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EFFECT OF THE MAGNETIC FIELD AND HEAT GENERATION ON THE FREE CONVECTION FLOW IN A TALL CAVITY FILLED WITH A POROUS MEDIUM[‡]

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Abstract. A analytical study of the steady magnetohydrodynamics (MFD) free convection in an tall cavity filled with a fluid-saturated porous medium and with internal heat generation has been performed. It is considered that the Darcy law model is used. It is assume that a uniform magnetic field normal to the walls of the cavity is externally imposed. The values of the governing parameters are as follows: Hartmann number $\mathrm{Ha} = 0, 10$ and 50, Rayleigh number $\mathrm{Ra} = 10^3$, and the aspect ratio $a = 0.01$. The velocity and temperature profile are determined. These profiles are presented graphically at the center line of the cavity. It is found that the analytical solution is in very good agreement with the numerical solution which is obtained by solving partial differential equations using a finite-difference method.

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1. INTRODUCTION

In many engineering applications, enclosures with fluid-saturated porous media are encountered in grain storage, high-performance insulation of buildings, geophysical problems, biological tissues, in the thermal insulation of buildings, in geothermal energy convection, in petroleum reservoirs, etc. These applications can be found in the recent books and review papers, such as, Nield and Bejan [1], Ingham and Pop [2], Vafai [3], Bejan et al. [4], etc. Buoyancy induced flow and heat transfer in enclosures of different geometries and filled with a fluid-saturated porous medium has been widely studied in literature using different numerical technique (Manole and Lage, [5]; Goyeau *et al.*, [6]; Saeid and Pop, [7]; Varol *et al.*, [8]). Recent demands in heat transfer engineering have requested researchers to develop various new types of heat transfer

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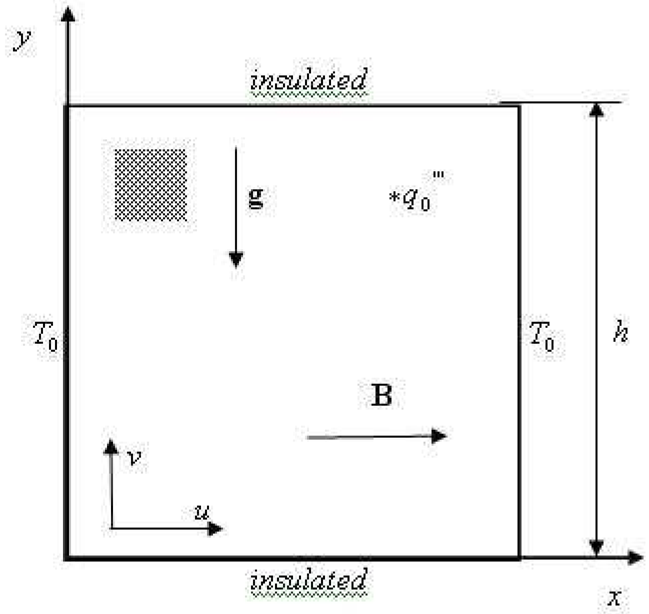

Fig. 1. Physical model and coordinates system.

equipments with superior performance, especially compact and light-weight ones. Increasing the need for small-size units, focuses have been cast on the effects of the interaction between the fluid flow in a cavity and a magnetic field which usually affects the flow development. This type of treatment is completely justified in many practical problems of the magnetohydrodynamics (MHD) generators. There have recently been published a number of research papers which investigate MHD flows of conducting fluids in cavities.

This paper presents an analytical solution of the effect of a magnetic field on the steady free convection in a tall cavity filled with a porous medium saturated with an electrically conducting fluid using a semi-analytical method proposed by Joshi et al., [9]. This analytical solution is compared with the numerical solution obtained in a previous paper by Groşan *et al.*, [10]. This type of problem arises in geophysics when a fluid saturates the earth's mantle in the presence of a geomagnetic field.

