ON A THEOREM OF Baire
ABOUT LOWER SEMICONtinuous FUNCTIONS

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Abstract. A theorem of Baire concerning the approximation of lower semicontinuous real valued functions defined on a metric space, by increasing sequences of continuous functions is extended to the “nonsymmetric” case, i.e. for quasi-metric spaces.


Keywords. Quasi-metric space, semi-Lipschitz function, approximation.

1. PRELIMINARIES

In the last years there have been an increasing interest for the study of quasi-metric spaces (spaces with asymmetric metric) motivated by their applications in various branches of mathematics, and especially in computer science. A direction of investigation is to study the possibility to extend to quasi-metric spaces known results in metric spaces (see, for example, [2]–[7]).

The following classical result of Baire is well known [8], [9]. Every lower semicontinuous real valued function defined on a metric space is the pointwise limit of an increasing sequence of continuous functions. Analyzing the proof of this result (see [9], Th. 2.2-23, p. 84), observe that every element of the increasing sequence is a Lipschitz function. This fact suggest to use the semi-Lipschitz functions defined in [10], [11], to obtain such a theorem for lower semi-continuous real valued functions defined on a quasi-metric space.

This short Note presents some notions connected with quasi-metric spaces and the result of Baire in this framework.

Let $X$ be a non-empty set. A function $d : X \times X \to [0, \infty)$ is called a quasi-metric on $X$ ([10]) if the following conditions hold:

- $Q_1$ $d(x,y) = d(y,x) = 0$ if, and only if, $x = y$;
- $Q_2$ $d(x,z) \leq d(x,y) + d(y,z)$, for all $x, y, z \in X$.

The function $\overline{d} : X \times X \to [0, \infty)$ defined by $\overline{d}(x,y) = d(y,x)$, for all $x, y \in X$ is also a quasi-metric on $X$, called the conjugate quasi-metric of $d$.

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The function \( d^*(x, y) = \max\{d(x, y), d(y, x)\} \) is a metric on \( X \). If \( d \) can take the value \(+\infty\), then it is called a quasi-distance on \( X \).

Each quasi-metric \( d \) on \( X \) induces a topology \( \tau(d) \) which has as a basis the family of open balls (called forward open balls in [6]):

\[
B^+(x, \varepsilon) := \{y \in X : d(x, y) < \varepsilon\}, \quad x \in X, \varepsilon > 0.
\]

This topology is called the forward topology of \( X \) ([3], [6]) and is denoted also by \( \tau^+ \).

Observe that the topology \( \tau^+ \) is a \( T_0 \)-topology [10]. If the conditions \( Q_1 \) is replaced by \( Q'_1 \):

\[
d(x, y) = 0 \text{ iff } x = y,
\]

then \( \tau^+ \) is a \( T_1 \)-topology ([10], [11]).

The pair \( (X, d) \) is called a quasi-metric space (\( T_0 \)-separated, respectively \( T_1 \)-separated). A sequence \( (x_n)_{n \geq 1} \) in the quasi-metric space \( (X, d) \) is called \( \tau^+ \)-convergent to \( x_0 \in X \) iff

\[
\lim_{n \to \infty} d(x_0, x_n) = 0.
\]

Similarly, the topology \( \tau(d) \) has as a basis the family of open balls:

\[
B^-(x, \varepsilon) := \{y \in X : d(y, x) < \varepsilon\}, \quad x \in X, \varepsilon > 0.
\]

This topology is denoted also by \( \tau^- \).

**Definition 1.** ([10]). Let \( (X, d) \) be a quasi-metric space. A function \( f : X \to \mathbb{R} \) is called \( d \)-semi-Lipschitz if there exists a number \( L \geq 0 \) (named a \( d \)-semi-Lipschitz constant for \( f \)) such that

\[
f(x) - f(y) \leq Ld(x, y),
\]

for all \( x, y \in X \).

A similar definition can be given for \( d \)-semi-Lipschitz functions.

**Definition 2.** The function \( f : X \to \mathbb{R} \) is called \( \leq d \)-increasing if

\[
f(x) \leq f(y), \text{ whenever } d(x, y) = 0.
\]

One denotes by \( R^X_{\leq d} \) the set of all \( \leq d \)-increasing functions on \( X \). The set \( \mathbb{R}^X_{\leq d} \) is a cone in the linear space \( \mathbb{R}^X \) of real valued functions defined on \( X \) [11].

