REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 37 (2008) no. 1, pp. 71–75
ictp.acad.ro/jnaat

ON A THEOREM OF BAIRE ABOUT LOWER SEMICONTINUOUS FUNCTIONS*

COSTICĂ MUSTĂŢA†

Abstract. A theorem of Baire concerning the approximation of lower semicontinuous real valued functions defined on a metric space, by increasing sequences of continuous functions is extended to the "nonsymmetric" case, i.e. for quasi-metric spaces.

MSC 2000. 41A30, 26A15, 26A16, 26A21, 54E25.

Keywords. Quasi-metric space, semi-Lipschitz function, approximation.

1. PRELIMINARIES

In the last years there have been an increasing interest for the study of quasi-metric spaces (spaces with asymmetric metric) motivated by their applications in various branches of mathematics, and especially in computer science. A direction of investigation is to study the possibility to extend to quasi-metric spaces known results in metric spaces (see, for example, [2]–[7]).

The following classical result of Baire is well known [8], [9]. *Every lower semicontinuous real valued function defined on a metric space is the pointwise limit of an increasing sequence of continuous functions.* Analyzing the proof of this result (see [9], Th. 2.2-23, p. 84), observe that every element of the increasing sequence is a Lipschitz function. This fact suggest to use the semi-Lipschitz functions defined in [10], [11], to obtain such a theorem for lower semi-continuous real valued functions defined on a quasi-metric space.

This short Note presents some notions connected with quasi-metric spaces and the result of Baire in this framework.

Let X be a non-empty set. A function $d : X \times X \to [0, \infty)$ is called a quasi-metric on X ([10]) if the following conditions hold:

Q_1) $d(x, y) = d(y, x) = 0$ iff $x = y$;

Q_2) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$.

The function $\overline{d} : X \times X \to [0, \infty)$ defined by $\overline{d}(x, y) = d(y, x)$, for all $x, y \in X$ is also a quasi-metric on X, called the *conjugate* quasi-metric of d.

*This work was supported by MEdC under grant 2CEEX06-11-96/19.09.2006.

†"T. Popoviciu" Institute of Numerical Analysis, P.O. Box 68-1, Cluj-Napoca, Romania, e-mail: cmustata@ictp.acad.ro, cmustata2001@yahoo.com.

The function $d^s(x,y) = \max\{d(x,y), \overline{d}(x,y)\}$ is a metric on X. If d can take the value $+\infty$, then it is called a quasi-distance on X.

Each quasi-metric d on X induces a topology $\tau(d)$ which has as a basis the family of open balls (called forward open balls in [6]):

$$B^+(x,\varepsilon) := \{y \in X : d(x,y) < \varepsilon\},\ x \in X,\ \varepsilon > 0.$$

This topology is called the forward topology of X ([3], [6]) and is denoted also by τ^+. Observe that the topology τ^+ is a T_0-topology [10]. If the conditions Q_1 is replaced by $Q_1' : d(x,y) = 0$ iff $x = y$, then τ^+ is a T_1-topology ([10], [11]).

The pair (X,d) is called a quasi-metric space (T_0-separated, respectively T_1-separated). A sequence $(x_n)_{n\geq 1}$ in the quasi-metric space (X,d) is called τ^+-convergent to $x_0 \in X$ iff $\lim\limits_{n\to\infty} d(x_0,x_n) = 0$.

Similarly, the topology $\tau(\overline{d})$ has as a basis the family of open balls:

$$B^-(x,\varepsilon) : \{y \in X : d(y,x) < \varepsilon\}, x \in X,\ \varepsilon > 0.$$

This topology is denoted also by τ^-.

Definition 1. ([10]). Let (X,d) be a quasi-metric space. A function $f : X \to \mathbb{R}$ is called d-semi-Lipschitz if there exists a number $L \geq 0$ (named a d-semi-Lipschitz constant for f) such that

$$f(x) - f(y) \leq L d(x,y), \tag{1}$$

for all $x, y \in X$.

A similar definition can be given for $\overline{d}$-semi-Lipschitz functions.

Definition 2. *The function $f : X \to \mathbb{R}$ is called $\leq_d$-increasing if $f(x) \leq f(y)$, whenever $d(x,y) = 0$.*

One denotes by $R^X_{\leq_d}$ the set of all $\leq_d$-increasing functions on X. The set $\mathbb{R}^X_{\leq_d}$ is a cone in the linear space $\mathbb{R}^X$ of real valued functions defined on X [11].

For a d-semi-Lipschitz function $f : X \to \mathbb{R}$, put

$$\|f|_d = \sup\left\{\tfrac{(f(x)-f(y))\vee 0}{d(x,y)} :\ d(x,y) > 0,\ x, y \in X\right\}. \tag{2}$$

Then $\|f|_d$ is the smallest d-semi-Lipschitz constant of f (see [7], [10], [11]).

