_2 p} \right)^{1/p} \tfrac{(\mathrm{e}^{\tau\lambda b q} - \mathrm{e}^{\tau\lambda x q})^{1/q}}{(\tau\lambda q)^{1/q}} \right]$$

$$\leq L_2^* \|y-z\|_B \left[\left(\tfrac{b^{1-\alpha_2 p}}{1-\alpha_2 p} \right)^{1/p} \tfrac{\mathrm{e}^{\tau x}}{(\tau\lambda q)^{1/q}} + \left(\tfrac{b^{1-\alpha_2 p}}{1-\alpha_2 p} \right)^{1/p} \tfrac{\mathrm{e}^{\tau\lambda b}}{(\tau\lambda q)^{1/q}} \right]$$

$$\leq L_2^* \|y-z\|_B \left[\left(\tfrac{b^{1-\alpha_2 p}}{1-\alpha_2 p} \right)^{1/p} \tfrac{\mathrm{e}^{\tau x}}{(\tau\lambda q)^{1/q}} + \left(\tfrac{b^{1-\alpha_2 p}}{1-\alpha_2 p} \right)^{1/p} \tfrac{\mathrm{e}^{\tau(b-x)} \mathrm{e}^{\tau x}}{(\tau\lambda q)^{1/q}} \right]$$

$$\leq L_2^* \|y-z\|_B \mathrm{e}^{\tau x} \left(\tfrac{b^{1-\alpha_2 p}}{1-\alpha_2 p} \right)^{1/p} \tfrac{1+\mathrm{e}^{\tau b}}{(\tau\lambda q)^{1/q}}.$$

It follows that

$$|T(y)(x) - T(z)(x)| \mathrm{e}^{-\tau x} \leq$$

$$\leq \left[L_1^* \left(\tfrac{b^{1-\alpha_1 p}}{1-\alpha_1 p} \right)^{1/p} + L_2^* (1+\mathrm{e}^{\tau b}) \left(\tfrac{b^{1-\alpha_2 p}}{1-\alpha_2 p} \right)^{1/p} \right] \tfrac{1}{(\tau\lambda q)^{1/q}} \|y-z\|_B.$$

So,

$$\|T(y) - T(z)\|_B \leq L_T \|y-z\|_B, \text{ for all } y, z \in C[0,b],$$

where

$$L_T = \tfrac{1}{(\tau\lambda q)^{1/q}} \left[L_1^* \left(\tfrac{b^{1-\alpha_1 p}}{1-\alpha_1 p} \right)^{1/p} + L_2^* (1+\mathrm{e}^{\tau b}) \left(\tfrac{b^{1-\alpha_2 p}}{1-\alpha_2 p} \right)^{1/p} \right].$$

So, the proof follows from Contraction principle. □

Remark 3.1. For $\lambda = 1$ in (2.1_λ) and (3.1_λ) we have the equations considered by Sz. András in [1] and [2]. □

Example 3.2. Consider (3.1_λ) in which $K_i(x,s) := -\frac{1}{|x-s|^{1/2}}$, $i = 1,2$, and $b := 1$, $\lambda := \frac{1}{2}$. □

We have

Theorem 3.2. *If there exist* $0 < p < 2$ *and* $\tau > 0$ *such that*

$$\left(\tfrac{2}{\tau}\right)^{\frac{p-1}{p}} \left(\tfrac{p-1}{p}\right)^{\frac{p-1}{p}} \left(\tfrac{2}{2-p}\right)^{\frac{1}{p}} (2 + \mathrm{e}^{\tau}) < 1,$$

then the equation

$$y(x) = f(x) - \int_0^x \tfrac{1}{(x-s)^{1/2}} y\left(\tfrac{s}{2}\right) \mathrm{d}s - \int_0^1 \tfrac{1}{|x-s|^{1/2}} y\left(\tfrac{s}{2}\right) \mathrm{d}s, \quad x \in [0,1], \tag{3.2}$$

where $f \in C[0,b]$, has in $C[0,1]$ a unique solution.

Remark 3.3. By choosing $p = \frac{11}{10}$ and $\tau = \frac{1}{10}$, the condition in Theorem 3.2 is satisfied. □

4. NUMERICAL MODEL

In this section we present a numerical model suitable to (3.1_λ) based on a global collocation method using approximating splines, in particular the so called Schoenberg variation-diminishing (SVD) splines [16].

In the following we recall the necessary background on SVD splines.

4.1. The SVD splines. Let $X_m := \{\alpha = x_0 < x_1 < \cdots < x_m < x_{m+1} = \beta\}$ be a partition of the interval $J := [\alpha, \beta]$ with $H_m := \max\limits_{0\le j\le m}(x_{j+1}-x_j), H_m \to 0$ as $m \to \infty$, and let $\{d_j : j = 0, \ldots, m+1\}$ be a vector of positive integers where $d_0 = d_{m+1} = p$ $(p > 1)$ and $d_j \le p-1$, $j = 1, \ldots, m$.

