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SOME CONE SEPARATION RESULTS AND APPLICATIONS

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Abstract. In this note we present some cone separation results in infinite dimensional spaces. Our approach is mainly based on two different types of cone outer approximation. Then we consider an application to vector optimization.

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1. INTRODUCTION AND PRELIMINARIES

The separation theorems are among the main tools in optimization theory and are especially used to obtain necessary optimality conditions. Starting with the classical convex separation results, a lot of such results have been used in the last decades in order to obtain new optimality results or in order to generalize the existing ones. In general, the functional separation by means of linear continuous maps is, by far, the most common way to separate two sets.

In this work we consider the method of "cone separation" having the same meaning as in the paper of Henig [6, Definition 2.1, Theorem 2.1]: consider two cones P, Q in a normed vector space with $P \cap Q = \{0\}$, and find another cone K which strictly separate them, i.e., $P \cap K = \{0\}$ and $Q \subset \operatorname{int} K$ (where int denotes the topological interior). Moreover, we want to specify the form of the cone K and we base our analysis on two models of outer approximation of cones: the Henig dilating cones and the plastering cones in the sense of Krasnosel'skiĭ [8]. Let us note that, in fact, the concept of Henig dilating cones was subsequently extracted by various authors from the proof of the main result in [6].

The cone separation technique we briefly described above has applications in multicriteria optimization because, once we can apply cone separation, we can treat the Pareto minima as weak Pareto minima and it is well-known that in vector optimization, in general, it is much more easier to work with weak minimality. We shall follow this line in the applications section of this paper.

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The main results of the paper are done in Section 2. Section 3 is dedicated to some applications of the results in Section 2 to the field of vector optimization. More exactly, we obtain some necessary optimality conditions for Pareto minima in reflexive Banach spaces.

Let us start with some preliminaries. Throughout Y is a normed vector space, unless otherwise stated. If $a \in Y$ and $\rho > 0$, we denote by $B(a, \rho)$ (resp. $D(a, \rho)$) the open (resp. closed) ball centered at a with radius ρ. The symbol $S(a, \rho)$ denotes the sphere centered at a with radius ρ. For simplicity we shall also use the notation $S_Y := S(0, 1)$. If $A, B \subset Y$ are two subsets, the distance from A to B is $d(A, B) := \inf\{\|a - b\| \mid a \in A, b \in B\}$. As usual, for a point $x \in Y$, $d(x, A) := d(\{x\}, A)$. The distance function will be denoted as $d_A : Y \to \mathbb{R}$, $d_A(x) := d(x, A)$.

Let $K \subset Y$ be a proper cone. The dual cone of the cone $K \subset Y$ is defined as

$$K^* := \{y^* \in Y^* \mid y^*(y) \geq 0, \forall y \in K\}$$

and the quasi-interior of K^* is

$$K^\sharp := \{y^* \in Y^* \mid y^*(y) > 0, \forall y \in K \setminus \{0\}\}.$$

where Y^* is the topological dual of Y. It is clear that if K and Q are two cones with $K \setminus \{0\} \subset \operatorname{int} Q$ then $Q^* \subset K^*$ and $Q^* \setminus \{0\} \subset K^\sharp$. A convex set B is said to be a base for the cone K if $0 \notin \operatorname{cl} B$ and $K = \operatorname{cone} B$, where cl denotes the topological closure and $\operatorname{cone} B := [0, \infty)B$ is the cone generated by B. A cone which admits a base is called based.

2. CONE SEPARATION RESULTS

In the setting of general vector spaces, the following result is well-known (see [2]) and it puts into light the main assumptions one should assume when dealing with cone separation results. We present a proof of it for the completeness and for easy comparison with the further results derived in this paper.