Denote also

$$d\text{-SlipX} := \left\{f \in \mathbb{R}^X_{\leq_d} : \|f|_d < \infty\right\}, \tag{3}$$

the subcone of the cone $\mathbb{R}^X_{\leq_d}$, of all d-semi-Lipschitz real valued functions on (X,d). If $\theta \in X$ is a fixed, but arbitrary element, denote

$$d\text{-Slip}_0\text{X} := \{f \in d\text{-SlipX} : f(\theta) = 0\}.$$

Then the functional $\|\cdot|_d : d\text{-Slip}_0\text{X} \to [0,\infty)$ is an asymmetric norm on d-Slip$_0$X, i.e. this functional is subadditive, positively homogeneous and $\|f|_d =$

0 iff $f = 0$. The pair $(d\text{-Slip}_0\text{X}, \|\cdot|_d)$ is called the “normed cone” of d-semi-Lipschitz real valued functions (vanishing at θ). The properties of this normed cone are studied in [10], [11].

DEFINITION 3. *Let (X, d) be a quasi-metric space and $f : X \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = [-\infty, +\infty]$ is equipped with the natural topology. The function f is called τ^+-lower semicontinuous (respectively τ^+-upper semicontinuous) (τ^+-l.s.c, respectively τ^+-u.s.c., in short) at the point $x_0 \in X$, if for every $\varepsilon > 0$ there exists $r > 0$ such that for all $x \in B^+(x_0, r)$, $f(x) > f(x_0) - \varepsilon$ (respectively $f(x) < f(x_0) + \varepsilon$)).*

Similar definitions can be given for τ^- -l.s.c. and τ^--u.s.c real valued functions on $(X, \overline{d})$.

Observe that $f : X \to \overline{\mathbb{R}}$ is τ^+-l.s.c iff $-f$ is τ^--u.s.c and f is τ^+-u.s.c iff $-f$ is τ^--l.s.c.

The result of Baire in this framework is:

THEOREM 4. (Baire). Let (X, d) be a quasi-metric space and $f : X \to \overline{\mathbb{R}}$ be a τ^+- l.s.c function on X. Then there exists a sequence $(F_n)_{n\geq 1}$, $F_n \in d$-SlipX such that $(F_n(x))_{n\geq 1}$, $x \in X$, is increasing and $\lim\limits_{n\to\infty} F_n(x) = f(x)$, $x \in X$.

Proof. a) Suppose firstly that $f(x) \geq 0$, for all $x \in X$. For $x \in X$ and $n \in \mathbb{N}$, let

$$F_n(x) = \inf\{f(z) + n \cdot d(x, z) : z \in X\}. \tag{4}$$

Obviously that

$$0 \leq F_n(x) \leq f(x) + nd(x, x) = f(x). \tag{5}$$

If $x, y, z \in X$, then

$$\begin{aligned} F_n(x) &\leq f(z) + nd(x, z) \leq f(z) + nd(x, y) + nd(y, z) \\ &= (f(z) + nd(y, z)) + nd(x, y). \end{aligned}$$

Taking the infimum with respect to z, one obtains

$$F_n(x) \leq F_n(y) + nd(x, y),$$

i.e.

$$F_n(x) - F_n(y) \leq n \cdot d(x, y),$$

for all $x, y \in X$. This means that $\|F_n|_d \leq n$ and $F_n \in d$-SlipX, for every $n = 1, 2, 3, ...$

If $n \leq m$, by the definition (4) it follows $F_n(x) \leq F_m(x)$, $x \in X$, so that the sequence $(F_n(x))_{n\geq 1}$ is increasing and bounded by $f(x)$, $x \in X$. Consequently there exists the limit $\lim\limits_{n\to\infty} F_n(x) = h(x)$, and $h(x) \leq f(x)$, $x \in X$.

In fact $h(x) = f(x)$, for every $x \in X$. Indeed, let $n \in \mathbb{N}$ and $x \in X$. By definition (4) of $F_n(x)$, for every $\varepsilon > 0$, there exists $z_n \in X$ such that

$$F_n(x) + \varepsilon > f(z_n) + nd(x, z_n) \geq nd(x, z_n),$$

and because $F_n(x)+\varepsilon \le f(x)+\varepsilon$, it follows

$$f(x)+\varepsilon \ge nd(x,z_n),$$

and then

$$d(x,z_n) \le \frac{1}{n}(f(x)+\varepsilon).$$

For $n\to\infty$ it follows that the sequence $(z_n)_{n\ge1}$ is τ^+-convergent to x, and because f is supposed τ^+-l.s.c.,

$$\lim_{n\to\infty}\inf f(z_n) \ge f(x).$$

(see [9], p. 127).

Consequently, for every $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that, for every $n\ge n_0$,

$$f(z_n)\ge f(x)-\varepsilon. \tag{6}$$

By (4) and (6) it follows

$$F_n(x) > f(z_n)-\varepsilon+nd(x,z_n) \ge f(z_n)-\varepsilon \ge f(x)-2\varepsilon,$$

for $n>n_0$. By (5) one obtains

$$0\le F_n(x)\uparrow f(x),$$

for all $x\in X$.

b) Now, let f be a bounded and τ^+-l.s.c function on X. Then there exists $M>0$ such that $|f(x)|\le M$, for all $x\in X$. Denoting $g(x)=f(x)+M$, $x\in X$, one obtains $g(x)\ge 0$, and g is τ^+-l.s.c., on X.

