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EXACT ORDERS IN SIMULTANEOUS APPROXIMATION BY COMPLEX BERNSTEIN-STANCU POLYNOMIALS[†]

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Abstract. In this paper the exact orders in approximation by the complex Bernstein-Stancu polynomials (depending on two parameters) and their derivatives on compact disks are obtained.

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1. INTRODUCTION

In the recent paper [1] the following upper estimates and Voronovskaja's theorem in approximation by complex Bernstein-Stancu polynomials depending on two parameters were proved.

THEOREM 1.1. *Let* $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ *be with* $R > 1$ *and let us suppose that* $f : \mathbb{D}_R \to \mathbb{C}$ *is analytic in* $\mathbb{D}_R$, *i.e.* $f(z) = \sum_{k=0}^{\infty} c_k z^k$, *for all* $z \in \mathbb{D}_R$. *Also, for* $0 \le \alpha \le \beta$ *(independent of* n*) let us define the complex Bernstein-Stancu polynomials by*

$$S_n^{(\alpha,\beta)}(f)(z) = \sum_{k=0}^{n} \binom{n}{k} z^k (1-z)^{n-k} f[(k+\alpha)/(n+\beta)],\ z \in \mathbb{C}.$$

(i) *For* $1 \le r < R$ *and* $n \in \mathbb{N}$, *we have*

$$\|S_n^{(\alpha,\beta)}(f) - f\|_r \le \frac{M_{2,r}^{(\beta)}(f)}{n+\beta},$$

where $0 < M_{2,r}^{(\beta)}(f) = 2r^2 \sum_{j=2}^{\infty} j(j-1)|c_j| r^{j-2} + 2\beta r \sum_{j=1}^{\infty} j|c_j| r^{j-1} < \infty$. *Here* $\|f\|_r = \sup\{|f(z)|; |z| \le r\}$.

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(ii) *If* $1 \le r < r_1 < R$, *then for all* $n, p \in \mathbb{N}$, *we have*

$$\|[S_n^{(\alpha,\beta)}(f)]^{(p)} - f^{(p)}\|_r \le \frac{M_{2,r_1}^{(\beta)}(f)p!r_1}{(n+\beta)(r_1-r)^{p+1}}.$$

(iii) *For all* $n \in \mathbb{N}$, *we have*

$$\left\|S_n^{(\alpha,\beta)}(f) - f + \frac{\beta e_1 - \alpha}{n+\beta} f' - \frac{n e_1(1-e_1)}{2(n+\beta)^2} f''\right\|_1 \le \frac{M_1^{(\alpha,\beta)}(f)}{(n+\beta)^2},$$

where $e_1(z) = z$ *and* $0 < M_1^{(\alpha,\beta)}(f) < \infty$ *depends only on* α, β *and* f.

Remark 1.2. Following exactly the lines in the proof of Theorem 1.1, (iii) in [1], it is immediate that in fact for any $1 \le r < R$ we have an upper estimate of the form

$$\left\|S_n^{(\alpha,\beta)}(f) - f + \frac{\beta e_1 - \alpha}{n+\beta} f' - \frac{n e_1(1-e_1)}{2(n+\beta)^2} f''\right\|_r \le \frac{M_r^{(\alpha,\beta)}(f)}{(n+\beta)^2},$$

where the constant $M_r^{(\alpha,\beta)}(f) > 0$ is independent of n and depends on f, r, α and β. This estimate will be useful in Section 3. □

The goal of this paper is to show that in Theorem 1.1, (i) and (ii), also lower estimates hold. Thus, in Section 2 we prove that if the analytic function f is not a polynomial of degree ≤ 0 and $1 \le r < R$, then we have $\|S_n^{(\alpha,\beta)}(f) - f\|_r \ge \frac{C_r^{(\alpha,\beta)}(f)}{n}$, $n \in \mathbb{N}$, that is in Theorem 1.1, (i), in fact the equivalence $\|S_n^{(\alpha,\beta)}(f) - f\|_r \sim \frac{1}{n}$ holds. In Section 3 we prove that for any $p \in \mathbb{N}$ and $1 \le r < R$, if f is not a polynomial of degree $\le p-1$ then we have $\|[S_n^{(\alpha,\beta)}(f)]^{(p)} - f^{(p)}\|_r \sim \frac{1}{n}$, where the constants in the equivalence depend only on f, α, β, r and p.

Since the case $\alpha = \beta = 0$ (i.e. the case of classical Bernstein polynomials) was already considered in [2], in the rest of the paper we will exclude it.

2. EXACT ORDER OF APPROXIMATION FOR COMPLEX BERNSTEIN-STANCU POLYNOMIALS

The main result of this section is the following.

