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PRECONDITIONING BY AN EXTENDED MATRIX TECHNIQUE FOR CONVECTION-DIFFUSION-REACTION EQUATIONS

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Abstract. In this paper we consider a preconditioning technique for the ill-conditioned systems arising from discretisations of nonsymmetric elliptic boundary value problems. The rectangular preconditioning matrix is constructed via the transfer operators between successive discretization levels of the initial problem. In this way we get an extended, square, singular, consistent, but mesh independent well-conditioned linear system. Numerical experiments are presented for a 2D convection-diffusion-reaction problem.

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1. INTRODUCTION

Let

$$Ax^* = b \tag{1.1}$$

be a linear system of equations, with A an $n \times n$ invertible matrix, $b \in \mathbb{R}^n$ and x^* its unique solution. If A is ill-conditioned a wide class of preconditioning techniques has been developed in the last 30 years (see [1], [3], [5], [7], [11] and references therein).

Usually these methods are of the form

$$Ax^* = b \Leftrightarrow (PAQ)(Q^{-1}x^*) = Pb, \tag{1.2}$$

where the $n \times n$ invertible matrices P and Q are constructed in an appropriate way which ensures a much smaller or independent on n condition number for the preconditioned matrix PAQ in (1.2). In this paper we shall consider a more general preconditioning technique which will include (1.2) as a particular case. For this, let $m \geq n$ and S be a full row rank $n \times m$ matrix. Then, the system (1.1) will be transformed as

$$\hat{A}\hat{x} = \hat{b}, \tag{1.3}$$

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with

$$\hat{A} = S^{\mathrm{t}}AS,\ \hat{b} = S^{\mathrm{t}}b. \tag{1.4}$$

The $m \times m$ matrix $\hat{A}$ is no more invertible for $m > n$ (because $\mathrm{rank}(\hat{A}) = n$), but the system (1.3) is consistent by construction. Moreover, for any solution $\hat{x} \in \mathbb{R}^m$ of (1.3), by multiplying on the left of (1.3) with S we get $SS^{\mathrm{t}}AS\hat{x} = SS^{\mathrm{t}}b \Leftrightarrow GAS\hat{x} = Gb \Leftrightarrow A(S\hat{x}) = b \Leftrightarrow S\hat{x} = x^*$, where G is the $n \times n$ symmetric and positive definite matrix (SPD, for short) defined by

$$G = SS^{\mathrm{t}}, \tag{1.5}$$

and the superscript t denotes the transpose of a matrix. For any square matrix B we shall denote by $\sigma^*(B)$ the set of all its nonzero eigenvalues and by $\mathrm{cond}(B)$ its *spectral condition number* defined by

$$\mathrm{cond(B)} = \sqrt{\frac{\max\{\lambda, \lambda \in \sigma^*(B^{\mathrm{t}}B)\}}{\min\{\lambda, \lambda \in \sigma^*(B^{\mathrm{t}}B)\}}}. \tag{1.6}$$

With respect to the preconditioning procedure (1.3)–(1.4) we are interested whether it exists a constant $c > 0$, independent on n and m such that

$$\mathrm{cond}(\hat{A}) \leq c. \tag{1.7}$$

A first answer to this question has been given by M. Griebel in [6]. In this respect, for symmetric elliptic boundary value problems (b.v.p.), he used for the construction of the preconditioner S in (1.4) the multigrid transfer operators between successive discretization levels of the initial problem. Unfortunately, Griebel's approach essentially uses the symetry of the b.v.p. In this paper, we extend Griebel's multilevel preconditioning procedure to nonsymmetric elliptic b.v.p. The paper is organized as follows: in section 2 we describe Griebel's preconditioner together with the related results in the case of symmetric eliptic b.v.p. In section 3 we introduce the general algebraic framework which gives us the possibility to extend Griebel's ideas to nonsymmetric b.v.p. and prove that the preconditioned matrix has a mesh independent condition number (Theorem 1). In section 4 we present numerical experiments with two versions of our preconditioning technique for a class of 2D convection-diffusion-reaction problems.