For a \( d \)-semi-Lipschitz function \( f : X \to \mathbb{R} \), put

\[
\|f\|_d = \sup \left\{ \frac{f(x) - f(y)}{d(x, y)} : d(x, y) > 0, \quad x, y \in X \right\}.
\]

Then \( \|f\|_d \) is the smallest \( d \)-semi-Lipschitz constant of \( f \) (see [7], [10], [11]).

Denote also

\[
d\text{-SlipX} := \left\{ f \in \mathbb{R}^X_{\leq d} : \|f\|_d < \infty \right\},
\]

the subcone of the cone \( \mathbb{R}^X_{\leq d} \), of all \( d \)-semi-Lipschitz real valued functions on \( (X, d) \). If \( \theta \in X \) is a fixed, but arbitrary element, denote

\[
d\text{-Slip}_0X := \{ f \in d\text{-SlipX} : f(\theta) = 0 \}.
\]

Then the functional \( \|\cdot\|_d : d\text{-Slip}_0X \to [0, \infty) \) is an asymmetric norm on \( d\text{-Slip}_0X \), i.e. this functional is subadditive, positively homogeneous and \( \|f\|_d = \)
0 if \( f = 0 \). The pair \((d\text{-}
abla \text{lip}_X, \| \cdot \|_d)\) is called the “normed cone” of \( d\text{-}
abla \text{lip}_X\) real valued functions (vanishing at \( \theta \)). The properties of this normed cone are studied in [10], [11].

**Definition 3.** Let \((X, d)\) be a quasi-metric space and \( f : X \to \mathbb{R} \), where \( \mathbb{R} = [-\infty, +\infty] \) is equipped with the natural topology. The function \( f \) is called \( \tau^+\) -lower semicontinuous (respectively \( \tau^+\) -upper semicontinuous) \((\tau^+\text{-u.s.c}, \text{respectively } \tau^+\text{-u.s.c.}, \text{in short})\) at the point \( x_0 \in X \), if for every \( \varepsilon > 0 \) there exists \( r > 0 \) such that for all \( x \in B^+(x_0, r) \), \( f(x) > f(x_0) - \varepsilon \) (respectively \( f(x) < f(x_0) + \varepsilon \)).

Similar definitions can be given for \( \tau^-\) -l.s.c. and \( \tau^-\) -u.s.c real valued functions on \((X, d)\).

Observe that \( f : X \to \mathbb{R} \) is \( \tau^+\) -l.s.c iff \(-f\) is \( \tau^-\) -u.s.c and \( f \) is \( \tau^+\) -u.s.c iff \(-f\) is \( \tau^-\) -l.s.c.

The result of Baire in this framework is:

**Theorem 4.** \((\text{Baire})\). Let \((X, d)\) be a quasi-metric space and \( f : X \to \mathbb{R} \) be a \( \tau^+\) -l.s.c function on \( X \). Then there exists a sequence \((F_n)_{n \geq 1}, F_n \in d\text{-}\nabla \text{lip}X\) such that \((F_n(x))_{n \geq 1}, x \in X\), is increasing and \( \lim_{n \to \infty} F_n(x) = f(x), \ x \in X \).

**Proof.** a) Suppose firstly that \( f(x) \geq 0 \), for all \( x \in X \). For \( x \in X \) and \( n \in \mathbb{N} \), let

\[
F_n(x) = \inf \{ f(z) + n \cdot d(x, z) : z \in X \}.
\]

Obviously that

\[
0 \leq F_n(x) \leq f(x) + nd(x, x) = f(x).
\]

If \( x, y, z \in X \), then

\[
F_n(x) \leq f(z) + nd(x, z) \leq f(z) + nd(x, y) + nd(y, z) = (f(z) + nd(y, z)) + nd(x, y).
\]

Taking the infimum with respect to \( z \), one obtains

\[
F_n(x) \leq F_n(y) + nd(x, y),
\]

i.e.

\[
F_n(x) - F_n(y) \leq n \cdot d(x, y),
\]

for all \( x, y \in X \). This means that \( \| F_{n,d} \| \leq n \) and \( F_n \in d\text{-}\nabla \text{lip}X \), for every \( n = 1, 2, 3, \ldots \).

If \( n \leq m \), by the definition \( [4] \) it follows \( F_n(x) \leq F_m(x), x \in X \), so that the sequence \((F_n(x))_{n \geq 1}\) is increasing and bounded by \( f(x), x \in X \). Consequently there exists the limit \( \lim_{n \to \infty} F_n(x) = h(x), h(x) \leq f(x), x \in X \).

