ON A DUAL CHARACTERIZATION
IN BEST APPROXIMATION PROBLEM

TEODOR PRECUPANU*

Abstract. We establish a dual characterization of solutions of Ky Fan best approximation problem and as consequence we obtain an existence criterium under conditions formulated for the weak topology.


Keywords. Ky Fan best approximation problem, fixed point, KKM-mapping, strongly continuous function.

1. INTRODUCTION AND PRELIMINARIES

Let $X$ be a linear normed space and $X^*$ its dual. Given a nonvoid set $A \subset X$ and a function $f : A \to X$, the associated best approximation problem of Ky Fan type is to find $y \in A$ such that

$$ (1.1) \quad \inf_{x \in A} \| f(y) - x \| = \| f(y) - y \|. $$

Obviously, if $f$ is a constant function we obtain the well known best approximation problem. Also, if the range of a function $f$ is contained in $A$ the solutions of Ky Fan best approximation problem are the fixed points of $f$. Generally, the set of all solutions of Ky Fan best approximation problem coincides with the set of all fixed points of the multivalued mapping $P_A \circ f$, where $P_A$ is the projection operator on $A$. On the other hand, any fixed point of $y$ is a solution of Ky Fan approximation problem, but it is possible to exist other solutions $y \in A$ such that $f(y) \notin A$ which, obviously are not fixed points of $f$. We denote

$$ (1.2) \quad P_A(f) = \{ y \in A; \| y - f(y) \| = d(f(y); A) \}, $$

where

$$ (1.3) \quad d(u; A) = \inf_{x \in A} \| x - u \|. $$

Therefore the best approximation problem (1.1) can be regarded as a unification of this two problems: the best approximation problem and the fixed point

*“Al.I. Cuza” University, Faculty of Mathematics, 11, Bd. Carol, 700506, Iași and Institute of Mathematics “O. Mayer”, 8, Bd. Carol, 700505, Iași, e-mail: tprecup@uaic.ro.
problem of a given function. The problem (1.1) is equivalent to the following variational inequality:

\[(1.4) \quad \|f(y) - y\| \leq \|f(y) - x\|, \quad \text{for all} \ x \in A. \]

In 1969 Ky Fan [4] was established an important criterion under hypotheses that the set \(A\) is nonvoid compact convex set and the function \(f\) is continuous on \(A\). This result generalizes the well known fixed point theorem of Browder. Many authors (see, for example, Kapoor [7], Ky Fan [5], Lin [10], Lin and Yen [12], Ding and Tan [2], Sehgal and Singh [16], Roux and Singh [15], Singh and Watson [17], Reich [14]) have obtained other interesting extensions of Ky Fan’s result. Also, several extensions and applications concerning Ky Fan best approximation problem are investigated by Singh, Watson and Srivastava in their monograph [18] (see also the references cited therein).

If we define the multivalued mapping \(G : A \Rightarrow A\) by

\[(1.5) \quad G(x) = \{y \in A : \|f(y) - y\| \leq \|f(y) - x\|\}, \]

then the solutions of Ky Fan best approximation problem are the elements of the intersection \(\bigcap_{x \in A} G(x)\).

Since \(G\) is a \(Knaster-Kuratowski-Mazurkiewicz\) mapping (KKM-mapping) we can obtain optimality criteria using special results for this mappings which assures that the above intersection is nonvoid. We recall that a multivalued mapping \(F : K \Rightarrow E\), where \(K\) is a nonvoid subset of a separated topological vector space \(E\), is called a KKM-mapping if

\[\text{co}\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^{n} F(x_i), \]

for any finite subset \(\{x_1, x_2, \ldots, x_n\}\) of \(K\) ([1], [8], [18]).

An important result concerning KKM-mappings was obtained by Ky Fan.

**Theorem 1.** ([3]) *Let \(K\) be a nonvoid subset in a separated topological space \(E\) and let \(F : K \Rightarrow E\) be a closed valued KKM-mapping. If \(F(x_0)\) is compact for at least one \(x_0 \in E\), then*

\[(1.6) \quad \bigcap_{x \in K} F(x) \neq \emptyset.\]

Consequently, if in the linear normed space \(X\) we consider \(KKM\)-mapping \(G\) defined by (1.5) we can obtain Ky Fan’s result above mentioned. In fact, the following two conditions are sufficient:

(i) \(f\) is continuous on every compact subset of \(A\);

(ii) there exists \(x_0 \in A\) such that the set \(\{y \in A : \|f(y) - y\| \leq \|f(y) - x_0\|\}\) is compact.

Now, if \(X\) is endowed with weak topology we get a result obtained by Kapoor [7]. The Kapoor’s hypotheses are:

(i') \(A\) is a nonempty weakly compact convex set;
(ii') \( f \) is a strongly continuous mapping.