THEOREM 1. *Let Y be a locally convex space and $P, S \subset Y$ be cones s.t. $P \cap S = \{0\}$. If P is closed and S has compact base B, then there exists a convex pointed cone K s.t. $S \setminus \{0\} \subset \operatorname{int} K$ and $P \cap K = \{0\}$.*

Proof. The relation $P \cap S = \{0\}$ implies that $0 \notin P - B$. Since P is closed and B is compact, the set $P - B$ is closed and, taking into account the topological separation properties of locally convex spaces, there exists a balanced convex neighborhood V of 0 s.t.

$$V \cap (P - B) = \emptyset. \tag{2.1}$$

This means that

$$0 \notin P - (B + V).$$

It is easy to see that the cone

$$K := \operatorname{cone}(B + V)$$

satisfies the requirements. □

The above proof shows that, in fact, for normed vector spaces the cone K can be taken in the form

$$K := \operatorname{cone}(\{y \in Y \mid d(y, B) \le \varepsilon\}).$$

The properties of this theoretical construction are presented in the next result.

Proposition 1. ([5, Lemma 3.2.51]) Let $K \subset Y$ be a closed convex cone with a base B and take $\delta = d(0, B) > 0$. For $\varepsilon \in (0, \delta)$, consider $B_\varepsilon = \{y \in Y \mid d(y, B) \le \varepsilon\}$ and $K_\varepsilon = [0, \infty)B_\varepsilon$, the cone generated by B_ε. Then

- (i) K_ε is a closed convex cone for every $\varepsilon \in (0, \delta)$;
- (ii) if $0 < \gamma < \varepsilon < \delta$, $K \setminus \{0\} \subset K_\gamma \setminus \{0\} \subset \operatorname{int} K_\varepsilon$;
- (iii) $K = \cap_{\varepsilon \in (0,\delta)} K_\varepsilon = \cap_{n \in \mathbb{N}} K_{\varepsilon_n}$ where $(\varepsilon_n) \subset (0, \delta)$ converges to 0.

The cones K_ε constructed in this way are termed as dilating cones or Henig dilating cones. It is important to note that the cone K_ε has a non-empty interior. Moreover, we can specify the exact form of the elements in the dual cone of such a dilating cone with respect to the elements in the dual cone of the original ordering cone. This was done in [3].

Proposition 2. *Let $K \subset Y$ be a closed convex cone with a base B. For every $\varepsilon \in (0, d(0, B))$,*

$$K_\varepsilon^* = \{y^* \in Y^* \mid \inf_{b \in B} y^*(b) \ge \varepsilon \|y^*\|\}.$$

If we analyze the proof of Theorem 1 we observe that one of the essential facts is that $P - B$ is closed. In order to generalize Theorem 1 we use a criterion for the closedness of the difference of two non-convex sets derived by Zălinescu in [11]. We present here only the particular case we are interested in and to this end we need the notion of asymptotically compact subset of Y in the general framework when Y is a locally convex space endowed with a topology τ compatible with the duality system (Y, Y^*). A subset A of Y is called τ-asymptotically compact (τ-a.c. for short) if there exists a neighborhood U of 0 in (Y, τ) s.t. $U \cap [0, 1]A$ is τ-relatively compact. In the case when τ is the strong topology, then we shall simply use the term "asymptotically compact". If the topology τ in question is the weak topology we call it "asymptotically weakly compact".

The asymptotic cone of a nonempty set $A \subset Y$ (with respect of τ) is defined by

$$A_\infty^\tau := \{u \in Y \mid \exists (t_n) \to 0_+, (a_n) \subset A : t_n a_n \xrightarrow{\tau} u\} = \bigcap_{t > 0} \operatorname{cl}_\tau([0, t]A)$$

and it is well known that, in general, A_∞^τ is a τ-closed cone. When we deal with the norm topology we omit the superscript.

Proposition 3. *Let* Y *be a normed vector space and* $C, D \subset Y$ *be nonempty (weakly) closed sets. If* $C_\infty \cap D_\infty = \{0\}$ $(C_\infty^w \cap D_\infty^w = \{0\})$, *and* C *or* D *is (weakly) a.c., then* $C - D$ *is a (weakly) closed set.*

The first cone separation result we would like to mention was mainly obtained in [3] and reads as follows.

