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ON A DUAL CHARACTERIZATION IN BEST APPROXIMATION PROBLEM

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Abstract. We establish a dual characterization of solutions of Ky Fan best approximation problem and as consequence we obtain an existence criterium under conditions formulated for the weak topology.

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1. INTRODUCTION AND PRELIMINARIES

Let X be a linear normed space and X^* its dual. Given a nonvoid set $A \subset X$ and a function $f : A \to X$, the associated *best approximation problem of Ky Fan type* is to find $y \in A$ such that

$$\inf_{x \in A} \|f(y) - x\| = \|f(y) - y\|. \tag{1.1}$$

Obviously, if f is a constant function we obtain the well known best *approximation problem*. Also, if the range of a function f is contained in A the solutions of Ky Fan best approximation problem are the *fixed points of* f. Generally, the set of all solutions of Ky Fan best approximation problem coincides with the set of all fixed points of the multivalued mapping $P_A \circ f$, where P_A is the projection operator on A. On the other hand, any fixed point of y is a solution of Ky Fan approximation problem, but it is possible to exist other solutions $y \in A$ such that $f(y) \notin A$ which, obviously are not fixed points of f. We denote

$$P_A(f) = \{y \in A; \|y - f(y)\| = d(f(y); A)\}, \tag{1.2}$$

where

$$d(u; A) = \inf_{x \in A} \|x - u\|. \tag{1.3}$$

Therefore the best approximation problem (1.1) can be regarded as a unification of this two problems: the best approximation problem and the fixed point

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problem of a given function. The problem (1.1) is equivalent to the following variational inequality:

$$(1.4) \qquad \|f(y) - y\| \leq \|f(y) - x\|, \text{ for all } x \in A.$$

In 1969 Ky Fan [4] was established an important criterium under hypotheses that the set A is nonvoid compact convex set and the function f is continuous on A. This result generalizes the well known fixed point theorem of Browder. Many authors (see, for example, Kapoor [7], Ky Fan [5], Lin [10], [11], Lin and Yen [12], Ding and Tan [2], Sehgal and Singh [16], Roux and Singh [15], Singh and Watson [17], Reich [14]) have obtained other interesting extensions of Ky Fan's result. Also, several extensions and applications concerning Ky Fan best approximation problem are investigated by Singh, Watson and Srivastava in their monograph [18] (see also the references cited therein).

If we define the multivalued mapping $G : A \rightrightarrows A$ by

$$(1.5) \qquad G(x) = \{y \in A; \|f(y)y\| \leq \|f(y) - x\|\},$$

then the solutions of Ky Fan best approximation problem are the elements of the intersection $\bigcap\limits_{x \in A} G(x)$.

Since G is a *Knaster-Kuratowski-Mazurkiewicz mapping* (KKM-mapping) we can obtain optimality criteria using special results for this mappings which assures that the above intersection is nonvoid. We recall that a multivalued mapping $F : K \rightrightarrows E$, where K is a nonvoid subset of a separated topological vector space E, is called a KKM-*mapping* if

$$co\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^{n} F(x_i),$$

for any finite subset $\{x_1, x_2, \ldots, x_n\}$ of K ([1], [8], [18]).

An important result concerning KKM-mappings was obtained by Ky Fan.

Theorem 1. ([3]) *Let K be a nonvoid subset in a separated topological space E and let $F : K \rightrightarrows E$ be a closed valued KKM-mapping. If $F(x_0)$ is compact for at least one $x_0 \in E$, then*

$$(1.6) \qquad \bigcap_{x \in K} F(x) \neq \emptyset.$$

Consequently, if in the linear normed space X we consider KKM-mapping G defined by (1.5) we can obtain Ky Fan's result above mentioned. In fact, the following two conditions are sufficient:

(i) f is continuous on every compact subset of A;
(ii) there exists $x_0 \in A$ such that the set $\{y \in A; \|f(y) - y\| \leq \|f(y) - x_0\|$ is compact.

Now, if X is endowed with weak topology we get a result obtained by Kapoor [7]. The Kapoor's hypotheses are:

(i′) A is a nonempty weakly compact convex set;

(ii′) f is a strongly continuous mapping.

2. DUAL CHARACTERIZATIONS

Firstly, we recall the well known dual characterization of best approximation elements with respect to a nonvoid convex set established by Garkavi [6].