2. BASIC EQUATION

The conservation equations for mass, momentum under the Darcy approximation, energy and electric transfer are give by

$$\nabla.\mathbf{V} = 0, \tag{1}$$

$$\mathbf{V} = \frac{K}{\mu}\left(-\nabla p + \rho\mathbf{g} + \mathbf{I}\times\mathbf{B}\right), \tag{2}$$

$$\left(\mathbf{V}.\nabla\right)T = \alpha_m \nabla^2 T + \frac{q_0'''}{\rho_0 c_p}, \tag{3}$$

$$\nabla.\mathbf{I} = 0, \tag{4}$$

$$\mathbf{I} = \sigma\left(-\nabla\phi + \mathbf{V}\times\mathbf{B}\right), \tag{5}$$

$$\rho = \rho_0[1 - \beta(T - T_0)], \tag{6}$$

where $\mathbf{V}$ is the fluid velocity vector, T is the fluid temperature, p is the pressure, $\mathbf{B}$ is the external magnetic field, $\mathbf{I}$ is the electric current, ϕ is the electric potential, $\mathbf{g}$ is the gravitational acceleration vector, K is the permeability of the porous medium, α_m is the effective thermal diffusivity, ρ is the density, μ is the dynamic viscosity, β is the coefficient of thermal expansion, c_p is the specific heat at constant pressure, σ is the electrical conductivity, ρ_0 is the reference density and $-\nabla\phi$ is the associated electric field. As discussed by Garandet *et al.* [11], Eqs. (4) and (5) reduce to $\nabla^2\phi = 0$. The unique solution is $\nabla\phi = 0$ since there is always an electrically insulating boundary around the enclosure. Thus, it follows that the electric field vanishes everywhere (see, Alchaar *et al.*, [12]).

Eliminating the pressure term in Eq. (2) in the usual way, the governing equations (1) to (3) can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{7}$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = -\frac{gK\beta}{\upsilon}\frac{\partial T}{\partial x} + \frac{\sigma K B_0^2}{\mu}\left(-\frac{\partial u}{\partial y}\sin^2\varphi + 2\frac{\partial v}{\partial y}\sin\varphi\cos\varphi + \frac{\partial v}{\partial x}\cos^2\varphi\right), \tag{8}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_m\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \frac{q_0'''}{\rho c_p}, \tag{9}$$

subject to the boundary conditions

$$\begin{aligned} &u = 0, T = T_0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = l, \quad 0 \le y \le h, \\ &v = 0, \frac{\partial T}{\partial y} = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = h, \quad 0 \le x \le l. \end{aligned} \tag{10}$$

Using non-dimensional variables

$$X = \frac{x}{l},\ Y = \frac{y}{h},\ U = \frac{h}{\alpha_m}u,\ V = \frac{l}{\alpha_m}v,\ \theta = \frac{(T-T_0)}{(q'''l^2/k)}, \tag{11}$$

Eqs. (8) and (9) can be written as

$$\frac{\partial^2\psi}{\partial x^2}+a^2\frac{\partial^2\psi}{\partial y^2}=-\mathrm{Ra}\frac{\partial\theta}{\partial x}-\mathrm{Ha}^2\left(a^2\frac{\partial^2\psi}{\partial y^2}\sin^2\varphi+2a\frac{\partial^2\psi}{\partial x\partial y}\sin\varphi\cos\varphi+\frac{\partial^2\psi}{\partial x^2}\cos^2\varphi\right), \tag{12}$$

$$\frac{\partial^2\theta}{\partial x^2}+a^2\frac{\partial^2\theta}{\partial y^2}+1=a\left(\frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}\right), \tag{13}$$

where the stream function ψ is defined as $U=\partial\psi/\partial Y$ and $V=-\partial\psi/\partial X$.

We consider a long cavity with $h\gg l$, with heat generation. In this case, it can be assumed that in the central region, the flow is only in vertical direction, that is, $u=0, v=v(x)$ and $\psi=\psi(x)$. Under this assumption, Eqs. (12) and (13) reduce to

$$A\frac{\mathrm{d}^2\psi}{\alpha x^2}=-\mathrm{Ra}\frac{\partial\theta}{\partial x}, \tag{14}$$

$$\frac{\partial^2\theta}{\partial x^2}+a^2\frac{\partial^2\theta}{\partial y^2}+1=-a\frac{\mathrm{d}\psi}{\mathrm{d}x}\frac{\partial\psi}{\partial y}, \tag{15}$$

where

$$A=1+\mathrm{Ha}^2\cos^2\varphi.$$

The left hand side of Eq. (14) is function of x only. Therefore, the right hand side term Ra $\partial\theta/\partial x$ must be a functional x only. Thus,

$$\frac{\partial\theta}{\partial x}=h(x)$$

and

$$\theta=\int h(x)\mathrm{d}x+g(y)=f(x)+g(y). \tag{16}$$

Substituting (16) into (14) and (15), we get

$$A\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}=-\mathrm{Ra}f'(x), \tag{17}$$

$$-ag'(y)\frac{\mathrm{d}\psi}{\mathrm{d}x}=f''(x)+a^2g''(y)+1. \tag{18}$$

Differentiating these equations with respect to x, we obtain

$$A\frac{\mathrm{d}^3\psi}{\mathrm{d}x^3}=-\mathrm{Ra}f'', \tag{19}$$

$$-ag'(y)\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}=f'''. \tag{20}$$