We set $n + p := \sum\limits_{j=0}^{m+1} d_j$ and define $\Pi_n = \{t_i : i = 1, 2, \ldots, n+p\}$ as the nondecreasing sequence obtained from X_m by repeating x_j exactly d_j times, $j = 0, \ldots, m+1$.

Π_n is assumed as mesh of the set of normalized B-splines $B_{i,p}$ $(i = 1, \ldots, n)$ of order p defined by the following recurrence relation:

$$B_{i,p}(x) = \frac{x - t_i}{t_{i+p-1} - t_i} B_{i,p-1}(x) + \frac{t_{i+p} - x}{t_{i+p} - t_{i+1}} B_{i+1,p-1}(x) \tag{4.1}$$

$$B_{i,1}(x) = \begin{cases} 1, & t_i \le x < t_{i+1} \\ 0, & \text{otherwise} \end{cases} \tag{4.2}$$

Let $\xi_i = \frac{t_{i+1}+\cdots+t_{i+p-1}}{p-1}$ $(i = 1, 2, \ldots, n)$ be a set of nodes, the so-called Schoenberg points, belonging to $[t_i, t_{i+p}]$ for $i = 1, 2, \ldots, n$.

For all $g \in C(J)$ we define the following spline operator:

$$W_n g := \sum_{i=1}^{n} g(\xi_i) B_{i,p}(x), \qquad \xi_i \in J \ (i = 1, 2, \ldots, n). \tag{4.3}$$

According to [9] W_n is a SVD spline operator. In [9] it is shown to be a projector operator.

5. NUMERICAL SOLUTION OF THE PROBLEM

Let us consider the function:

$$y_n(x) := \sum_{i=1}^{n} \alpha_i B_{i,p}(x),$$

where α_i $(i = 1, 2, \ldots, n)$ are chosen to satisfy the so called generalized Nystrom collocation system. Precisely, we introduce $y_n(\lambda x)$ instead of $y(\lambda x)$ in (3.1_λ) obtaining:

$$y_n(x) = f(x) + \int_0^x K_1(x,s) y_n(\lambda s)\mathrm{d}s + \int_0^b K_2(x,u) y_n(\lambda u)\mathrm{d}u, \tag{5.1}$$

$$x \in [0,b],\ \lambda \in (0,1].$$

We can rewrite (5.1) as:

$$y_n(x) = f(x) + \tfrac{1}{\lambda} \int_0^{\lambda x} K_1(x, \tfrac{s}{\lambda}) y_n(s)\mathrm{d}s + \tfrac{1}{\lambda} \int_0^{\lambda b} K_2(x, \tfrac{u}{\lambda}) y_n(u)\mathrm{d}u, \tag{5.2}$$

$$x \in [0,b],\ \lambda \in (0,1].$$

Let $J := [0,b]$, we choose in J a set of collocation points τ_k $(k = 1, 2, \ldots, n)$, decoupled from the set of the ξ_i $(i = 1, 2, \ldots, n)$. Consequently from (5.2) we obtain the following collocation system:

(5.3)

$$\sum_{i=1}^{n} \alpha_i B_{i,p}(\tau_k) - \tfrac{1}{\lambda} \sum_{i=1}^{n} \alpha_i \left[\int_0^{\lambda\tau_k} K_1(\tau_k, \tfrac{s}{\lambda}) B_{i,p}(s)\mathrm{d}s + \int_0^{\lambda b} K_2(\tau_k, \tfrac{u}{\lambda}) B_{i,p}(u)\mathrm{d}u \right]$$
$$= f(\tau_k),\quad \tau_k \in J\ (k = 1, 2, \ldots, n),\ \lambda \in (0,1].$$

The evaluation of the singular integrals

$$I(K_1, B_{i,p}) = \int_0^{\lambda\tau_k} K_1(\tau_k, \tfrac{s}{\lambda}) B_{i,p}(s)\mathrm{d}s$$

and

$$I(K_2, B_{i,p}) = \int_0^{\lambda b} K_2(\tau_k, \tfrac{u}{\lambda}) B_{i,p}(u)\mathrm{d}u$$

is carried out by a recurrence formula analogous to (4.1).

The basis integrals

$$I(K_1, B_{1,p}) = \int_0^{\lambda\tau_k} K_1(\tau_k, \tfrac{s}{\lambda}) s^{p-1} B_{1,p}(\tau_k)\mathrm{d}s$$

and

$$I(K_2, B_{1,p}) = \int_0^{\lambda b} K_2(\tau_k, \tfrac{u}{\lambda}) u^{p-1} B_{1,p}(u)\mathrm{d}u$$

are evaluated by a closed analytical formula.