### 2. DUAL CHARACTERIZATIONS

Firstly, we recall the well known dual characterization of best approximation elements with respect to a nonvoid convex set established by Garkavi \([6]\).

**Theorem 2.** ([6]) Let \( A \) be a nonvoid convex set in a linear normed space \( X \). Then, an element \( \overline{x} \in A \) is a best approximation of an element \( u \in X \) from elements of \( A \) if and only if there exists \( x_0^* \in X^* \) such that

\[
(2.1) \quad \|x_0^*\| = \|\overline{x} - u\|,
\]

\[
(2.2) \quad x_0^*(x - u) \geq \|\overline{x} - u\|^2, \quad \text{for all } x \in A.
\]

See also \([1], [9], [13]\).

Because an element \( y \in A \) is a solution of Ky Fan best approximation problem if and only if \( y \in \overline{P}_A(f(y)) \), by Theorem 2 we obtain the following dual characterization.

**Theorem 3.** Let \( A \) be a nonvoid convex set in \( X \). Then an element \( y \in A \) is a solution of Ky Fan best approximation problem if and only if there exists \( y^* \in X^* \) such that

\[
(2.3) \quad \|y^*\| \leq 1,
\]

\[
(2.4) \quad y^*(x - f(y)) \geq \|y - f(y)\|, \quad \text{for all } x \in A.
\]

**Remark 4.** It is obvious that (2.3), (2.4) are fulfilled whenever \( y \) is a fixed point of \( f \) taking \( y^* = 0 \). If \( y \) is not a fixed point of \( f \), i.e. \( \|y - f(y)\| > 0 \), then we have necessarily \( \|y^*\| = 1 \). \( \square \)

**Remark 5.** If \( A \) is a closed linear subspace of \( X \) and \( y \) is not a fixed point of \( f \), then \( \|y^*\| = 1 \) and (2.4) is equivalent with the following two conditions

\[
(2.4') \quad y^*(x) = 0 \quad \text{for all } x \in A;
\]

\[
(2.4'') \quad y^*(f(y)) = \|y - f(y)\|.
\]

\( \square \)

The conditions (2.3), (2.4) can be equivalently rewrite in a special minimax form or in a variational form.

**Theorem 6.** Let \( A \) be a nonvoid convex set in \( X \). Then \( y \in \overline{P}_A(f) \) if and only if there exists \( y^* \in X^* \) such that the pair \((y^*, y)\) is a saddle point of the minimax equality

\[
(2.5) \quad \max_{\|x^*\| \leq 1} \min_{x \in A} x^*(f(y) - x) = \min_{x \in A} \max_{\|x^*\| \leq 1} x^*(f(y) - x),
\]
or equivalently, the following variational inequality is fulfilled

\[(2.6)\qquad y^*(f(y) - x) - x^*(f(y) - y) \geq 0,\]

for all \((x^*, x) \in \mathcal{F}_{X^*}(0; 1) \times A.\]

Proof. The minimax equality (2.5) says that \((y^*, y)\) is a saddle point of the function \(\emptyset_y(x^*, x) = x^*(f(y) - x)\) and so we have

\[(2.7)\qquad \emptyset_y(x^*, y) \leq \emptyset_y(y^*, y) \leq \emptyset_y(y^*, x),\]

for all \((x^*, x) \in \mathcal{F}_{X^*}(0; 1) \times A,\)

which is just the variational inequality (2.6). Therefore (2.5) and (2.6) are equivalent. On the other hand, from (2.6) it follows

\[y^*(f(y) - x) \geq \sup_{x^* \in \mathcal{F}_{X^*}(0; 1)} x^*(f(y) - y) = \|f(y) - y\|,\]

which proves that \(-y^*\) has the properties (2.3) and (2.4). Conversely, it is easily to prove that (2.4) implies (2.6) for \(-y^*\).

Now, we denote

\[(2.8)\qquad F(x^*, x) = \{(y^*, y) \in \mathcal{F}_{X^*}(0; 1) \times A;\]

\[y^*(f(y) - x) - x^*(f(y) - y) \geq 0\},\]

for every \((x^*, x) \in \mathcal{F}_{X^*}(0; 1) \times A.\) Obviously,

\[(2.9)\qquad (x^*, x) \in F(x^*, x) \text{ for every } (x^*, x) \in \mathcal{F}_{X}(0; 1) \times A.\]