Since in this problem $v = -\frac{\mathrm{d}\psi}{\mathrm{d}x}$ is not constant on the vertical walls of the cavity it implies that $\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} \neq 0$ for all values of x. Hence, from Eq. (20), one can write

$$-g'(y) = \frac{f'''(x)}{a\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}}.$$

The left hand side of this equation is a functional of y only, while the right hand side is a function of x only. Therefore, we must have

$$g'(y) = \gamma, \tag{21}$$

where γ is a constant. It can be conducted from Eqs. (16) and (21) that the temperature gradient in the vertical direction is constant. Using (21) and (18), Eqs. (19) reduced to

$$A\frac{\mathrm{d}^3\psi}{\mathrm{d}x^3} - a\mathrm{Ra}\gamma\frac{\mathrm{d}\psi}{\mathrm{d}x} - \mathrm{Ra} = 0. \tag{22}$$

We define

$$\psi_1 = \frac{\psi}{\mathrm{Ra}},\ \gamma_1 = \gamma\mathrm{Ra},\ v_1 = -\frac{v}{\mathrm{Ra}} \tag{23}$$

and Eq. (22) become

$$A\frac{\mathrm{d}^2 v_1}{\mathrm{d}x^2} - a\gamma_1 v_1 - 1 = 0, \tag{24}$$

the general solution of this equation is of the from

$$v_1(x) = c_1\mathrm{e}^{mx} + c_2\mathrm{e}^{-mx} - \frac{1}{Am^2}, \tag{25}$$

where c_1 and c_2 are constants of integration and m is given by

$$m = \left(\frac{\gamma_1}{A}\right)^{1/2} = \left(\frac{\gamma a\mathrm{Ra}}{A}\right)^{1/2}.$$

It should be noted that in the present problem $v_2(0) \neq 0$ and $v_1(1) \neq 0$ so that we have to use other conditions to determine the constants c_1 and c_2. A condition is the conservation of mass

$$\int_0^1 v_1 \mathrm{d}x = 0 \tag{26}$$

which gives

$$\psi_1(0) = \psi_1(1) \neq 0 \tag{27}$$

if (23) is used and the fact that $\psi = \psi(x)$ is not valid near the boundary of the cavity. Using (27) it gives

$$c_1\left(\mathrm{e}^{m}-1\right)-c_2\left(\mathrm{e}^{-m}-1\right)=\tfrac{1}{Am}. \tag{28}$$

Using the symmetry of the velocity profile about the centerline, some of the constants, can be delimited, we have

$$v_1(x)=v_1(1-x)$$

which gives

$$c_1\mathrm{e}^{m}=c_2. \tag{29}$$

Solving the system of Eqs. (28) and (29) we obtain the constant c_1 and c_1:

$$c_1=\tfrac{\mathrm{e}^{-m}}{2mA(1-\mathrm{e}^{-m})},\ c_2=\tfrac{1}{2mA(1-\mathrm{e}^{-m})}.$$

In order to determine θ we use Eqs. (16), (17), (21) and (23). Thus, from these equations, we have

$$A\tfrac{\mathrm{d}^2\psi}{\mathrm{d}x^2}=-\mathrm{Ra}f\,'(x)\quad\text{or}\quad -A\tfrac{\mathrm{d}v_1}{\mathrm{d}x}=f\,'(x) \tag{30}$$

which gives

$$f(x)=Av_1+\delta_1,$$

where δ_1 is a constant. Then, from (21), we get

$$g(y)=\gamma y+\delta_2,$$

where δ_2 is a constant. Therefore, θ given by (16), can be written as

$$\theta(x,y)=-Av_1+\tfrac{Am^2}{a\mathrm{Ra}}y+\delta,$$

where δ is a constant. Substitute v_1 from (25), we get

$$\theta(x,y)=-A(c_1\mathrm{e}^{mx}+c_1\mathrm{e}^{-mx})+\tfrac{Am^2}{a\mathrm{Ra}}y+\tfrac{1}{m^2}+\delta. \tag{31}$$