Theorem 2. *Let* $P, Q \subset Y$ *be (weakly) closed cones such that* Q *admits a (weakly) closed base* B. *Moreover, suppose that* P *or* B *is (weakly) a.c. If* $P \cap Q = \{0\}$, *then there exists* $\varepsilon \in (0, d(0, B))$ *s.t.* $P \cap Q_\varepsilon = \{0\}$.

Proof. Since $P \cap Q = \{0\}$, one deduces that $P \cap B = \emptyset$, which is equivalent to $0 \notin P - B$. On the other hand, one has that $P_\infty^\tau \subset P, Q_\infty^\tau \subset Q$, because P, Q are τ-closed cones (in both cases of weak or strong convergence). Therefore,

$$\{0\} \subset P_\infty^\tau \cap B_\infty^\tau \subset P \cap Q_\infty^\tau = P \cap Q = \{0\},$$

whence, from Proposition 3, one deduces that $P - B$ is a (weakly) closed set, therefore in any case it is norm closed whence $\alpha := d(0, P - B) > 0$. Let us take $0 < \varepsilon < \min(\alpha/2, d(0, B))$ and we prove that $0 \notin P - B_\varepsilon$ (in the notations of Proposition 1). This is obvious, because otherwise there would exist $b_\varepsilon \in B_\varepsilon \cap P$ and $b \in B$ s.t. $\|b_\varepsilon - b\| < 2\varepsilon$. Then

$$d(0, P - B) \le \|b_\varepsilon - b\| < 2\varepsilon < \alpha,$$

a contradiction. Consequently, $P \cap -B_\varepsilon = \emptyset$, so $P \cap -Q_\varepsilon = \{0\}$, and the thesis is proved. □

Since a τ-compact set is automatically τ-a.c. one has the following corollary.

Corollary 1. *Let* $P, Q \subset Y$ *be (weakly) closed cones such that* Q *admits a (weakly) closed base* B. *Moreover, suppose that* P *or* B *is (weakly) a.c. If* $P \cap Q = \{0\}$ *then there exists* $\varepsilon \in (0, d(0, B))$ *s.t.* $P \cap Q_\varepsilon = \{0\}$.

If one compares Theorem 1 with Theorem 2 one can observe (apart from the specific setting) that in the latter result the compactness of the base is replaced by asymptotically compactness. But, if one looks at the proofs, one can observe that Theorem 1 works for any closed set P (not necessarily a cone), while in Theorem 2 one uses that P is a (τ-closed) cone in the relation $P_\infty^\tau \subset P$.

Next, we use another method to approximate an original cone by a larger one (having nonempty interior). For a given cone K, the following conical ε-enlargement ($\varepsilon > 0$) is studied in the literature (see [8], [1]):

$$K^\varepsilon = \{u \in X \mid d(u, K) \le \varepsilon \|u\|\}.$$

It is clear that the so-defined K^ε is a cone (since for every $x \in X$ and $t \ge 0$, $d(tx, K) = td(x, K)$) which contains K. It is also clear that for $\varepsilon \ge 1$, $K_\varepsilon = X$. It is shown in [1, Proposition 3.2.1] that a closed pointed convex cone K admits a convex ε-enlargement if and only if K has a bounded base. Other properties of this construction are listed below.

Proposition 4. *Let $K \subset Y$ be a cone and K^ε be the cone defined above. Then:*

(i) $K^\varepsilon = \operatorname{cone} B^\varepsilon$, *where* $B^\varepsilon := \{u \in S_Y \mid d(u,K) \leq \varepsilon\}$;
(ii) *for every* $y \in K$, $D(y, (1+\varepsilon)^{-1}\varepsilon \|y\|) \subset K^\varepsilon$;
(iii) *if* $y^* \in (K^\varepsilon)^*$ *then for every* $y \in K$, $y^*(y) \geq (1+\varepsilon)^{-1}\varepsilon \|y\| \|y^*\|$.