In fact \( h(x) = f(x) \), for every \( x \in X \). Indeed, let \( n \in \mathbb{N} \) and \( x \in X \). By definition \( [4] \) of \( F_n(x) \), for every \( \varepsilon > 0 \), there exists \( z_n \in X \) such that

\[
F_n(x) + \varepsilon > f(z_n) + nd(x, z_n) \geq nd(x, z_n),
\]
and because \( F_n(x) + \epsilon \leq f(x) + \epsilon \), it follows
\[
f(x) + \epsilon \geq nd(x, z_n),
\]
and then
\[
d(x, z_n) \leq \frac{1}{n} (f(x) + \epsilon).
\]
For \( n \to \infty \) it follows that the sequence \((z_n)_{n \geq 1}\) is \( \tau^+\)-convergent to \( x \), and because \( f \) is supposed \( \tau^+\)-l.s.c.,
\[
\liminf_{n \to \infty} f(z_n) \geq f(x).
\]
(see [9], p. 127).
Consequently, for every \( \epsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) such that, for every \( n \geq n_0 \),
\[
(6) \quad f(z_n) \geq f(x) - \epsilon.
\]
By (4) and (6) it follows
\[
F_n(x) > f(z_n) - \epsilon + nd(x, z_n) \geq f(z_n) - \epsilon \geq f(x) - 2\epsilon,
\]
for \( n > n_0 \). By (5) one obtains
\[
0 \leq f_n(x) \uparrow f(x),
\]
for all \( x \in X \).

b) Now, let \( f \) be a bounded and \( \tau^+\)-l.s.c function on \( X \). Then there exists \( M > 0 \) such that \( |f(x)| \leq M \), for all \( x \in X \). Denoting \( g(x) = f(x) + M, x \in X \), one obtains \( g(x) \geq 0 \), and \( g \) is \( \tau^+\)-l.s.c., on \( X \).

From the part a), there exists the sequence \((G_n)_{n \geq 1}\), \( G_n \in d\text{-Slip}X, n = 1, 2, 3, \ldots \) such that \( 0 \leq G_n(x) \uparrow g(x) = f(x) + M \). This means that the sequence \((F_n)_{n \geq 1}\), \( F_n = G_n - M, n = 1, 2, 3, \ldots \) is increasing and converges in every point \( x \) of \( X \) to \( f(x) \). Moreover \( |F_n(x)| \leq M, x \in X, n = 1, 2, 3, \ldots \).

c) Now, we consider the general case, i.e., \( f : X \to \mathbb{R} \) is an arbitrary \( \tau^+\)-l.s.c. function.

Using the Baire function (see [11]) \( \varphi : [-\infty, +\infty] \to [-1, 1] \),
\[
\varphi(x) = \begin{cases} 
-1, & \text{for } x = -\infty, \\
\frac{x}{1 + |x|}, & \text{for } -\infty < x < \infty, \\
1, & \text{for } x = +\infty
\end{cases}
\]
which is a Lipschitz increasing isomorphism, it follows that \( \varphi \circ f : \mathbb{R} \to [-1, 1] \) is bounded and \( \tau^+\)-l.s.c. on \( X \).

By the previous point b), there exists a sequence \((H_n)_{n \geq 1}\) with \( H_n \in d\text{-Slip}X \) such that
\[
H_n(x) \uparrow (\varphi \circ f)(x), \quad x \in X.
\]
Consequently, the sequence \((F_n)_{n \geq 1}\), \( F_n(x) = (\varphi^{-1} \circ H_n)(x), x \in X \), is increasing and \( \lim_{n \to \infty} F_n(x) = f(x), x \in X \).

□
For \( \tau^+\)-u.s.c functions on \( X \), one obtains:

**Theorem 5.** Let \((X, d)\) be a quasi-metric space and \( f : X \to \mathbb{R} \) a \( \tau^+\)-u.s.c. function. Then there exists a sequence \( (G_n)_{n \geq 1} \), \( G_n \in d\text{-Slip}X \), \( n = 1, 2, 3, \ldots \) such that \( (G_n(x))_{n \geq 1}, x \in X \), is monotonically decreasing and \( \lim_{n \to \infty} G_n(x) = f(x), x \in X \).

**Proof.** If \( f \) is \( \tau^+\)-u.s.c., then \(-f\) is \( \tau^-\)l.s.c on \( X \). By Theorem 1, there exists a sequence \( (F_n)_{n \geq 1} \) in \( \mathcal{D}\text{-Slip}X \), monotonically increasing and pointwise convergent to \(-f\). Then the sequence \( G_n = -F_n, n = 1, 2, \ldots \), has the following properties: \( G_n = -F_n \in d\text{-Slip}X \), and \( G_n(x) \downarrow f(x), \lim_{n \to \infty} G_n = f(x), x \in X \). □

**REFERENCES**


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