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INPUT-OUTPUT CONDITIONS FOR EXPONENTIAL TRICHOTOMY OF DYNAMICAL SYSTEMS[†]

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Abstract. The aim of the paper is to provide new properties concerning the property of uniform exponential trichotomy on the real line. We obtain a characterization for uniform exponential trichotomy in terms of the solvability of an associated integral equation.

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1. INTRODUCTION

Exponential trichotomy is a fundamental concept in the asymptotic theory of evolution equations, which proceed from the central manifold theorem. We say that an evolutionary system has exponential trichotomy if the main space is decomposed at every moment intro three invariant subspaces: the stable, unstable and neutral subspace, such that the solution on the first subspace is exponentially stable, on the second subspace is exponentially expansive and on the third subspace is bounded. The concept of exponential trichotomy for differential equations has the origin in the remarkable works of Elaydi and Hájek (see [1], [2]) and of Sacker and Sell (see [4]). Elaydi and Hájek studied the exponential trichotomy of differential systems and of nonlinear differential systems, respectively, proving a number of interesting properties in these cases (see [1], [2]). The case of linear differential systems described by linear skew-product flows on $X \times \Theta$ was considered by Sacker and Sell in [4], where the authors gave characterizations of the exponential trichotomy for the case when X is a finite dimensional Banach space. The case of nonlinear difference equations was considered by Elaydi and Janglajew in [3], where the authors introduced new concepts of exponential dichotomy and exponential trichotomy, using two different methods. In their first approach the authors used the tracking method, while in their second approach they introduced a

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discrete analogue of dichotomy and trichotomy in variation. The case of evolution families on the real line was considered for the first time in [8], where we gave necessary and sufficient conditions for the existence of exponential trichotomy. The concept considered in [8] does not assume that the families of projections are uniformly bounded. Thus, we obtained that the p-admissibility of the pair $(C_b(\mathbb{R}, X), C_c(\mathbb{R}, X))$ implies the decomposition of the space at every moment into stable, unstable and neutral subspace according to Definition 2.2 in [8] (see Theorem 3.8 in [8]). The main problem was that the converse implication was valid only under additional hypotheses (see Theorem 3.10 in [8]). Taking into account that exponential trichotomy generalizes the concept of exponential dichotomy for the case when the neutral subspace contains only the zero vector, and that the family of the dichotomy projections is uniformly bounded (see [6], Lemma 4.1), it is natural to work with families of trichotomy projections which are uniformly bounded.

The aim of this paper is to obtain several important properties concerning exponential trichotomy on the real line. First we consider a concept of exponential trichotomy which is a direct generalization of the concept of exponential dichotomy (see e.g. [6], Definition 3.2). According to our study made in [8] we point out that if an evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ is exponentially trichotomic with respect to three families of projections $\{P_k(t)\}_{t\in\mathbb{R}},\ k \in \{1,2,3\}$, then the range of $P_1(t)$ is the linear subspace of all vectors $x \in X$ with the property that the corresponding orbit tends to zero at infinity, the range of $P_2(t)$ is the subspace of all vectors $x \in X$ with the property that admit bounded negative continuation and the corresponding orbit is bounded, and the range of $P_3(t)$ is the linear subspace of all vectors $x \in X$ with the property that admit a negative continuation which tends to zero at minus infinity. In what follows, we prove that an evolution family is uniformly exponentially trichotomic if and only if there is $p \in (1,\infty)$ such that the pair $(C_b(\mathbb{R}, X), C_c(\mathbb{R}, X))$ is uniformly p-admissible for it.

2. TRICHOTOMY OF EVOLUTION FAMILIES AND THE STRUCTURE THEOREM

The aim of this section is to recall some basic definitions and notations and to establish the general context of our study.

Definition 1. *A family $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ of bounded linear operators on X is called an evolution family if the following properties hold:*

(i) $U(t,t) = I$ *(the identity operator), for all* $t \in \mathbb{R}$*;*
(ii) $U(t,s)U(s,t_0) = U(t,t_0)$*, for all* $t \geq s \geq t_0$*;*
(iii) *there exist* $M \geq 1$ *and* $\omega > 0$ *such that* $||U(t,t_0)|| \leq M e^{\omega(t-t_0)}$*, for all* $t \geq t_0$*;*
(iv) *for every* $x \in X$ *and every* $t_0 \in \mathbb{R}$ *the mapping* $s \mapsto U(s,t_0)x$ *is continuous on* $[t_0,\infty)$ *and the mapping* $t \mapsto U(t_0,t)x$ *is continuous on* $(-\infty,t_0]$.