Theorem 2.1. *Let* $R > 1$, $0 \le \alpha \le \beta$ *with* $\alpha + \beta > 0$, $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ *and let us suppose that* $f : \mathbb{D}_R \to \mathbb{C}$ *is analytic in* $\mathbb{D}_R$, *that is we can write* $f(z) = \sum_{k=0}^{\infty} c_k z^k$, *for all* $z \in \mathbb{D}_R$. *If* f *is not a polynomial of degree* 0 *and* $1 \le r < R$, *then we have*

$$\|S_n^{(\alpha,\beta)}(f) - f\|_r \ge \frac{C_r^{(\alpha,\beta)}(f)}{n+\beta}, n \in \mathbb{N},$$

where the constant $C_r^{(\alpha,\beta)}(f)$ *depends only on* f, r, α *and* β.

Proof. For all $z \in \mathbb{D}_R$ and $n \in \mathbb{N}$ we have

$$S_n^{(\alpha,\beta)}(f)(z) - f(z) = \frac{1}{n+\beta}\left\{-(\beta z - \alpha) f'(z) + \frac{z(1-z)}{2} f''(z) + \frac{1}{n+\beta} \cdot\right.$$
$$\left. \cdot \left[(n+\beta)^2 \left(S_n^{(\alpha,\beta)}(f)(z) - f(z) + \frac{\beta z - \alpha}{n+\beta} f'(z) - \frac{n z(1-z)}{2(n+\beta)^2} f''(z)\right) - \frac{\beta z(1-z)}{2} f''(z)\right]\right\}.$$

Note that in the case $\alpha=\beta=0$ in [2], necessarily f was supposed to be not a polynomial of degree ≤ 1.

In what follows we will apply to the above identity the following obvious property:

$$\|F+G\|_r \geq |\, \|F\|_r - \|G\|_r \,| \geq \|F\|_r - \|G\|_r.$$

It follows

$$\|S_n^{(\alpha,\beta)}(f)-f\|_r \geq \tfrac{1}{n+\beta}\left\{\left\|-(\beta e_1-\alpha)f' + \tfrac{e_1(1-e_1)}{2}f''\right\|_r - \tfrac{1}{n+\beta}\cdot\right.$$
$$\left.\cdot\left[\left\|(n+\beta)^2\left(S_n^{(\alpha,\beta)}(f)-f+\tfrac{\beta e_1-\alpha}{n+\beta}f'-\tfrac{ne_1(1-e_1)}{2(n+\beta)^2}f''\right)-\tfrac{\beta e_1(1-e_1)}{2}f''\right\|_r\right]\right\}.$$

Since by Remark 1.2 we have

$$\left\|(n+\beta)^2\left(S_n^{(\alpha,\beta)}(f)-f+\tfrac{\beta e_1-\alpha}{n+\beta}f'-\tfrac{ne_1(1-e_1)}{2(n+\beta)^2}f''\right)-\tfrac{\beta e_1(1-e_1)}{2}f''\right\|_r \leq$$
$$\leq M_r^{(\alpha,\beta)}(f)+\beta\|f''\|_r,$$

and denoting $H(z)=-(\beta z-\alpha)f'(z)+\frac{z(1-z)}{2}f''(z)$, if we prove that $\|H\|_r>0$, then it is clear that there exists an index n_0 depending only on f, α and β, such that

$$\|S_n^{(\alpha,\beta)}(f)-f\|_r \geq \frac{1}{n+\beta}\cdot\frac{\|H\|_r}{2}, \forall n\geq n_0.$$

For $n\in\{1,2,\ldots,n_0-1\}$ we have $\|S_n^{(\alpha,\beta)}(f)-f\|_r \geq \frac{A_{n,r}^{(\alpha,\beta)}(f)}{n+\beta}$ with $A_{n,r}^{(\alpha,\beta)}(f)=(n+\beta)\cdot\|S_n^{(\alpha,\beta)}(f)-f\|_r>0$, which finally implies $\|S_n^{(\alpha,\beta)}(f)-f\|_r\geq\frac{C_r^{(\alpha,\beta)}(f)}{n+\beta}$ for all $n\in\mathbb{N}$, with $C_r^{(\alpha,\beta)}(f)=\min\left\{A_{1,r}^{(\alpha,\beta)}, A_{2,r}^{(\alpha,\beta)}(f),\ldots,A_{n_0-1,r}^{(\alpha,\beta)}(f),\frac{\|H\|_r}{2}\right\}$.

Therefore it remains to show that $\|H\|_r>0$. Indeed, suppose that $\|H\|_r=0$. We have two possibilities: 1) $0=\alpha<\beta$ or 2) $0<\alpha\leq\beta$.