THEOREM 2. ([6]) *Let A be a nonvoid convex set in a linear normed space X. Then, an element $\overline{x} \in A$ is a best approximation of an element $u \in X$ from elements of A if and only if there exists $x_0^* \in X^*$ such that*

$$\|x_0^*\| = \|\overline{x} - u\|, \tag{2.1}$$

$$x_0^*(x - u) \geq \|\overline{x} - u\|^2, \quad \textit{for all}\, x \in A. \tag{2.2}$$

See also [1], [9], [13].

Because an element $y \in A$ is a solution of Ky Fan best approximation problem if and only if $y \in P_A(f(y))$, by Theorem 2 we obtain the following dual characterization.

THEOREM 3. *Let A be a nonvoid convex set in X. Then an element $y \in A$ is a solution of Ky Fan best approximation problem if and only if there exists $y^* \in X^*$ such that*

$$\|y^*\| \leq 1, \tag{2.3}$$

$$y^*(x - f(y)) \geq \|y - f(y)\|, \quad \textit{for all}\ x \in A. \tag{2.4}$$

REMARK 4. It is obvious that (2.3), (2.4) are fulfilled whenever y is a fixed point of f taking $y^* = 0$. If y is not a fixed point of f, i.e. $\|y - f(y)\| > 0$, then we have necessarily $\|y^*\| = 1$. □

REMARK 5. If A is a closed linear subspace of X and y is not a fixed point of f, then $\|y^*\| = 1$ and (2.4) is equivalent with the following two conditions

$$y^*(x) = 0 \;\text{ for all}\; x \in A; \tag{2.4′}$$

$$y^*(f(y)) = \|y - f(y)\|. \tag{2.4″}$$

□

The conditions (2.3), (2.4) can be equivalently rewrite in a special minimax form or in a variational form.

THEOREM 6. *Let A be a nonvoid convex set in X. Then $y \in P_A(f)$ if and only if there exists $y^* \in X^*$ such that the pair $(y^*, y) \in \overline{S}_{X^*}(0; 1) \times A$ is a saddle point of the minimax equality*

$$\max_{\|x^*\|\leq 1} \min_{x\in A} x^*(f(y) - x) = \min_{x\in A} \max_{\|x^*\|\leq 1} x^*(f(y) - x), \tag{2.5}$$

or equivalently, the following variational inequality is fulfilled

$$y^*(f(y)-x) - x^*(f(y)-y) \geq 0, \quad \textit{for all } (x^*,x) \in \overline{S}_{X^*}(0;1) \times A. \tag{2.6}$$

Proof. The minimax equality (2.5) says that (y^*, y) is a saddle point of the function $\emptyset_y(x^*, x) = x^*(f(y) - x)$ and so we have

$$\emptyset_y(x^*, y) \leq \emptyset_y(y^*, y) \leq \emptyset_y(y^*, x), \quad \text{for all } (x^*,x) \in \overline{S}_{X^*}(0;1) \times A, \tag{2.7}$$

which is just the variational inequality (2.6). Therefore (2.5) and (2.6) are equivalent. On the other hand, from (2.6) it follows

$$y^*(f(y)-x) \geq \sup_{x^* \in \overline{S}_{X^*}(0;1)} x^*(f(y)-y) = \|f(y)-y\|,$$

which proves that $-y^*$ has the properties (2.3) and (2.4). Conversely, it is easily to prove that (2.4) implies (2.6) for $-y^*$.

Now, we denote

$$F(x^*,x) = \{(y^*,y) \in \overline{S}_{X^*}(0;1) \times A; \; y^*(f(y)-x) - x^*(f(y)-y) \geq 0\}, \tag{2.8}$$

for every $(x^*,x) \in \overline{S}_{X^*}(0;1) \times A$. Obviously,

$$(x^*,x) \in F(x^*,x) \text{ for every } (x^*,x) \in \overline{S}_X(0;1) \times A. \tag{2.9}$$