Proof. (i) If $u \in K^\varepsilon$, then $d(u,K) \leq \varepsilon \|u\|$, whence $d(\|u\|^{-1} u, K) \leq \varepsilon$. It is clear that $v := \|u\|^{-1} u \in B^\varepsilon$ and since $u = \|u\| v$, one has that $u \in \operatorname{cone} B^\varepsilon$. Conversely, if $u \in \operatorname{cone} B^\varepsilon$ then there exists $v \in B^\varepsilon$ and $\alpha \geq 0$ s.t. $u = \alpha v$. Therefore,

$$d(u,K) = d(\alpha v, K) = \alpha d(v,K) \leq \alpha\varepsilon = \varepsilon \|u\|$$

and the first part is proved.

(ii) Let $y \in K$ and take $v \in D(y, (1+\varepsilon)^{-1}\varepsilon \|y\|)$. It is obvious that

$$\begin{aligned} \|v\| &\geq \|y\| - (1+\varepsilon)^{-1}\varepsilon \|y\| \\ &= (1+\varepsilon)^{-1} \|y\| \end{aligned}$$

and, on the other hand,

$$d(v,K) \leq \|v - y\| \leq (1+\varepsilon)^{-1}\varepsilon \|y\| \leq \varepsilon \|v\|.$$

This shows that $v \in K^\varepsilon$.

(iii) Let $y^* \in (K^\varepsilon)^*$ and $y \in K$. Following the preceding item, $y^*(v) \geq 0$ for every $v \in D(y, (1+\varepsilon)^{-1}\varepsilon \|y\|)$. In particular,

$$y^*(u) \geq (1+\varepsilon)^{-1}\varepsilon \|y\| \, y^*(u)$$

for every $u \in S_Y$. This is the conclusion and the proof is complete. □

We put into relation the two types of enlargements in a particular situation (see also [1, Propositions 2.1.1, 3.2.1]).

Proposition 5. *Let $K \subset Y$ be a closed convex cone with a bounded base B. Then for every $\varepsilon > 0$ there is a $\delta > 0$ s.t. $K_\delta \subset K^\varepsilon$.*

Proof. Since $0 \notin \operatorname{cl} B$, there exists $\mu := d(0,B) > 0$ s.t. $\|b\| \geq \mu$ for every $b \in B$. Then, from Proposition 4, (ii) we have that for every $b \in B$, $D(b, (1+\varepsilon)^{-1}\varepsilon\mu) \subset K^\varepsilon$, whence $B_{(1+\varepsilon)^{-1}\varepsilon\mu/2} \subset K^\varepsilon$, i.e. the conclusion for $\delta = (1+\varepsilon)^{-1}\varepsilon\mu/2$. □

Definition 1. *Let $K \subset Y$ be a cone. One says that K has the property $(\mathcal{S})$ if there exist $x_1^*, x_2^*, ..., x_n^* \in Y^*$ $(n \in \mathbb{N} \setminus \{0\})$ s.t.*

$$K \subset \{u \in Y \mid \|u\| \leq \max_{i=\overline{1,n}} \{x_i^*(u)\}.$$

Observe that in the case $n = 1$ in the above definition one gets the definition of the supernormal cone. It is well known that a cone in a normed vector space is supernormal if and only if it has a bounded base (see, e.g., [5, p. 37]).

However, is it easy to see that there are cones with the property $(\mathcal{S})$ which are not supernormal: take, for example,

$$K := \{(x_n) \in l^2 \mid (x_n) = (x_1, 0, 0, 0...), x_1 \in \mathbb{R}\}.$$

We are interested in considering this property because if a cone K has the property $(\mathcal{S})$ then it is clear that

$$(y_n) \subset K,\ y_n \xrightarrow{w} 0 \Rightarrow y_n \to 0.$$

We are now able to present the main result of this note.