2. GRIEBEL'S MULTILEVEL PRECONDITIONING

In [5] M. Griebel considered the following boundary value problem

$$\begin{cases} Lu = f, & \text{on } \Omega \\ u = g, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega = (0,1)^d$, (usually $d = 1, 2, 3$) and L is a second order elliptic differential operator on Ω. The problem (2.1) can be discretized by appropriate finite differences (see [3]) or finite element techniques (see [7], [9]). In the later case we may also suppose that a variational formulation is available in the following form: find $u \in U(\Omega)$ with

$$a(u, v) = \langle f, v \rangle_{L^2}, \quad \forall v \in V(\Omega), \tag{2.2}$$

where $U(\Omega), V(\Omega)$ are Hilbert spaces of real valued functions defined on Ω, $a : U(\Omega) \times V(\Omega) \longrightarrow \mathbb{R}$ is a bilinear functional, $f \in L^2(\Omega)$ and $\langle \cdot, \cdot \rangle_{L^2}, \| \cdot \|_{L^2}$ are the $L^2(\Omega)$ scalar product and norm. We shall also suppose that the bilinear functional a is such that the variational formulation (2.2) has a unique solution $u \in U(\Omega)$. In [5] M. Griebel considered a symmetric elliptic b.v.p. of the form (2.1)–(2.2) for which $U(\Omega) = V(\Omega) = H_0^1(\Omega)$ and a bilinear, symmetric, bounded and coercive, i.e.

$$|a(u, v)| \leq M \parallel u \parallel_{H_0^1} \parallel v \parallel_{H_0^1}, \ a(u, u) \geq \mu \parallel u \parallel_{H_0^1}^2, \quad \forall u, v \in H_0^1(\Omega), \tag{2.3}$$

where $\parallel v \parallel_{H_0^1(\Omega)}^2 = \sum_{i=1}^{d} \parallel \frac{\partial v}{\partial x_i} \parallel_{L^2}^2 + \parallel v \parallel_{L^2}^2$.

Remark 2.1. For $g = 0$ there is a straightforward way to obtain a variational formulation as (2.2)–(2.3), whereas in the nonhomogeneous case, $g \neq 0$, we may use the approach from Chapter 7 in [7]. □

Let $k \geq 2$ be a given integer, $n_k = (2^k - 1)^d$ and $B_k = \{\varphi_1^{(k)}, \ldots, \varphi_{n_k}^{(k)}\}$ a standard finite element basis (e.g. piecewise d-linear, see [9]). Then the linear system associated to (2.2) is

$$A_k x_k = b_k, \tag{2.4}$$

where

$$(A_k)_{ij} = a(\varphi_j^{(k)}, \varphi_i^{(k)}), \ (b_k)_i = \langle f, \varphi_i^{(k)} \rangle_{L^2}, \ i, j = 1, \ldots, n_k. \tag{2.5}$$

From the properties of the bilinear functional a it results that the matrix A_k is SPD. Let now

$$V_1 \subset V_2 \subset \cdots \subset V_k \tag{2.6}$$

be a sequence of spaces of piecewise d-linear functions associated to a sequence of uniform, equidistant, nested grids

$$\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_k, \tag{2.7}$$

$n_q = (2^q - 1)^d$ the dimension of V_q, $q = 1, 2, \ldots$ and $B_q = \{\varphi_1^{(q)}, \ldots, \varphi_{n_q}^{(q)}\}$ the finite element basis in V_q. Let also $\hat{B}_k \subset V_k$ and m_k be given by

$$\hat{B}_k = B_1 \cup B_2 \cup \cdots \cup B_k, \ m_k = n_1 + n_2 + \cdots + n_k. \tag{2.8}$$

The functions from $\hat{B}_k$ are linearly dependent and generate the subspace V_k. Each function $\varphi_j^{(q)} \in V_q \subset V_{q+1}$ has a unique representation as an element of

V_{q+1}, of the form

$$\varphi_j^{(q)} = \sum_{i=1}^{n_{q+1}} c_{ij} \varphi_i^{(q+1)},\ j = 1, \ldots, n_q. \tag{2.9}$$

We consider the $n_{q+1} \times n_q$ grid transfer matrix I_q^{q+1} given by

$$\left(I_q^{q+1}\right)_{ij} = c_{ij} \tag{2.10}$$

and for $q = 1, 2, \ldots, k-1$ define the $n_k \times n_q$ matrices S_q^k by

$$S_q^k = I_{k-1}^k I_{k-2}^{k-1} \ldots I_q^{q+1} \tag{2.11}$$

and the $n_k \times m_k$ matrix S_k (in block notation)

$$S_k = \left[\begin{array}{c|c|c|c|cccc} & & & & 1 & & & \\ & & & & & 1 & & \\ S_1^k & S_2^k & \ldots & S_{k-1}^k & & & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{array} \right], \tag{2.12}$$

in which the last $n_k \times n_k$ block is the unit matrix. We then consider the