We assume $y_n(x)$ as approximated solution of (3.1_λ). Now the problem is to analyze the convergence of $y_n(x)$ to $y(x)$.

5.1. Convergence analysis. In order to carry out the error analysis for the proposed method we can rewrite (3.1_λ) as

$$y(x) = f(x) + \int_0^b \widetilde{K}(x,s)y(\lambda s)\mathrm{d}s, \quad x \in [0,b],\ \lambda \in (0,1] \tag{5.4}$$

and (5.1) as

$$y_n(x) = f(x) + \int_0^b \widetilde{K}(x,s)y_n(\lambda s)\mathrm{d}s, \quad x \in [0,b],\ \lambda \in (0,1], \tag{5.5}$$

where

$$\widetilde{K}(x,s) = \widetilde{K_1}(x,s) + K_2(x,s)$$

and

$$\widetilde{K_1}(x,s) = \begin{cases} K_1(x,s) & \text{if} \quad 0 \le s \le x \\ 0 & \text{if} \quad s > x \end{cases}$$

THEOREM 5.1. *Let W_n as in (4.3) and let*

$$W_n\ \widetilde{K}g = \tfrac{1}{\lambda}\int_0^{\lambda b} \widetilde{K}(x,\tfrac{s}{\lambda})W_n g(s)\mathrm{d}s, \quad x \in [0,b],\ \lambda \in (0,1].$$

Then

$$\|y - y_n\| \le \|I - W_n\widetilde{K}\|^{-1}\|(I - W_n)y\|. \tag{5.6}$$

Proof. We can transform (5.4) and (5.5) in the operator form

$$(I - \widetilde{K})y = f \tag{5.7}$$

$$(I - \widetilde{K})y_n = f \tag{5.8}$$

where

$$Iy = y, \quad \widetilde{K}y = \tfrac{1}{\lambda}\int_0^{\lambda b} \widetilde{K}(x,\tfrac{s}{\lambda})y(s)\mathrm{d}s, \quad \widetilde{K}y_n = \tfrac{1}{\lambda}\int_0^{\lambda b} \widetilde{K}(x,\tfrac{s}{\lambda})y_n(s)\mathrm{d}s,$$

$$x \in [0,b],\ \lambda \in (0,1].$$

Applying the operator (4.3) to (5.7) we obtain

$$W_n(I - \widetilde{K})y = W_n f, \tag{5.9}$$

that is equivalent to

$$(I - W_n\widetilde{K})y = W_n f + (I - W_n)y. \tag{5.10}$$

Analogously for (5.8) we obtain

$$(I - W_n\widetilde{K})y_n = W_n f + (I - W_n)y_n. \tag{5.11}$$

From (5.10) and (5.11) and taking account that W_n is a projector operator, it follows

$$(I - W_n\widetilde{K})(y - y_n) = (I - W_n)y. \tag{5.12}$$

Then (5.6) holds. □

COROLLARY 5.2. *Let W_n and $\widetilde{K}$ be the operators as in (5.12). Then for n sufficiently large, say $n \geq N$, the operator $(I - W_n\widetilde{K})^{-1}$ from $C(I)$ to $C(I)$ exists. Moreover it is uniformly bounded, i.e.:*

$$sup_{n\geq N} \|I - W_n\widetilde{K}\|^{-1} \leq M < \infty.$$

Proof. From Theorem 1 and Corollary 2 in [14] it follows that $\|\widetilde{K} - W_n\widetilde{K}\| \to 0$ as $n \to \infty$. Consequently, by the proof of Theorem 1 in [16], the Corollary 5.2 is proved. □

REMARK 5.1. From (5.12) and from the Corollary 5.2 it follows that $\|y - y_n\| \to 0$ exactly with the same rate of convergence as $\|y - W_n y\|$ does. □

6. NUMERICAL RESULTS

In what follows we present some numerical results for some Volterra-Fredholm integral equation, by using the numerical method presented above. In particular, the exactness of the method for polynomial functions till third degree is tested. In all examples the hypotheses of existence and uniqueness of the solution are guaranteed.

We consider the following equation:

$$y(x) = f(x) + \int_0^x K_1(x,s)y(\lambda s)\mathrm{d}s + \int_0^b K_2(x,u)y(\lambda u)\mathrm{d}u,\ x \in [0,b], \lambda \in (0,1],$$

where y is the unknown function and K_1, K_2, f are given functions.

Table 1.