THEOREM 3. *Let Y be a reflexive Banach space, $P, Q \subset Y$ be cones such that P, Q are weakly closed and Q has the property $(\mathcal{S})$. If $P \cap Q = \{0\}$ then there exists $\varepsilon > 0$ s.t. $P \cap Q^\varepsilon = \{0\}$.*

Proof. We proceed by contradiction. Suppose that for every $n \in \mathbb{N} \setminus \{0\}$ there exists $u_n \in P \cap Q^{1/n}$, $u_n \neq 0$. Since $u_n \in Q^{1/n}$ we have

$$d(u_n, Q) \leq n^{-1} \|u_n\|$$

i.e.

$$d(\|u_n\|^{-1} u_n, Q) \leq n^{-1}.$$

By the definition of the distance function, for every $n \in \mathbb{N} \setminus \{0\}$, there exists $v_n \in Q$ s.t.

$$\left\| \|u_n\|^{-1} u_n - v_n \right\| \leq 2n^{-1}.$$

Taking into account that Y is reflexive and the sequence $(\|u_n\|^{-1} u_n)_n$ is bounded there exists a subsequence of it (denoted in the same way) weakly convergent towards an element $u \in Y$. We show that $u \neq 0$. Indeed, in the contrary case $\|u_n\|^{-1} u_n \xrightarrow{w} 0$ and since $\|u_n\|^{-1} u_n - v_n \to 0$ (in norm topology) we get that $v_n \xrightarrow{w} 0$. Since Q has the property $(\mathcal{S})$, we obtain that (v_n) is strongly convergent towards 0, whence $\|u_n\|^{-1} u_n \to 0$ and this is not possible because $\|u_n\|^{-1} u_n$ has the norm 1 for every $n \in \mathbb{N} \setminus \{0\}$. Consequently, $u \neq 0$. Moreover, since P is weakly closed, $u \in P \setminus \{0\}$. Finally, $v_n \xrightarrow{w} u$, so $u \in Q$ and we arrive at a contradiction. The thesis is proved. $\square$

Note that a necessary and sufficient condition to have the conclusion of above theorem (i.e. $P \cap Q = \{0\}$ implies that there is an $\varepsilon > 0$ s.t. $P \cap Q^\varepsilon = \{0\}$) are given in [10] and it can be written in our notation as: there exists an $\varepsilon > 0$ s.t. $P^\varepsilon \cap Q^\varepsilon = \{0\}$. In particular, this holds if P, Q are (norm) closed and P is (norm) locally compact. We are interested here to consider somehow different conditions in view of the applications we envisage and in order to avoid the local compactness condition.

We record the following consequences.

COROLLARY 2. *Let Y be a reflexive Banach space, $P, Q \subset Y$ be cones such that P, Q are weakly closed and Q has a bounded base. If $P \cap Q = \{0\}$ then:*

(i) *there exists $\varepsilon > 0$ s.t. Q^ε is convex, pointed and $P \cap Q^\varepsilon = \{0\}$.*

(ii) *there exists* $\varepsilon > 0$ *s.t.* $P \cap Q_\varepsilon = \{0\}$.

Proof. (i) Since Q has a bounded base, then it admits a convex ε-enlargement. Taking a smaller ε, if necessary, and using Theorem 3 we get the conclusion.

(ii) This part follows easily from Proposition 5. We would like to present as well a proof based on Corollary 1. First, since Q has a base, it is convex. Moreover, since it is closed and its base B is bounded, then $\operatorname{cl} B$ is a base as well, so we can consider that Q has a closed bounded base. Since in a reflexive Banach space a bounded weakly closed set is weakly compact we can apply Corollary 1 to get the conclusion. □

In order to illustrate the application field of this corollary let us consider $Y = L_2(\Omega)$ where Ω is a non-empty subset of $\mathbb{R}^n$ and $Q := L_2(\Omega)$ is the well known space of square Lebesgue-integrable functions $f : \Omega \to \mathbb{R}$. The space $L_2(\Omega)$ is a Hilbert space and hence $(L_2(\Omega))^* = L_2(\Omega)$ and the natural ordering cone in $L_2(\Omega)$ is given as

$$L_2^+(\Omega) = \{f \in L_2(\Omega) \mid f(x) \geq 0 \text{ almost everywhere on } \Omega\}.$$

It is interesting to note that $(L_2^+(\Omega))^* = L_2^+(\Omega)$ and $L_2^+(\Omega)^\sharp$ is non-empty (see for example [7]). Thus by considering an element $x^* \in L_2^+(\Omega)^\sharp$ then

$$B = \{x \in L_2^+(\Omega) \mid x^*(x) = 1\}$$

is a base for $L_2^+(\Omega)$ and it is is weakly compact and hence bounded.