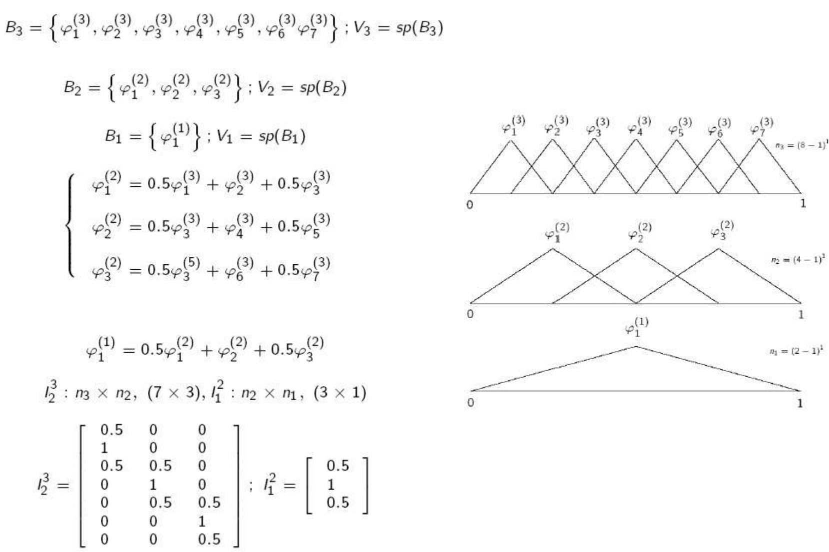

Fig. 1. Example of the one-dimensional functions.

preconditioned version of (2.4) (of the form (1.3)–(1.4)) (see Figure 1 for 1D case and 3 successive level of discretization)

$$\hat{A}_k \hat{x}_k = \hat{b}_k, \tag{2.13}$$

$$\hat{A}_k = S_k^t A_k S_k,\ \hat{b}_k = S_k^t b_k. \tag{2.14}$$

It results that the $m_k \times m_k$ matrix $\hat{A}_k$ is symmetric and positive semidefinite. M. Griebel proved (see [5] and [6] references therein) that the spectral condition number of the preconditioned matrix $\hat{A}_k$ is mesh independent. He's proof is essentially based on the symmetry of A_k (and $\hat{A}_k$) and the fact that the $n_k \times n_k$ matrix G_k defined as in (1.5), i.e.

$$G_k = S_k S_k^{\mathrm{t}}, \tag{2.15}$$

is spectrally equivalent (see the definition (3.1) in section 3) with the inverse of the standard discretized Laplacian Δ_k (5-point stencil, see [7]), which for picewise finite element basis functions and a variational formulation on $H_0^1(\Omega)$ as (2.2)–(2.3) is defined by

$$(\Delta_k)_{ij} = \langle \varphi_j^{(k)}, \varphi_i^{(k)} \rangle_{H_0^1(\Omega)}, \ i, j = 1, \ldots, n_k. \tag{2.16}$$

In the next section we shall present an extension of this procedure for non-symmetric problems as (2.1).

3. THE GENERAL ALGEBRAIC FORMULATION

We shall start the presentation of our extension by a brief replay of some results related to spectrally equivalent matrices. In this sense we introduce the following notations: for an $n \times n$ SPD matrix A we shall denote by $\sigma(A), \rho(A)$ its spectrum and spectral radius and by $\lambda_{\min}(A), \lambda_{\max}(A)$ its smallest and biggest eigenvalue. For an arbitrary $m \times n$ matrix T, $N(T), R(T)$ will denote its null space and range, respectively; the notations $\langle \cdot, \cdot \rangle, \| \cdot \|$ will be used for the euclidean scalar product and norm on some vector space $\mathbb{R}^q$.

