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DOUBLE INEQUALITIES FOR QUADRATURE FORMULA OF GAUSS TYPE WITH TWO NODES

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Abstract. In this paper upper and lower error bonds for Gauss's quadrature rule with two nodes are given.

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1. INTRODUCTION

In a series of papers (see [2], [3], [8]) the authors establish bounds for the quadrature rules such as the trapezoid, Simpson and Newton quadrature rules.

In this work we will consider Gauss's quadrature rule with two nodes:

$$\int_a^b f(x)\mathrm{d}x = \frac{b-a}{2}[f(x_1)+f(x_2)] + R[f] \tag{1}$$

where $f:[a,b]\to\mathbb{R}$, $x_1 = \frac{a+b}{2} - \frac{b-a}{2}\cdot\xi$, $x_2 = \frac{a+b}{2} + \frac{b-a}{2}\cdot\xi$, $\xi = \frac{1}{\sqrt{3}} = 0,57735027\ldots$

If $f\in C^4[a,b]$, the error $R[f]$ from the formula (1) is given by:

$$R[f] = \int_a^b \varphi(x) f^{(4)}(x)\mathrm{d}x \tag{2}$$

(see [7], pp. 137–138 and 283–284), where the function φ has the form:

$$\varphi(x) = \begin{cases} \frac{(x-a)^4}{4!} & \text{if} \quad x\in[a,x_1], \\ \frac{(x-a)^4}{4!} - \frac{b-a}{2}\frac{(x-x_1)^3}{3!} & \text{if} \quad x\in]x_1,x_2[, \\ \frac{(b-x)^4}{4!} & \text{if} \quad x\in[x_2,b]. \end{cases} \tag{3}$$

It is easy to see that the function φ has the following properties:

a) $\varphi\in C^4[a,b]$;

b) $\varphi\left(\frac{a+b}{2}-h\right) = \varphi\left(\frac{a+b}{2}+h\right)$, for any $h\in\left[0,\frac{b-a}{2}\right]$;

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c) $\int_a^b \varphi(x)\mathrm{d}x = \frac{1}{135}\left(\frac{b-a}{2}\right)^5.$

2. MAIN RESULTS

In the following theorem double inequalities for $-R[f]$ are presented where $R[f]$ is the error in the quadrature formula (1) given by the relation (2).

THEOREM 1. *If $f \in C^4[a,b]$ then*

$$\begin{aligned}(4)\qquad &\tfrac{1}{17280}(b-a)^5(41\gamma_4 - 45S_3 + 180\xi^3 S_3 - 180\xi^3\gamma_4) \leq \\ &\leq \tfrac{b-a}{2}[f(x_1)+f(x_2)] - \int_a^b f(x)\mathrm{d}x \\ &\leq \tfrac{1}{17280}(b-a)^5(41\Gamma_4 - 45S_3 + 180\xi^3 S_3 - 180\xi^3\Gamma_4)\end{aligned}$$

where $\gamma_4, \Gamma_4 \in \mathbb{R}$, $\gamma_4 \leq f^{(4)}(x) \leq \Gamma_4$, for all $x \in [a,b]$ and $S_3 = \frac{f'''(b)-f'''(a)}{b-a}$. Moreover,

$$(5)\qquad \gamma_4 = \min_{x\in[a,b]} f^{(4)}(x), \quad \Gamma_4 = \max_{x\in[a,b]} f^{(4)}(x),$$

and the inequalities (4) *are sharp.*

Proof. From (2), using the properties of the function φ and integrating by parts, we obtain:

$$(6)\qquad \int_a^b \varphi(x)f^{(4)}(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x - \tfrac{b-a}{2}[f(x_1)+f(x_2)].$$

By using the equality c) in the formula (6) and the assumptions of the theorem, we have:

$$\begin{aligned}(7)\qquad \int_a^b [f^{(4)}(x)-\gamma_4]\varphi(x)\mathrm{d}x &= \int_a^b f(x)\mathrm{d}x - \tfrac{b-a}{2}[f(x_1)+f(x_2)] \\ &\quad - \tfrac{1}{135}\left(\tfrac{b-a}{2}\right)^5\gamma_4\end{aligned}$$

and

$$\begin{aligned}(8)\qquad \int_a^b [\Gamma_4 - f^{(4)}(x)]\varphi(x)\mathrm{d}x &= -\int_a^b f(x)\mathrm{d}x + \tfrac{b-a}{2}[f(x_1)+f(x_2)] \\ &\quad + \tfrac{1}{135}\left(\tfrac{b-a}{2}\right)^5\Gamma_4\end{aligned}$$

On the other hand, we have:

$$(9)\qquad \int_a^b [f^{(4)}(x)-\gamma_4]\varphi(x)\mathrm{d}x \leq \max_{x\in[a,b]}|\varphi(x)| \int_a^b |f^{(4)}(x)-\gamma_4|\mathrm{d}x,$$

where

$$(10)\qquad \max_{x\in[a,b]}|\varphi(x)| = \tfrac{1}{384}(1-4\xi^3)(b-a)^4 = \tfrac{9-4\sqrt{3}}{3456}(b-a)^4,$$

and

$$\int_a^b |f^{(4)}(x)-\gamma_4|\mathrm{d}x = \int_a^b (f^{(4)}(x)-\gamma_4)\mathrm{d}x = f'''(b)-f'''(a)-\gamma_4(b-a) = (S_3-\gamma_4)(b-a). \tag{11}$$