Since $\psi=\psi(x)$ is not valid near the boundaries of the cavity, the expression (33) for θ cannot satisfy all the boundary conditions. The value of δ can be calculated in terms of m by substituting $\theta(x,y)=0$ at the bottom corner of the cavity $x=0$ and $y=0$ in Eq. (33). This gives

$$\delta=A(c_1+c_2)-\tfrac{1}{m^2}.$$

In this analysis, the value of m, which depend on the vertical temperature gradient γ, remains undetermined. It can be, however, determined from the fact that under natural convection, the maximum temperature attained must be lower than maximum temperature in conduction region, that is

$$\tfrac{\partial^2\theta}{\partial x^2}+\tfrac{\partial^2\theta}{\partial y^2}+1=0, \tag{32}$$

in the central of the cavity the boundary conditions is

$$\theta=0\text{ at }x=0\text{ and }x=1. \tag{33}$$

Solution at this problem is given by

$$\theta = \tfrac{x(1-x)}{2}.$$

We notice that the temperature is maximum at $x = 1/2$ and the maximum value $\theta_{\max} = 1/8$

$$\theta = \tfrac{1}{2}\left[-\left(x-\tfrac{1}{2}\right)^2 + \tfrac{1}{4}\right].$$

Similarly, the temperature cannot be less than the wall temperature, i.e. $\theta_{\min} = 0$. Therefore, substituting $\theta_{\max} = 1/8$ in (33), we get

$$\tfrac{1}{8} = -A(c_1 \mathrm{e}^{mx} + c_2 \mathrm{e}^{-mx})_{\max} + \tfrac{Am^2}{a^2 \mathrm{Ra}} - A(c_1 + c_2).$$

This expression gives the value of m as a function of $a^2\mathrm{Ra}$ and A.

$$\tfrac{1}{8} = \tfrac{1}{2mA(1-\mathrm{e}^{-m})}\left[-2\mathrm{e}^{-m/2} + \mathrm{e}^{-m}\right] + \tfrac{Am^2}{a^2\mathrm{Ra}}$$

3. RESULTS AND DISCUSSIONS

The obtained analytical solutions in the centerline on the cavity is shown in Figs. 2 to 5 for the temperature and velocity profiles. The values of the parameters considered are the Rayleigh number $\mathrm{Ra} = 10^3$, Hartmann number $\mathrm{Ha} = 0, 10, 50$ and aspect ration $a = 0.01$. These solution is shown by circle. In additions the numerical results are included in these figures, which are shown by full line. It is seen that there is a very good agreement between the analytical and numerical solutions obtained. Therefore, we are confident that the present analytical solution is accurate. We notice that the effect of the Hartmann number is not significant for the temperature profile, while it's effect is more significative for the velocity profiles. This is because the magnetic term thus not directly enter in the energy equation. Figs. 3 to 5 shows that the vertical component of the velocity is positive in the center of the cavity and negative close to the vertical walls. This is because of the conservation of mass, Eq. (26).

4. CONCLUSION

In this paper analytical expression are obtained to describe the free convection in a tall cavity filled with the porous medium and with an applied magnetic field and volumetric heat generation. Because of the volumetric generation, the central region is of high temperature, and therefore the flow is driven upwards due to buoyancy, in the central part of the cavity and driven downwards in the relatively colder boundary regions.

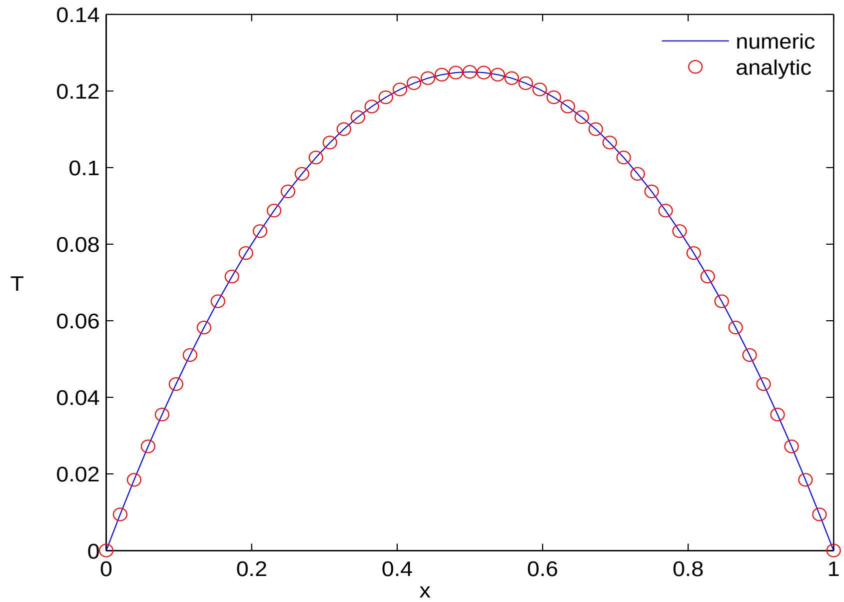

Fig. 2. Temperature profiles for Ra = 1000 and Ha = 0 at the centerline of the cavity ($y = 1/2$).