Definition 3.1. *For A and B two $n \times n$ SPD matrices, we shall say that A is spectrally equivalent with B if there exist two positive constants $\alpha_1, \alpha_2 > 0$, independent on n such that*

$$\alpha_1 \leq \tfrac{\langle Ax, x \rangle}{\langle Bx, x \rangle} \leq \alpha_2, \ \forall x \in \mathbb{R}^n, \ x \neq 0. \tag{3.1}$$

If we write this by $A \approx B$ the following results are known (see e.g. [1, 3, 4]*).*

Proposition 3.2. (i) *The matrix A is spectrally equivalent with B if and only if for a decomposition of the form $B = CC^{\mathrm{t}}$ we have*

$$\alpha_1 \leq \lambda_{\min}(C^{-1} A C^{-\mathrm{t}}) \leq \lambda_{\max}(C^{-1} A C^{-\mathrm{t}}) \leq \alpha_2, \tag{3.2}$$

for some a_1, a_2 independent on n. Moreover, inequalities (3.2) *are independent on the decomposition of B.*

(ii) *If $A \approx B$ then $B \approx A$ and $A^{-1} \approx B^{-1}$;*

(iii) *If $A \approx B$ and $B \approx C$, then $A \approx C$.*

Remark 3.3. Relation (3.2) makes the connection between the spectral equivalence and preconditioning. Indeed, by taking into account that the spectral condition number of an SPD matrix T is defined as (see also (1.6))

$$\mathrm{cond}(T) = \frac{\lambda_{\max}(T)}{\lambda_{\min}(T)}, \tag{3.3}$$

from (3.2) it results that, if the matrices A and B are spectrally equivalent, then B is a "good preconditioner" for A, i.e.

$$\mathrm{cond}(C^{-1}AC^{-\mathrm{t}}) \leq \frac{\alpha_2}{\alpha_1}. \tag{3.4}$$

let us know consider a general (nonsymmetric) system of the form (1.1) and let M, R be the matrices defined by

$$M = \frac{1}{2}(A + A^{\mathrm{t}}),\ R = \frac{1}{2}(A - A^{\mathrm{t}}). \tag{3.5}$$

Because

$$M^{\mathrm{t}} = M,\ R^{\mathrm{t}} = -R \tag{3.6}$$

it results that both matrices are normal (see e.g. [1]). We shall suppose that M is positive definite, i.e.

$$\langle Mx, x\rangle > 0,\ \forall x \neq 0. \tag{3.7}$$

□

Our extension of the previous Griebel's preconditioning procedure is based on the following assumptions.

Assumption A1. The matrix M is spectrally equivalent with G^{-1} (G from (1.5) and S as in (1.4)), i.e. there exist the positive constants α_1, α_2, independent on n, such that

$$\alpha_1 \leq \frac{\langle Mx, x\rangle}{\langle G^{-1}x, x\rangle} \leq \alpha_2,\ \forall x \in \mathbb{R}^n,\ x \neq 0. \tag{3.8}$$

Assumption A2. It exists a constant $\beta \geq 0$, independent on n and m, such that

$$\rho(GR) \leq \beta. \tag{3.9}$$

The following results shows that the preconditioned matrix $\hat{A}$ from (1.4) is well-conditioned.

Theorem 3.4. *In the above hypothesis and under the assumptions* A1 *and* A2 *we have for the matrix* $\hat{A}$ *in* (1.3)–(1.4)

$$\mathrm{cond}(\hat{A}) \leq \frac{\alpha_2 + \beta}{\alpha_1}. \tag{3.10}$$

For proving Theorem 1 we need two auxiliary results which will be presented in what follows.

Lemma 3.5. *We have the equality*

$$\sigma^*(\hat{A}) = \sigma(GA). \tag{3.11}$$

Proof. Let first $\lambda \in \sigma^*(\hat{A})$. Then, for some nonzero vector z we have the following sequence of equalities, in which the last one shows that $\lambda \in \sigma(GA)$.

$$\hat{A}z = \lambda z \Longrightarrow S^{\mathrm{t}}ASz = \lambda z \Longrightarrow SS^{\mathrm{t}}ASz = \lambda Sz \Longrightarrow GA(Sz) = \lambda Sz.$$

Conversely, let $\lambda \in \sigma(GA)$. Then, $\lambda \neq 0$ and for some nonzero vector y we have $GAy = \lambda y$. Using the invertibility of the SPD matrix G we define $w = G^{-1}y \neq 0$ and get

$$GAGw = \lambda Gw \Longleftrightarrow SS^{\mathrm{t}}ASS^{\mathrm{t}}w = \lambda SS^{\mathrm{t}}w.$$