Case 1). We obtain $H(z)=-\beta zf'(z)+\frac{z(1-z)}{2}f''(z)=0$, for all $|z|\leq r$ and denoting $y(z)=f'(z)$, it follows that $y(z)$ is an analytic function in $\mathbb{D}_R$, solution of the differential equation $-\beta zy(z)+\frac{z(1-z)}{2}y'(z)=0, |z|\leq r$, which after simplification with $z\neq 0$ becomes $-\beta y(z)+\frac{(1-z)}{2}y'(z)=0, |z|\leq r$. Now, seeking $y(z)$ in the form $y(z)=\sum_{k=0}^{\infty}b_kz^k$ and replacing it in the differential equation, by the identification of the coefficients we easily obtain $b_k=0$ for all $k=0,1,\ldots$. Therefore $y(z)=0$ for all $|z|\leq r$, which by the identity theorem on analytic (holomorphic) functions implies $y(z)=0$ for all $z\in\mathbb{D}_R$ and the contradiction that f is a polynomial of degree ≤ 0.

Case 2). Denoting $y(z)=f'(z)$ by hypothesis it follows that $y(z)$ is an analytic function in $\mathbb{D}_R$ solution of the differential equation $(-\beta z+\alpha)y(z)+\frac{z(1-z)}{2}y'(z)=0, |z|\leq r$.

Taking $z=0$ it follows $\alpha y(0)=0$, which means $y(0)=0$. Seeking $y(z)$ in the form $y(z)=\sum_{k=1}^{\infty}b_kz^k$ and replacing it in the differential equation, by the identification of the coefficients we easily obtain $b_k = 0$ for all $k = 1, 2, \ldots$, which finally leads to the contradiction that f is a constant. $\square$

Combining now Theorem 2.1 with Theorem 1.1, (i), we immediately get the following.

COROLLARY 2.2. *Let $R > 1$, $0 \le \alpha \le \beta$ with $\alpha + \beta > 0$, $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ and let us suppose that $f : \mathbb{D}_R \to \mathbb{C}$ is analytic in $\mathbb{D}_R$. If f is not a polynomial of degree 0 and $1 \le r < R$, then we have*

$$\|S_n^{(\alpha,\beta)}(f) - f\|_r \sim \frac{1}{n+\beta}, n \in \mathbb{N},$$

where the constants in the equivalence depend on f, r, α and β.

3. EXACT ORDERS OF APPROXIMATION FOR DERIVATIVES OF COMPLEX BERNSTEIN-STANCU POLYNOMIALS

The main result of this section is the following.

THEOREM 3.1. *Let $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ be with $R > 1$, $0 \le \alpha \le \beta$ with $\alpha + \beta > 0$ and let us suppose that $f : \mathbb{D}_R \to \mathbb{C}$ is analytic in $\mathbb{D}_R$, i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$. Also, let $1 \le r < r_1 < R$ and $p \in \mathbb{N}$ be fixed. If f is not a polynomial of degree $\le p - 1$, then we have*

$$\|[S_n^{(\alpha,\beta)}(f)]^{(p)} - f^{(p)}\|_r \sim \frac{1}{n+\beta},$$

where the constants in the equivalence depend on f, α, β, r, r_1 and p.

Proof. Taking into account Theorem 1.1, (ii), it remains only to prove the lower estimate for $\|[S_n^{(\alpha,\beta)}(f)]^{(p)} - f^{(p)}\|_r$.

Denoting by Γ the circle of radius $r_1 > r$ (with $r \ge 1$) and center 0, by the Cauchy's formulas it follows that for all $|z| \le r$ and $n \in \mathbb{N}$ we have

$$[S_n^{(\alpha,\beta)}(f)]^{(p)}(z) - f^{(p)}(z) = \frac{p!}{2\pi \mathrm{i}} \int_\Gamma \frac{S_n^{(\alpha,\beta)}(f)(v) - f(v)}{(v-z)^{p+1}} \mathrm{d}v,$$

where we have the inequality $|v - z| \ge r_1 - r$ valid for all $|z| \le r$ and $v \in \Gamma$.