From the part a), there exists the sequence $(G_n)_{n\ge1}$, $G_n\in d$-SlipX, $n=1,2,3,\dots$ such that $0\le G_n(x)\uparrow g(x)=f(x)+M$. This means that the sequence $(F_n)_{n\ge1}$, $F_n=G_n-M$, $n=1,2,3\dots$ is increasing and converges in every point x of X to $f(x)$. Moreover $|F_n(x)|\le M$, $x\in X$, $n=1,2,3,\dots.$

c) Now, we consider the general case, i.e., $f:X\to\overline{\mathbb{R}}$ is an arbitrary τ^+-l.s.c. function.

Using the Baire function (see [1]) $\varphi:[-\infty,+\infty]\to[-1,1]$,

$$\varphi(x)=\begin{cases} -1, \text{ for } x=-\infty,\\ \frac{x}{1+|x|}, \text{ for } -\infty<x<\infty,\\ 1, \text{ for } x=+\infty\end{cases}$$

which is a Lipschitz increasing isomorphism, it follows that $\varphi\circ f:\overline{\mathbb{R}}\to[-1,1]$ is bounded and τ^+-l.s.c. on X.

By the previous point b), there exists a sequence $(H_n)_{n\ge1}$ with $H_n\in d$-SlipX such that

$$H_n(x)\uparrow(\varphi\circ f)(x),\ x\in X.$$

Consequently, the sequence $(F_n)_{n\ge1}$, $F_n(x)=(\varphi^{-1}\circ H_n)(x)$, $x\in X$, is increasing and $\lim\limits_{n\to\infty} F_n(x)=f(x)$, $x\in X$.

□

For τ^+-u.s.c functions on X, one obtains:

THEOREM 5. *Let (X, d) be a quasi-metric space and $f : X \to \overline{\mathbb{R}}$ a τ^+-u.s.c. function. Then there exists a sequence $(G_n)_{n\geq 1}$, $G_n \in d$-SlipX, $n = 1, 2, 3, \ldots$ such that $(G_n(x))_{n\geq 1}$, $x \in X$, is monotonically decreasing and $\lim_{n\to\infty} G_n(x) = f(x)$, $x \in X$.*

Proof. If f is τ^+-u.s.c., then $-f$ is τ^--l.s.c on X. By Theorem 1, there exists a sequence $(F_n)_{n\geq 1}$ in $\overline{d}$-SlipX, monotonically increasing and pointwise convergent to $-f$. Then the sequence $G_n = -F_n, n = 1, 2, \ldots$, has the following properties: $G_n = -F_n \in d$-SlipX, and $G_n(x) \downarrow f(x)$, $\lim_{n\to\infty} G_n = f(x)$, $x \in X$. □

REFERENCES

[1] BAIRE, R., *Leçon sur les fonctions discontinues,* Paris, Collection Borel, 1905, pp. 121–123.

[2] BORODIN, P.A., *The Banach-Mazur theorem for spaces with asymmetric norm and its applications in convex analysis*, Mat. Zametki, **69**, no. 3, pp. 329–337, 2001.

[3] COLLINS, J. and ZIMMER, J. *An asymmetric Arzèla-Ascoli theorem,* Topology Appl., **154**, no. 11, pp. 2312–2322, 2007.

[4] KÜNZI, H.P.A., *Nonsymmetric distances and their associated topologies: About the origin of basic ideas in the area of asymmetric topology*, in: Handbook of the History of General Topology, edited. by C.E. Aull and R. Lower, vol. **3**, Kluwer Acad. Publ., Dordrecht, pp. 853–968, 2001.

[5] MCSHANE, E.T., *Extension of range of fucntions*, Bull. Amer. Math. Soc., **40**, pp. 837–842, 1934.

[6] MENUCCI, A., *On asymmetric distances,* Technical Report, Scuola Normale Superiore, Pisa, 2004.

[7] MUSTĂŢA, C., *Extensions of semi-Lipschitz functions on quasi-metric spaces*, Rev. Anal. Numér. Théor. Approx., **30**, no. 1, pp. 61–67, 2001.

[8] NICOLESCU, M., *Mathematical Analysis*, Vol. II, Editura Tehnică, Bucharest, p. 119, 1958 (in Romanian).

[9] PRECUPANU, A., *Mathematical Analysis: Measure and Integration*, Vol. I., Editura Universităţii A.I. Cuza Iaşi, 2006 (Romanian).

[10] ROMAGUERA, S. and SANCHIS, M., *Semi-Lipschitz functions and best-approximation in quasi-metric spaces*, J. Approx. Theory, **103**, pp. 292–301, 2000.

[11] ROMAGUERA, S. and SANCHIS, M., *Properties of the normed cone of semi-Lipschitz functions.* Acta Math. Hungar., **108**, no. 1-2, pp. 55–70, 2005.

Received by the editors: March 27, 2008.