Thus, because the application $S^{\mathrm{t}} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is injective, the vector $z = S^{\mathrm{t}}w$ will be nonzero and $S\hat{A}z = \lambda Sz$, i.e.

$$\hat{A}z - \lambda z \in N(S). \tag{3.12}$$

On the other hand, from (1.4) we get

$$\hat{A}z - \lambda z = S^{\mathrm{t}}ASz - \lambda S^{\mathrm{t}}w \in R(S^{\mathrm{t}}),$$

which together with (3.12) gives us $\hat{A}z - \lambda z = 0$, i.e. $\lambda \in \sigma^*(\hat{A})$. □

LEMMA 3.6. *Let $G = CC^{\mathrm{t}}$ with C an $n \times n$ invertible matrix and $\bar{A}, \bar{M}, \bar{R}$ defined by*

$$\bar{A} = C^{\mathrm{t}}AC,\ \ \bar{M} = C^{\mathrm{t}}MC,\ \ \bar{R} = C^{\mathrm{t}}RC, \tag{3.13}$$

with A, M, R from (3.5). *Then*

$$\lambda_{\min}(\bar{M}) \geq \alpha_1,\ \ \lambda_{\max}(\bar{M}) \leq \alpha_2. \tag{3.14}$$

Proof. Because M (see (3.6)–(3.7)) is SPD, so will be $\bar{M}$ from (3.13). Then, by also using Assumption A1 we get

$$\lambda_{\min}(\bar{M}) = \min_{x\neq 0} \tfrac{\langle \bar{M}x,x\rangle}{\langle x,x\rangle} = \min_{x\neq 0} \tfrac{\langle MCx,Cx\rangle}{\langle x,x\rangle} =$$

$$\min_{y\neq 0} \tfrac{\langle My,y\rangle}{\langle C^{-1}y,C^{-1}y\rangle} = \min_{y\neq 0} \tfrac{\langle My,y\rangle}{\langle G^{-1}y,y\rangle} \geq \alpha_1.$$

The proof of the second inequality in (3.14) results by a similar procedure. □

Proof of Theorem 1. According to [3] (Theorem 1), from (3.13)–(3.14) we obtain

$$\lambda_{\min}(\bar{A}^{\mathrm{t}}\bar{A}) \geq \left(\lambda_{\min}(\bar{M})\right)^2,\ \ \lambda_{\max}(\bar{A}^{\mathrm{t}}\bar{A}) \leq \left(\lambda_{\max}(\bar{M}) + \rho(\bar{R})\right)^2. \tag{3.15}$$

Moreover, from (3.13) and (3.9) it follows that

$$\rho(\bar{R}) = \rho(C^{\mathrm{t}}RC) = \rho(CC^{\mathrm{t}}R) = \rho(GR) \leq \beta. \tag{3.16}$$

Now, by using (3.11) and some well-known properties of the spectrum and spectral radius of matrices, we successively get

$$\sigma^*(\hat{A}^{\mathrm{t}}\hat{A}) = \sigma^*(SA^{\mathrm{t}}SS^{\mathrm{t}}AS) = \sigma^*(S^{\mathrm{t}}(A^{\mathrm{t}}GA)S) = \sigma(G(A^{\mathrm{t}}GA)) =$$

$$\sigma(CC^{\mathrm{t}}(A^{\mathrm{t}}GA)) = \sigma(C^{\mathrm{t}}(AGA)C) = \sigma(C^{\mathrm{t}}A^{\mathrm{t}}CC^{\mathrm{t}}AC) = \sigma(\bar{A}^{\mathrm{t}}\bar{A}). \tag{3.17}$$

Then, from (3.15), (3.17), (1.6), (3.14) and (3.9), we obtain (3.10) and the proof is complete. □

REMARK 3.7. We have to observe that, beside the specific properties mentioned in the above assumptions A1 and A2, our preconditioning method requests only the invertibility of the system matrix in (1.1) and the fact that its symmetric part is SPD. These properties are fulfilled by a large class of finite element or finite differences discretizations of elliptic boundary value problems (b.v.p.), see e.g. [8]). □