From the relations (7), (9), (10) and (11) we obtain:

$$\int_a^b f(x)\mathrm{d}x - \tfrac{b-a}{2}[f(x_1)+f(x_2)] \le -\tfrac{1}{17280}(b-a)^5(41\gamma_4 - 45S_3 + 180\xi^3 S_3 - 180\xi^3\gamma_4). \tag{12}$$

In the same way we have:

$$\int_a^b [\Gamma_4 - f^{(4)}(x)]\varphi(x)\mathrm{d}x \le \max_{x\in[a,b]}|\varphi(x)| \int_a^b |\Gamma_4 - f^{(4)}(x)|\mathrm{d}x \tag{13}$$

and

$$\int_a^b |\Gamma_4 - f^{(4)}(x)|\mathrm{d}x = \int_a^b (\Gamma_4 - f^{(4)}(x))\mathrm{d}x = \Gamma_4(b-a) - f'''(b) + f'''(a) = (\Gamma_4 - S_3)(b-a). \tag{14}$$

Using (8), (10), (13) and (14) we obtain the inequality:

$$\int_a^b f(x)\mathrm{d}x - \tfrac{b-a}{2}[f(x_1)+f(x_2)] \ge -\tfrac{1}{17280}(b-a)^5(41\Gamma_4 - 45S_3 + 180\xi^3 S_3 - 180\xi^3\gamma_4). \tag{15}$$

Inequalities (4) follow from the inequalities (12) and (15).

To prove the second part of the theorem we consider the function $f(x) = (x-a)^4$. It is easy to show that all the three members of the double inequality (4) have in common the value $-\frac{1}{180}(b-a)^5$. This completes the proof. □

In the following theorem double inequalities for $R[f]$ are presented.

THEOREM 2. *If the function $f \in C^4[a,b]$ then:*

$$\tfrac{1}{17280}(b-a)^5(49\gamma_4 - 45S_3 + 180\xi^3 S_3 - 180\xi^3\gamma_4) \le \int_a^b f(x)\mathrm{d}x - \frac{b-a}{2}[f(x_1)+f(x_2)] \le \tfrac{1}{17280}(b-a)^5(49\Gamma_4 - 45S_3 + 180\xi^3 S_3 - 180\xi^3\Gamma_4), \tag{16}$$

where γ_4, Γ_4, ξ and S_3 are given in Theorem 1. Moreover,

$$\gamma_4 = \min_{x\in[a,b]} f^{(4)}(x), \quad \Gamma_4 = \max_{x\in[a,b]} f^{(4)}(x),$$

and the inequalities (16) are sharp.

Proof. By using the relations (7), (9), (10) and (11) it follows:

$$(17)\qquad -\int_a^b f(x)\mathrm{d}x + \tfrac{b-a}{2}[f(x_1)+f(x_2)] \leq -\tfrac{1}{17280}(b-a)^5(49\gamma_4 - 45S_3 + 180\xi^3 S_3 - 180\xi^3\gamma_4).$$

Analogously, using the relations (8), (13), (14) and (15) we obtain:

$$(18)\qquad \int_a^b f(x)\mathrm{d}x - \tfrac{b-a}{2}[f(x_1)+f(x_2)] \leq \tfrac{1}{17280}(b-a)^5(49\Gamma_4 - 45S_3 + 180\xi^3 S_3 - 180\xi^3\Gamma_4).$$

From the relations (17) and (18) result the inequalities (16). To prove that the double inequalities (16) are exact we follow the steps of the proof for Theorem 1. □

Theorem 3 gives us the inequalities which do not depend on S_3.

THEOREM 3. *In the assumptions of Theorem* 1*, we have:*

$$(19)\qquad \tfrac{1}{34560}(b-a)^5(41\gamma_4 - 49\Gamma_4 - 180\xi^3\gamma_4 + 180\xi^3\Gamma_4) \leq \tfrac{b-a}{2}[f(x_1)+f(x_2)] - \int_a^b f(x)\mathrm{d}x \leq \tfrac{1}{34560}(b-a)^5(41\Gamma_4 - 49\gamma_4 + 180\xi^3\gamma_4 - 180\xi^3\Gamma_4).$$

If

$$\gamma_4 = \min_{x\in[a,b]} f^{(4)}(x), \quad \Gamma_4 = \max_{x\in[a,b]} f^{(4)}(x),$$

then the inequalities (19) *are sharp.*

Proof. Multiplying the inequality (16) with (-1) and adding it with the inequality (4) we obtain the double inequalities (19). Considering the function $f(x) = (x-a)^4$ and calculating all the three members of the inequality (19) the value obtained is $-\frac{1}{180}(b-a)^5$. The double inequalities are sharp. This completes the proof. □

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