As in the proof of Theorem 2.1 (keeping the notation for H), for all $v \in \Gamma$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} & S_n^{(\alpha,\beta)}(f)(v) - f(v) = \\ & = \tfrac{1}{n+\beta} \Big\{ H(v) + \tfrac{1}{n+\beta} \Big[(n+\beta)^2 \Big(S_n^{(\alpha,\beta)}(f)(v) - f(v) + \\ & \tfrac{\beta v - \alpha}{n+\beta} f'(v) - \tfrac{nv(1-v)}{2(n+\beta)^2} f''(v) \Big) - \tfrac{\beta v(1-v)}{2} f''(v) \Big] \Big\} \end{aligned}$$

which replaced in the above Cauchy's formula implies

$$[S_n^{(\alpha,\beta)}(f)]^{(p)}(z) - f^{(p)}(z) = \frac{1}{n+\beta}\left\{H^{(p)}(z) + \frac{1}{n+\beta}\cdot\right.$$

$$\cdot\left[\frac{p!}{2\pi \mathrm{i}}\int_\Gamma \frac{(n+\beta)^2\left(S_n^{(\alpha,\beta)}(f)(v)-f(v)+\frac{\beta v-\alpha}{n+\beta}f'(v)-\frac{nv(1-v)}{2(n+\beta)^2}f''(v)\right)}{(v-z)^{p+1}}\mathrm{d}v-\right.$$

$$\left.\left.-\ \frac{p!}{2\pi \mathrm{i}}\int_\Gamma \frac{\beta v(1-v)}{2(v-z)^{p+1}}f''(v)\mathrm{d}v\right]\right\}.$$

Passing now to absolute value, for all $|z| \le r$ and $n \in \mathbb{N}$ it follows

$$|[S_n^{(\alpha,\beta)}(f)]^{(p)}(z) - f^{(p)}(z)| \ge \frac{1}{n+\beta}\left\{|H^{(p)}(z)| - \frac{1}{n+\beta}\cdot\right.$$

$$\left[\left|\frac{p!}{2\pi \mathrm{i}}\int_\Gamma \frac{(n+\beta)^2\left(S_n^{(\alpha,\beta)}(f)(v)-f(v)+\frac{\beta v-\alpha}{n+\beta}f'(v)-\frac{nv(1-v)}{2(n+\beta)^2}f''(v)\right)}{(v-z)^{p+1}}\mathrm{d}v-\right.\right.$$

$$\left.\left.\left.\frac{p!}{2\pi \mathrm{i}}\int_\Gamma \frac{\beta v(1-v)}{2(v-z)^{p+1}}f''(v)\mathrm{d}v\right|\right]\right\},$$

where by using the Remark 1.2, for all $|z| \le r$ and $n \in \mathbb{N}$ we get

$$\left|\frac{p!}{2\pi \mathrm{i}}\int_\Gamma \frac{(n+\beta)^2\left(S_n^{(\alpha,\beta)}(f)(v)-f(v)+\frac{\beta v-\alpha}{n+\beta}f'(v)-\frac{nv(1-v)}{2(n+\beta)^2}f''(v)\right)}{(v-z)^{p+1}}\mathrm{d}v-\right.$$

$$\left.-\ \frac{p!}{2\pi \mathrm{i}}\int_\Gamma \frac{\beta v(1-v)}{2(v-z)^{p+1}}f''(v)\mathrm{d}v\right| \le \frac{p!}{2\pi}\cdot\frac{2\pi r_1 M_r^{(\alpha,\beta)}}{(r_1-r)^{p+1}} + \frac{p!}{2\pi}\cdot\frac{2\pi r_1\beta r_1(1+r_1)\|f''\|_{r_1}}{2(r_1-r)^{p+1}}.$$

Denoting now $F_p(z) = H^{(p)}(z)$, we prove that $\|F_p\|_r > 0$. Indeed, if we suppose that $\|F_p\|_r = 0$ then it follows that f satisfies the differential equation

$$-\beta z f'(z) + \frac{z(1-z)}{2}f''(z) = Q_{p-1}(z), \forall |z| \le r,$$

where $Q_{p-1}(z)$ is a polynomial of degree $\le p-1$. Simplifying with z, making the substitution $y(z) = f'(z)$, searching $y(z)$ in the form $y(z) = \sum\limits_{k=0}^{\infty} b_k z^k$ and then replacing in the differential equation, by simple calculations we easily obtain that $b_k = 0$ for all $k \ge p-1$, that is $y(z)$ is a polynomial of degree $\le p-2$. This implies the contradiction that f is a polynomial of degree $\le p-1$.

Continuing exactly as in the proof of Theorem 2.1 (with $\|S_n^{(\alpha,\beta)}(f) - f\|_r$ replaced by $\|[S_n^{(\alpha,\beta)}(f)]^{(p)} - f^{(p)}\|_r$), finally there exists an index $n_0 \in \mathbb{N}$ depending on f, r, r_1 and p, such that for all $n \ge n_0$ we have

$$\|[S_n^{(\alpha,\beta)}(f)]^{(p)} - f^{(p)}\|_r \ge \frac{1}{n}\cdot\frac{C_0}{2}.$$

Also, the cases when $n \in \{1, 2, \ldots, n_0 - 1\}$ are similar with those in the proof of Theorem 2.1. □

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