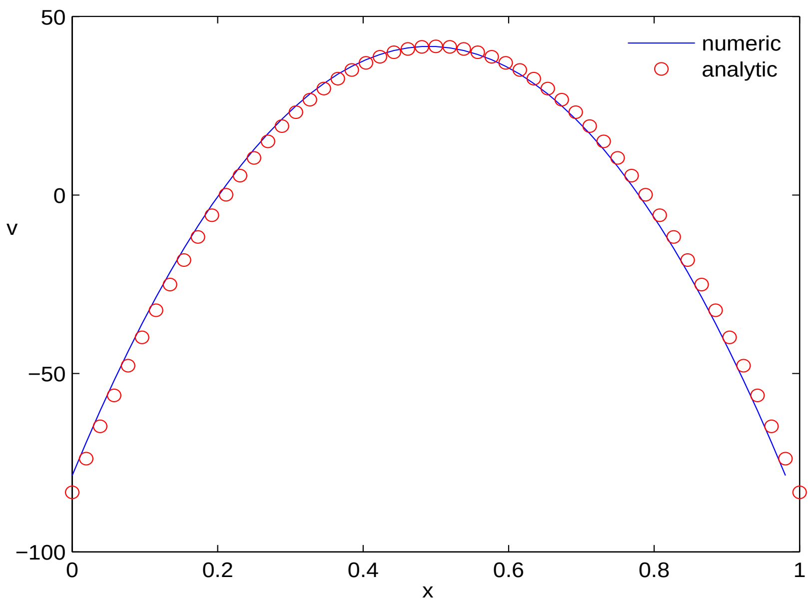

Fig. 3. Velocity profiles for Ra = 1000 and Ha = 0 at the centerline of the cavity ($y = 1/2$).

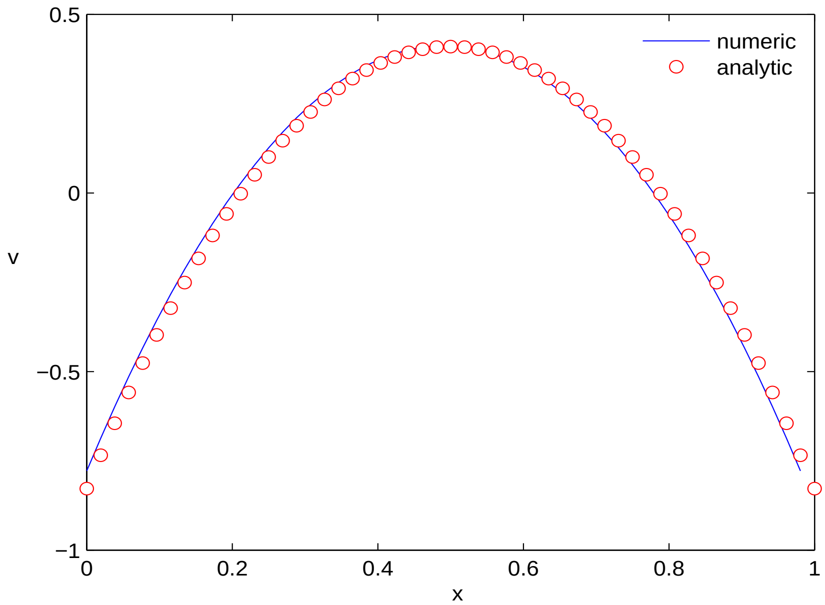

Fig. 4. Velocity profiles for Ra = 1000 and Ha = 10 at the centerline of the cavity ($y = 1/2$).