$f(x)$	$y(x)$	$\lambda = 1$	$\lambda = 0.5$
$1 + 2[\sqrt{b-x} + 2\sqrt{x}]$	1	$1.43 \cdot 10^{-16}$	$2.39 \cdot 10^{-15}$
$x + 2\lambda/3[\sqrt{b-x}(b+2x) + 4x^{3/2}]$	x	$4.45 \cdot 10^{-16}$	$7.78 \cdot 10^{-16}$
$x^2 + 2\lambda^2/5[\sqrt{b-x}(b^2 + 4x(b+2x)/3)$ $+16x^{5/2}/3]$	x^2	$6.16 \cdot 10^{-17}$	$5.50 \cdot 10^{-17}$
$x^3 + 2\lambda^3/7[\sqrt{b-x}(b^3 + 6xb^2/5$ $+8x^2(b+2x)/5) + 32x^{7/2}/5]$	x^3	$9.08 \cdot 10^{-17}$	$3.60 \cdot 10^{-16}$
$x^4 + 2\lambda^4/9[\sqrt{b-x}(b^4 + 8xb^3/7$ $+48x^2b^2/35 + 64x^3(b+2x)/35)$ $+256x^{7/2}/35]$	x^4	$1.88 \cdot 10^{-5}$	$1.29 \cdot 10^{-5}$

In all cases the interval $[0,b]$ has been divided by $m = 11$ equispaced simple nodes $x_j = (0.1)jb$, $(j = 0,1,\ldots,10)$, except for x_0 and x_{10} of multiplicity $p = 4$. The corresponding vector $\mathbf{t}$ has $n + p = 17$ components.

The unknown function is approximated in $n = 13$ nodes belonging to $[0, b]$.

In Table 1 we show the results obtained with the choice $K_1(x,t) = K_2(x,t) = -|t-x|^{-1/2}$, $b = 1$, $\lambda = 1$ and $\lambda = 0.5$. For $\lambda = 0.5$ we have the equation (3.2), considered in Example 3.2. For brevity we indicate the mean of the absolute values of the errors evaluated in the interval. Our computer programs are written in MATLAB 7.3.

REFERENCES

[1] András, Sz., *Weakly singular Volterra and Fredholm-Volterra integral equations*, Studia Univ. Babeş-Bolyai, Mathematica, **48**, no. 3, pp. 147–155, 2003.

[2] András, Sz., *Fredholm-Volterra integral equations*, EDP, Bucureşti, 2005 (in Romanian).

[3] Anselone, P.M., *Nonlinear integral equations*, The University of Wisconsin Press, 1964.

[4] Caliò, F., Marchetti, E. and Rabinowitz, P., *On the numerical solution of the generalized Prandtl equation using variation-diminishing splines*, J. Comput. Appl. Math., **60**, pp. 297–307, 1995.

[5] Chakrabarti, A. and Vanden Berge, G., *Numerical solutions of singular integral equations*, Elsevier Preprint, 2002.

[6] Dagnino, C., Demichelis, V. and Santi, E., *A nodal spline collocation method for weakly singular Volterra integral equations*, Studia Univ. "Babeş-Bolyai", **48**, no. 3, pp. 71–81, 2003.

[7] Estrada, R. and Kanwal, R.P., *Singular integral equations*, Birkhauser, 2000.

[8] Ionescu, D.V., *Differential and integral equations*, EDP, Bucureşti, 1972 (in Romanian).

[9] Lyche, T. and Schumaker, L.L., *Local spline approximation methods*, J. Approximation Theory, **15**, pp. 294–325, 1975.

[10] Mureşan, V., *Existence, uniqueness and data dependence for the solutions of some integro-differential equations of mixed type in Banach space*, Journal for Analysis and its Applications, **23**, no. 1, pp. 205–216, 2004.

[11] Naroşi, I., *A remark on Fredholm-Volterra integral equations*, Babeş-Bolyai Univ., Preprint, **3**, pp. 21–23, 1986.

[12] Petruşel, A., *Fredholm-Volterra integral equations and Maia's theorem*, Seminar on Fixed Point Theory, Preprint, **3**, pp. 79–82, 1988.

[13] Pogorzelski, W., *Integral equations and their applications*, Pergamon Press, 1966.

[14] Rabinowitz, P., *Numerical integration based on approximating splines*, J. Comput. Appl., **33**, pp. 73–83, 1990.

[15] Scherbakova, A.G., *The solvability and properties of solutions of an integral convolutional equation*, Buletinul Academiei de Ştiinte a Republicii Moldova, Matematica, **2**, no. 51, pp. 87–94, 2006.

[16] Schumaker, A.G., *Spline functions: basic theory*, Wiley, New York, 1981.

[17] Srivastava, H.M. and Buschman, R.G., *Theory and applications of convolution integral equations*, Kluwer Academic Publishers, 1992.

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