At the end of this section, we would like to present another possible enlargement, smaller than K^ε, but which can be obtained in a similar way. Let $K \subset Y$ be a cone. Based on Proposition 4 let us define

$$A^\varepsilon := \{u \in S_Y \mid d(u, S_Y \cap K) \leq \varepsilon\}.$$

and

$$K_1^\varepsilon := \operatorname{cone} A^\varepsilon.$$

Since $d(u, K) \leq d(u, S_Y \cap K)$ for every $u \in Y$, it is clear that $A^\varepsilon \subset B^\varepsilon$, whence $K_1^\varepsilon \subset K^\varepsilon$.

Let us prove now that $K \setminus \{0\} \subset K_1^\varepsilon$. For this, it is enough to prove that $K \cap S_Y \subset K_1^\varepsilon$. Take $\overline{y} \in K \cap S_Y$. Clearly, $S_Y \cap D(\overline{y}, \varepsilon) \subset A^\varepsilon$. Suppose, by way of contradiction that there exists a sequence $(y_n) \to \overline{y}$, $y_n \notin K_1^\varepsilon$ for every $n \geq 1$. Therefore,

$$\|y_n\|^{-1} y_n \to \|\overline{y}\|^{-1} \overline{y} = \overline{y}$$

whence $v_n := \|y_n\|^{-1} y_n \in S_Y, v_n \to \overline{y}$. Since $S_Y \cap D(\overline{y}, \varepsilon) \subset A^\varepsilon$, $v_n \in A^\varepsilon$ for n large enough. This shows that $y_n \in K_1^\varepsilon$ and this is a contradiction. Using these observations and Theorem 3 we get the next result.

Proposition 6. *Let Y be a reflexive Banach space, $P, Q \subset Y$ be cones such that P, Q are weakly closed and Q has the property $(\mathcal{S})$. If $P \cap Q = \{0\}$ then there exists $\varepsilon > 0$ s.t. $P \cap Q_1^\varepsilon = \{0\}$.*

3. AN APPLICATION

In this section we present some applications to vector optimization, mainly based on Corollary 2 (i), i.e. on the separation with enlargement cones. Note that several applications based on the separation with dilating cones have been done in [3].

Let K be a proper closed convex cone. It is well known that such a cone induces a partial order relation $\leq_K$ on Y by $y_1 \leq_K y_2$ if and only if $y_2 - y_1 \in K$ and, with respect to this relation, one can consider different types on minimum points. If $A \subset Y$ is a nonempty set, then a point $\overline{y} \in A$ is called a Pareto minimum point for A with respect to K if $(A - \overline{y}) \cap -K \subset K$. In particular, if K is pointed, this means that $(A - \overline{y}) \cap -K = \{0\}$. Observe that, in fact the condition $(A - \overline{y}) \cap -K \subset K$ (resp. $(A - \overline{y}) \cap -K = \{0\}$) is equivalent by $\operatorname{cone}(A - \overline{y}) \cap -K \subset K$ (resp. $\operatorname{cone}(A - \overline{y}) \cap -K = \{0\}$). If $\operatorname{int} K \neq \emptyset$, then a point $\overline{y} \in A$ is called weak Pareto minimum point of A with respect to K if $(A - \overline{y}) \cap - \operatorname{int} K = \emptyset$, i.e. it is a minimum point for A with respect to the cone $\operatorname{int} K \cup \{0\}$. One denotes by $\operatorname{Min}(A \mid K)$ $(\operatorname{WMin}(A \mid K)$, respectively) the sets of Pareto (weak Pareto, respectively) minima of A with respect to K. It is well-known that the main difficulty which arise when dealing with Pareto minima consist of the emptiness of the interior of the underlying ordering cone. Unfortunately, this situation is a common one for the majority of the natural ordering cones of the usual Banach spaces. In contrast, when the cone has nonempty interior one has several tools to handle the (weak) Pareto minima. One of this tools is the following scalarizing lemma (see [5, Section 2.3] and [4]). The symbol ∂ denotes the Fenchel subdifferential of a convex function.