Moreover, if \(A\) is a convex set, then the multivalued function \(F : \mathcal{F}_{X^*}(0; 1) \times A \mapsto \mathcal{F}_{X^*}(0; 1) \times A\) is a KKM-mapping. Indeed, if \((y_i^*, y_i) \in \mathcal{F}_{X^*}(0; 1) \times A, i = \overline{1, n},\) and we suppose that \((\sum_{i=1}^{n} \lambda_i y_i^*, \sum_{i=1}^{n} \lambda_i y_i) \notin F(y_j^*, y_j),\) for all \(j = \overline{1, n},\)

where \(\lambda_i \geq 0\) such that \(\sum_{i=1}^{n} \lambda_i = 1,\) then we obtain that

\[(\sum_{i=1}^{n} \lambda_i y_i^*)(f(\sum_{i=1}^{n} \lambda_i y_i) - y_j) - y_j^*(f(\sum_{i=1}^{n} \lambda_i y_i) - \sum_{i=1}^{n} \lambda_i y_i) < 0,\]

for all \(j = \overline{1, n}.\) Consequently, we obtain

\[
(\sum_{i=1}^{n} \lambda_i y_i^*)(f(\sum_{i=1}^{n} \lambda_i y_i) - \sum_{j=1}^{n} \lambda_j y_j) - (\sum_{j=1}^{n} \lambda_j y_j^*)(f(\sum_{i=1}^{n} \lambda_i y_i) - \sum_{i=1}^{n} \lambda_i y_i) < 0,
\]

i.e. \((y^*, y) \notin F(y^*, y),\) where \(y^* = \sum_{j=1}^{n} \lambda_j y_j^*, y = \sum_{i=1}^{n} \lambda_i y_i,\) which contradicts (2.9).

\[\square\]

Theorem 7. Let \(A\) be a nonempty weakly compact convex set in \(X\) and let \(f : A \to X\) be a mapping having the following continuity property:

\[(2.10)\quad (y^*, y) \to y^*(f(y)) \text{ is } w^* \times w\text{-continuous on } \mathcal{F}_{X^*}(0; 1) \times A.
\]

Then there exists at least one element \(y \in P_A(f).\)
Proof. Let us denote $K = \overline{S}_{X^*}(0; 1) \times A$ in $X^* \times X$ endowed with $w^* \times w$ topology. Obviously, $K$ is a convex $w^* \times w$ compact and $F$ is a KKM-mapping.

Moreover, $F$ is the compact valued. Indeed, $F(x, x^*)$ is a closed set in the compact $K$ for every $(x^*, x) \in \overline{S}_{X^*}(0; 1) \times A$, because if we consider two nets $(y_i)_{i \in I}, (y'_i)_{i \in I}$ such that $y_i \xrightarrow{w} y, y'_i \xrightarrow{w^*} y^*$ and $y_i^*(f(y_i) - x) - x^*(f(y_i) - y'_i) \geq 0$ for all $i \in I$, then we have $(y_i^* - x^*)(f(y_i)) \geq y_i^*(x) - x^*(y_i), i \in I$. Taking into account the hypothesis of continuity we obtain

$$
(y^* - x^*)(f(y)) \geq \lim(y_i^* - x^*)(f(y_i)) \\
\geq \lim(y_i^*(x) - x^*(y_i)) \\
= y^*(x) - x^*(y),
$$

i.e., $(y^*, y) \in F(x^*, x)$. Therefore, according to Theorem 1 there exists at least one element $(y^*, y) \in F(x^*, x)$ for all $(x^*, x) \in \overline{S}_{X^*}(0; 1) \times A$. Hence, by Theorem 3 we obtain that $y \in PA(f)$, as claimed. \medskip

Remark 8. If the function $f$ is strongly continuous on $A$ then the continuity property (2.10) is fulfilled, and so we obtain the result of Kapoor [7]. Indeed, if $f$ is strongly continuous, i.e. $f(y_i) \xrightarrow{\|\|} f(y)$, whenever $(y_i)_{i \in I} \xrightarrow{w} y$ in $A$, then for a given net $(y_i^*)_{i \in I} \xrightarrow{w^*} y^*$ in $\overline{S}_{X^*}(0; 1)$ we have

$$
|y_i^*(f(y_i)) - y^*(f(y))| \leq |(y_i^* - y^*)(f(y))| + \|f(y_i) - f(y)\|,
$$

and so, the condition (2.10) is fulfilled.

In the special case of real Hilbert spaces the inequality (1.6) becomes

$$
(2.11) \quad \langle f(y), x - y \rangle \leq \frac{1}{2}(\|x\|^2 - \|y\|^2), \text{ for all } x \in A.
$$

Also, by Theorem 2 it follows that an element $y \in A$ is a solution of variational inequality (2.11) if and only if there exists an element $z \in X$ such that

$$
(2.12) \quad \|z\| = \|f(y) - y\|,
$$

$$
(2.13) \quad \langle z, x - f(y) \rangle \geq \|y - f(y)\|^2, \text{ for all } x \in A.
$$

Thus, if $A \perp f(A)$ then the solutions of Ky Fan best approximation problem are just the minimum elements of the set $A$.

REFERENCES


Received by the editors: June 4, 2008.