Moreover, if A is a convex set, then the multivalued function $F : \overline{S}_{X^*}(0;1) \times A \rightrightarrows \overline{S}_{X^*}(0;1) \times A$ is a KKM-mapping. Indeed, if $(y_i^*, y_i) \in \overline{S}_{X^*}(0;1) \times A$, $i = \overline{1,n}$, and we suppose that $(\sum\limits_{i=1}^{n} \lambda_i y_i^*, \sum\limits_{i=1}^{n} \lambda_i y_i) \notin F(y_j^*, y_j)$, for all $j = \overline{1,n}$, where $\lambda_i \geq 0$ such that $\sum\limits_{i=1}^{n} \lambda_i = 1$, then we obtain that

$$(\sum_{i=1}^{n} \lambda_i y_i^*)(f(\sum_{i=1}^{n} \lambda_i y_i) - y_j) - y_j^*(f(\sum_{i=1}^{n} \lambda_i y_i) - \sum_{i=1}^{n} \lambda_i y_i) < 0,$$

for all $j = \overline{1,n}$. Consequently, we obtain

$$(\sum_{i=1}^{n} \lambda_i y_i^*)(f(\sum_{i=1}^{n} \lambda_i y_i) - \sum_{j=1}^{n} \lambda_j y_j) - (\sum_{j=1}^{n} \lambda_j y_j^*)(f(\sum_{i=1}^{n} \lambda_i y_i) - \sum_{i=1}^{n} \lambda_i y_i) < 0,$$

i.e. $(y^*, y) \notin F(y^*, y)$, where $y^* = \sum_{j=1}^{n} \lambda_j y_j^*$, $y = \sum_{i=1}^{n} \lambda_i y_i$, which contradicts (2.9). □

Theorem 7. *Let A be a nonempty weakly compact convex set in X and let $f : A \to X$ be a mapping having the following continuity property:*

$$(y^*, y) \to y^*(f(y)) \textit{ is } w^* \times w\textit{-continuous on } \overline{S}_{X^*}(0;1) \times A. \tag{2.10}$$

Then there exists at least one element $y \in P_A(f)$.

Proof. Let us denote $K = \overline{S}_{X^*}(0;1) \times A$ in $X^* \times X$ endowed with $w^* \times w$ topology. Obviously, K is a convex $w^* \times w$ compact and F is a KKM-mapping.

Moreover, F is the compact valued. Indeed, $F(x, x^*)$ is a closed set in the compact K for every $(x^*, x) \in \overline{S}_{X^*}(0;1) \times A$, because if we consider two nets $(y_i)_{i\in I}, (y_i^*)_{i\in I}$ such that $y_i \xrightarrow{w} y, y_i^* \xrightarrow{w^*} y^*$ and $y_i^*(f(y_i)-x)-x^*(f(y_i)-y_i) \geq 0$ for all $i \in I$, then we have $(y_i^* - x^*)(f(y_i)) \geq y_i^*(x) - x^*(y_i), i \in I$. Taking into account the hypothesis of continuity we obtain

$$\begin{aligned}(y^* - x^*)(f(y)) &\geq \lim(y_i^* - x^*)(f(y_i)) \\ &\geq \lim(y_i^*(x) - x^*(y_i)) \\ &= y^*(x) - x^*(y),\end{aligned}$$

i.e., $(y^*, y) \in F(x^*, x)$. Therefore, according to Theorem 1 there exists at least one element $(y^*, y) \in F(x^*, x)$ for all $(x^*, x) \in \overline{S}_{X^*}(0;1) \times A$. Hence, by Theorem 3 we obtain that $y \in P_A(f)$, as claimed. □

REMARK 8. If the function f is strongly continuous on A then the continuity property (2.10) is fulfilled, and so we obtain the result of Kapoor [7]. Indeed, if f is strongly continuous, i.e. $f(y_i) \xrightarrow{\|\cdot\|} f(y)$, whenever $(y_i)_{i\in I} \xrightarrow{w} y$ in A, then for a given net $(y_i^*)_{i\in I} \xrightarrow{w^*} y^*$ in $\overline{S}_{X^*}(0;1)$ we have

$$|y_i^*(f(y_i)) - y^*(f(y))| \leq |(y_i^* - y^*)(f(y))| + \|f(y_i) - f(y)\|,$$

and so, the condition (2.10) is fulfilled.

In the special case of real Hilbert spaces the inequality (1.6) becomes

$$\langle f(y), x - y\rangle \leq \frac{1}{2}(\|x\|^2 - \|y\|^2), \text{ for all } x \in A. \tag{2.11}$$

□

Also, by Theorem 2 it follows that an element $y \in A$ is a solution of variational inequality (2.11) if and only if there exists an element $z \in X$ such that

$$\|z\| = \|f(y) - y\|, \tag{2.12}$$

$$\langle z, x - f(y)\rangle \geq \|y - f(y)\|^2, \text{ for all } x \in A. \tag{2.13}$$

Thus, if $A \perp f(A)$ then the solutions of Ky Fan best approximation problem are just the minimum elements of the set A.

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