4. NUMERICAL EXPERIMENTS

We considered in our experiments the convection-diffusion-reaction problem

$$\begin{cases} -\Delta u + \gamma \frac{\partial u}{\partial x} + \delta u & = f, \text{ in } \Omega = (0,1)^2 \\ u & = 0 \text{ on } \partial(\Omega), \end{cases} \tag{4.1}$$

where $\gamma \in (0, \infty)$ and the right hand side f such that the unique exact solution is $u(x,y) = xy(1-x)(1-y)\mathrm{e}^{xy}$ (see [3]). The problem was discretized using the classical 5-point stencil finite differences on k successive grids. On each grid, the number of interior nodes is $n_q = (2^q - 1)^2, q = k, \ldots, 1$, such that the system on the finest level (2.4) has dimension n_k. We used in our experiments two types of intergrid transfer operators, I_q^{q+1} and J_q^{q+1}, $q = k-1, \ldots, 1$ defined (in stencil notation) by

$$\left[I_q^{q+1}\right] = \frac{1}{4}\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}_h, \quad \left[J_q^{q+1}\right]_h = \frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}_h. \tag{4.2}$$

The preconditioning $n_k \times m_k$ matrix S_k was constructed as in (2.11)–(2.12) and the extended preconditioned system and matrix G_k as in (2.13)–(2.15). For solving the initial system (2.4) and the preconditioned one (2.13)–(2.14) we used the CGLS (Conjugate Gradient for Least Squares) algorithm from [1], with the stopping rules, respectively

$$\| A_k x_k - b_k \| \leq 10^{-6}, \ \| \hat{A}_k \hat{x}_k - \hat{b}_k \| \leq 10^{-6}. \tag{4.3}$$

The results (number of iterations versus the dimension of the finest grid n_k and the convection coefficient γ (for the reaction coefficient $\delta = 0$)) are presented in Tables 1–3. We observe in Tables 2–3 the mesh independence behaviour of the preconditioned system for both choices of intergrid transfer operators, together with γ-dependence for fixed k and n_k. Moreover in Table 4 we can see that $\hat{A}_k$ remains a sparse matrix and (because of the mesh independence of the preconditioning) the computational time for solving the system (2.13)–(2.14) (constructed with I_q^{q+1} from (4.2) is much less, for bigger dimensions, than for solving the nonpreconditioned one (2.4).

Table 1. No preconditioning.

n_k/γ	0	50	100	200
225	120	109	96	143
961	489	342	249	239
3969	1940	1290	849	602
16129	7599	4737	3230	2120

Table 2. Preconditioning with I_q^{q+1} from (4.2).

m_k/γ	0	50	100	200
284	25	56	111	274
1245	28	61	109	255
5214	30	60	103	228
21343	33	61	107	214

Table 3. Preconditioning with J_q^{q+1} from (4.2).

m_k/γ	0	50	100	200
284	34	64	115	295
1245	46	71	116	250
5214	54	71	114	222
21343	61	71	105	199

Table 4. Sparsity and computational time (for J_q^{q+1}).

k	n_k	m_k	spa(A)	spa($\widehat{A}_k$)	$Time(A_k; b_k)$	$Time(\widehat{A}_k; \widehat{b}_k)$
3	49	59	9%	19%	0.01	0.03
4	225	284	2%	6.6%	0.09	0.10
5	961	1245	0.5%	2.1%	0.73	0.56
6	3969	5214	1.2%	6.8%	10	3.59
7	16129	21343	0.03%	0.2%	204	19
8	65025	86368	0.007%	0.06%	3370	102

5. CONCLUSIONS AND FURTHER WORK

In this paper we presented an extension of Griebel's preconditioning technique from [5], for symmetric elliptic boundary value problems to nonsymmetric problems if the form (4.1). The extension, formulated in Section 3 in a very general algebraic form is based on the assumptions A1 and A2, which controls the symmetric and, respectively skew-symmetric part of the system matrix A. We applied this extension to the (nonsymmetric) boundary value problem (4.1), using a finite differences discretization and a classical coarsening. We used the two versions of intergrid transfer operators from (4.2). In both cases, the results presented in Tables 2 and 3 show a mesh-independence behaviour. For I_{k-1}^k the symmetric part M_k of A_k is exactly the 5-point stencil Laplacian Δ_k (see [3], [7]), thus the assumption A1 holds. Unfortunately we have not yet a similar result for the J_{k-1}^k operators and also for the second assumption A2, in both cases from (4.2). Work is in progress on this direction.

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