Lemma 1. *Let $K \subset Y$ be a closed convex cone with nonempty interior and let $e \in \operatorname{int} K, M \subset Y, \overline{y} \in M$. Define the functional $\varphi_e : Y \to \mathbb{R}$ as*

$$\varphi_e(y) = \inf\{\lambda \in \mathbb{R} \mid y \in \lambda e - K\}.$$

This map is continuous, convex, strictly-$\operatorname{int} K$-monotone, $d(e, \operatorname{bd}(Q))^{-1}$-Lipschitz (where $\operatorname{bd} Q$ denotes the topological boundary of Q) and for every $\lambda \in \mathbb{R}$

$$\{y \mid \varphi_e(y) \leq \lambda\} = \lambda e - K, \quad \{y \mid \varphi_e(y) < \lambda\} = \lambda e - \operatorname{int} K.$$

Moreover, for every $u \in Y$,

$$\partial \varphi_e(u) = \{v^* \in K^* \mid v^*(e) = 1, v^*(u) = \varphi_e(u)\}.$$

The point $\overline{y}$ is a weak minimum point for M with respect to K if and only if $\overline{y}$ is a minimum point for $\varphi_e(\cdot - \overline{y})$ on M.

The other major tools we use in this section are the generalized differentiation objects introduced and developed by Mordukhovich and his collaborators (see [9, Vol. I]). We restrict ourselves to the case of Asplund spaces because we shall work in the setting of reflexive Banach spaces, a subset of the collections of Asplund spaces.

Definition 2. *Let X be an Asplund space and $S \subset X$ be a non-empty closed subset of X and let $x \in S$.*

(i) *The basic (or limiting, or Mordukhovich) normal cone to S at x is*

$$N_M(S,x) := \{x^* \in X^* \mid \exists x_n \xrightarrow{S} x, x_n^* \xrightarrow{w^*} x^*, x_n^* \in N_F(S,x_n)\}$$

where $N_F(S,z)$ denotes the Fréchet normal cone to S at a point $z \in S$, given as

$$N_F(S,z) := \{x^* \in X^* \mid \limsup_{u \in S, u \to z} \tfrac{x^*(u-z)}{\|u-z\|} \leq 0\}.$$

(ii) *Let $f : X \to \overline{\mathbb{R}}$ be finite at $\overline{x} \in X$; the Fréchet subdifferential of f at $\overline{x}$ is the set*

$$\hat{\partial} f(\overline{x}) := \{x^* \in X^* \mid (x^*, -1) \in N_F(\operatorname{epi} f, (\overline{x}, f(\overline{x})))\}$$

and the basic (or limiting, or Mordukhovich) subdifferential of f at $\overline{x}$ is

$$\partial_M f(\overline{x}) := \{x^* \in X^* \mid (x^*, -1) \in N_M(\operatorname{epi} f, (\overline{x}, f(\overline{x})))\},$$

where epi f *denotes the epigraph of f.*

On the Asplund spaces one has

$$\partial_M f(\overline{x}) = \limsup_{x \xrightarrow{f} \bar{x}} \hat{\partial} f(x),$$