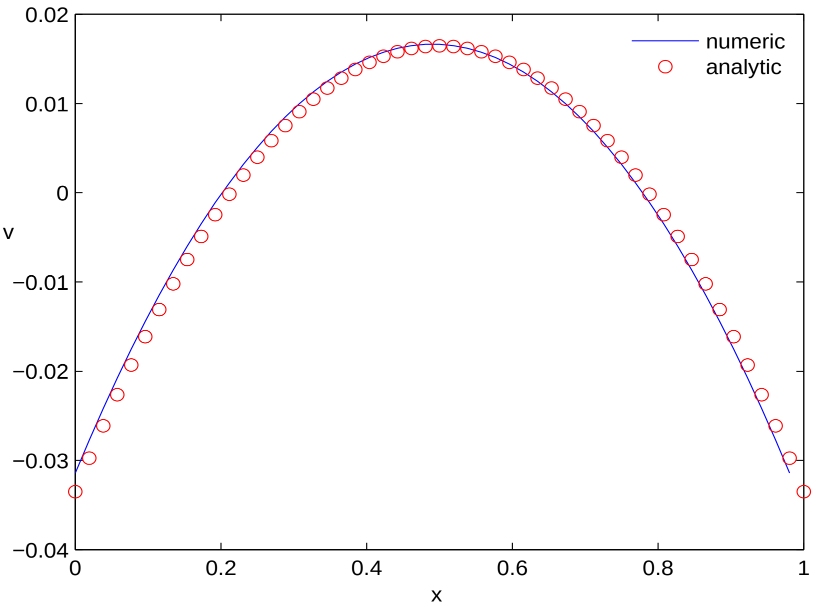

Fig. 5. Velocity profiles for Ra = 1000 and Ha = 50 at the centerline of the cavity ($y = 1/2$).

REFERENCES

[1] Nield, D.A. and Bejan, A., *Convection in Porous Media*, 3rd edition, Springer, New York, 2006.

[2] Ingham, D.B. and Pop, I. (eds.), *Transport Phenomena in Porous Media III*, Elsevier, Oxford, 2005.

[3] Vafai, K., *Hand book of Porous Media*, 2nd edition, Taylor and Francis, New York, 2005.

[4] Bejan, A., Dincer, I., Lorente, S., Miguel, A.F. and Reis, A.H., *Porous and Complex Flow Structures in Modern Technologies,* Springer, New York, 2004.

[5] Manole, D.M. and Lage, J.L., *Numerical benchmark results for natural convection in a porous medium cavity,* in: Heat and Mass Transfer in Porous Media, ASME Conference, HTD-vol. **216**, pp. 55–60, 1992.

[6] Goyeau, B., Songbe, J.P. and Gobin, D., *Numerical study of double-diffusive natural convection in a porous cavity using the Darcy-Brinkman formulation,* Int. J. Heat Mass Transfer, **39**, pp. 1363–1378, 1996.

[7] Saeid, N.H. and Pop, I., *Natural convection from a discrete heater in a square cavity filled with a porous medium,* J. Porous Media, **8**, pp. 55–63, 2005.

[8] Varol, Y., Oztop, H.F. and Pop, I., *Numerical analysis of natural convection for a porous rectangular enclosure with sinusoidally varying temperature profile on the bottom wall*, International Communications in Heat and Mass Transfer, **35**, pp. 56–64, 2008.

[9] Joshi, M.V., Gaitonde, U.N. and Mitra, S.K., Analytical study of natural convetion in a cavity with volumetric heat generation, Preceedings of HTFED04, 11-15 July, 2004.

[10] Groşan, T., Revnic, C., Ingham, D.B., and Pop, I., *Magnetic field and internal heat generation effects on the free convection in a rectangular cavity filled with a porous medium*, in progress.

[11] Garandet, J.P., Albussoiere, T. and Moreau, R., *Buoyancy driven convection in a rectangular enclosure with a transverse magnetic field*, International journal of heat and mass transfer, **35**, pp. 741–748, 1992.

[12] Alchaar, S., Vasseur, P. and Bilgen, E., *Natural convection heat transfer in a rectangular enclosure with a transverse magnetic field*, J. Heat Transfer, **117**, pp. 668–673, 1995.

[13] Varol, Y., Oztop, H.F. and Pop, I., *Influence of inclination angle on buoyancy-driven convection in triangular enclosure filled with a fluid-saturated porous medium*, Heat and Mass Transfer, **44**, pp. 617–624, 2008.

[14] Varol, Y., Oztop, H.F. and Pop, I., *Natural convection flow in porous enclosures with heated and cooled on adjacent walls and divided by a triangular massive partition*, International Communications in Heat and Mass Transfer, **35**, pp. 476–491, 2008.

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