and, in particular, $\hat{\partial} f(\overline{x}) \subset \partial_M f(x)$. Of course, if a function f attains a local minimum at a point $\overline{x}$ then $0 \in \partial_M f(\overline{x})$. If δ_Ω denotes the indicator function associated with a nonempty set $\Omega \subset X$ (i.e. $\delta_\Omega(x) = 0$ if $x \in \Omega$, $\delta_\Omega(x) = \infty$ if $x \notin \Omega$), then for any $\overline{x} \in \Omega$, $\partial_M \delta_\Omega(\overline{x}) = N_M(\Omega, \overline{x})$. In contrast with the Fréchet subdifferential, the basic subdifferential satisfies a robust calculus rule: if X is Asplund, f_1 is Lipschitz around $\overline{x}$ and f_2 is l.s.c. around this point, then

$$\partial_M (f_1 + f_2)(\overline{x}) \subset \partial_M f_1(\overline{x}) + \partial_M f_2(\overline{x}). \tag{3.1}$$

Theorem 4. *Let Y be a reflexive Banach space and $Q \subset Y$ be a weakly closed cone with bounded base. Let $A \subset Y$ be a set and $\overline{y} \in \operatorname{Min}(A \mid K)$ s.t.* $\operatorname{cone}(A - \overline{y})$ *is weakly closed. Then there exists $\varepsilon > 0$ such that for every $e \in K \setminus \{0\}$ there exists $y^* \in -N_M(A, \overline{y})$ with $y^*(e) = 1$ and $y^*(y) \geq (1+\varepsilon)^{-1} \varepsilon \|y\| \|y^*\|$ for every $y \in Y$.*

Proof. In our conditions, $\operatorname{cone}(A - \overline{y}) \cap K = \{0\}$, whence, following Corollary 2 (i), there exists a positive ε s.t. $\operatorname{cone}(A - \overline{y}) \cap K^\varepsilon = \{0\}$, i.e. $\overline{y} \in \operatorname{Min}(A \mid K^\varepsilon) \subset \operatorname{WMin}(A \mid K^\varepsilon)$. We can apply Lemma 1 for the cone $Q := K^\varepsilon$ and an element $e \in K \setminus \{0\} \subset \operatorname{int} K_\varepsilon$. Then $\overline{y}$ is a minimum point over A for the functional $s_e(\cdot - \overline{y})$ and then, by the infinite penalization method, $\overline{y}$ is a minimum point without constraints for $s_e(\cdot - \overline{y}) + \delta_A$. Therefore,

$$0 \in \partial_M (s_e(\cdot - \overline{y}) + \delta_A)(\overline{y})$$

and, since the first function is locally Lipschitz and the second one is lower-semicontinuous, we have

$$0 \in \partial_M(s_e(\cdot - \overline{y}))(\overline{y}) + \partial_M \delta_A(\overline{y}).$$

Moreover, the functional $s_e(\cdot - \overline{y})$ is sublinear and hence by using again Lemma 1 and Proposition 4 we obtain

$$\begin{aligned}\partial_M(s_e(\cdot - \overline{y}))(\overline{y}) = \partial s_e(0) &= \{y^* \in (K^\varepsilon)^* \mid y^*(e) = 1\} \\ &= \{y^* \in Y^* \mid y^*(y) \geq (1+\varepsilon)^{-1}\varepsilon \|y\| \|y^*\|, \forall y \in Y, y^*(e) = 1\}.\end{aligned}$$

On the other hand, $\partial_M \delta_A(\overline{y}) = N_M(A, \overline{y})$, whence the conclusion. □

Note that in this result the original cone K can have empty interior. If we denote by B the base of K, the property $y^*(y) \geq (1+\varepsilon)^{-1}\varepsilon \|y\| \|y^*\|$ for every $y \in Y$ fulfilled by y^* yields, in particular,

$$\inf_{b \in B} y^*(b) \geq (1+\varepsilon)^{-1}\varepsilon \|y^*\| d(0, B)$$

(compare with [3, Theorem 4.1]).

Following the technique developed in [4], one can derive optimality conditions for vector optimization problems governed by single-valued or set-valued maps as well. However, in order to keep this note short, we restrict our attention to the above case only.

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