DEFINITION 2. *An evolution family* $\mathcal{U} = \{U(t,s)\}_{t \geq s}$ *is said to be uniformly exponentially trichotomic if there exist three families of projections* $\{P_k(t)\}_{t\in\mathbb{R}} \subset \mathcal{B}(X)$, $k \in \{1,2,3\}$ *and two constants* $K \geq 1$ *and* $\nu > 0$ *such that:*

(i) $P_k(t)P_j(t) = 0$, *for all* $t \in \mathbb{R}$ *and* $k \neq j$;
(ii) $P_1(t) + P_2(t) + P_3(t) = I$, *for all* $t \in \mathbb{R}$;
(iii) $\sup\limits_{t\in\mathbb{R}} ||P_k(t)|| < \infty$, *for all* $k \in \{1,2,3\}$;
(iv) $U(t,t_0)P_k(t_0) = P_k(t)U(t,t_0)$, *for all* $t \geq t_0$ *and* $k \in \{1,2,3\}$;
(v) $||U(t,t_0)x|| \leq K\mathrm{e}^{-\nu(t-t_0)}||x||$, *for all* $x \in \mathrm{Im}\ P_1(t_0)$ *and all* $t \geq t_0$;
(vi) $\frac{1}{K}\ ||x|| \leq ||U(t,t_0)x|| \leq K||x||$, *for all* $x \in \mathrm{Im}\ P_2(t_0)$ *and all* $t \geq t_0$;
(vii) $||U(t,t_0)x|| \geq \frac{1}{K}\ \mathrm{e}^{\nu(t-t_0)}||x||$, *for all* $x \in \mathrm{Im}\ P_3(t_0)$ *and all* $t \geq t_0$;
(viii) *the restriction* $U_k(t,t_0)_{|} : \mathrm{Im}\ P_k(t_0) \to \mathrm{Im}\ P_k(t)$ *is an isomorphism, for all* $t \geq t_0$ *and all* $k \in \{2,3\}$.

Let $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ be an evolution family on X. For every $t_0 \in \mathbb{R}$, we define

$$X_1(t_0) = \{x \in X : \lim_{t\to\infty} U(t,t_0)x = 0\}.$$

We denote by $\mathcal{F}_{\mathcal{U}}(t_0)$ the set of all functions $\varphi : \mathbb{R}_- \to X$ with the property that

$$\varphi(t) = U(t+t_0, s+t_0)\varphi(s), \quad \forall s \leq t \leq 0.$$

Let $X_2(t_0)$ be the linear subspace of all $x \in X$ with $\sup\limits_{t\geq t_0} ||U(t,t_0)x|| < \infty$ and there is a function $\varphi_x \in \mathcal{F}_{\mathcal{U}}(t_0)$ such that $\varphi_x(0) = x$ and $\sup\limits_{s\leq 0} ||\varphi_x(s)|| < \infty$.

Let $X_3(t_0)$ be the linear subspace of all $x \in X$ with the property that there is a function $\lambda_x \in \mathcal{F}_{\mathcal{U}}(t_0)$ such that $\lambda_x(0) = x$ and $\lim\limits_{s\to-\infty} \lambda_x(s) = 0$.

LEMMA 3. $U(t,t_0)X_k(t_0) \subset X_k(t)$, *for all* $t \geq t_0$ *and* $k \in \{1,2,3\}$.

Proof. See Lemma 2.1 in [8]. □

THEOREM 4. *(The structure theorem) Let* $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ *be an evolution family on* X. *If* $\mathcal{U}$ *is uniformly exponentially trichotomic with respect to the families of projections* $\{P_k(t)\}_{t\in\mathbb{R}}$, $k \in \{1,2,3\}$, *then*

$$\mathrm{Im}\ P_k(t) = X_k(t), \quad \forall t \in \mathbb{R}, \forall k \in \{1,2,3\}.$$

Proof. This follows from Theorem 3.9 in [8] and Definition 2 (iii). □

REMARK 5. From Theorem 4 it follows that if an evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ is uniformly exponentially trichotomic with respect to three families of projections $\{P_k(t)\}_{t\in\mathbb{R}}$, $k \in \{1,2,3\}$, then these families are uniquely determined by the conditions from Definition 2. □

REMARK 6. For every $t_0 \in \mathbb{R}$, $X_1(t_0)$ is called *the stable subspace* at the point t_0, $X_2(t_0)$ is called *the neutral subspace* at the point t_0 and, respectively, $X_3(t_0)$ is called *the unstable subspace* at the point t_0. □

3. ADMISSIBILITY AND UNIFORM EXPONENTIAL TRICHOTOMY

In this section we will obtain necessary and sufficient conditions for the existence of uniform exponential trichotomy in terms of the solvability of an associated integral equation.

Let X be a real or complex Banach space.

NOTATIONS Let $p \in [1, \infty)$ and let $L^p(\mathbb{R}, X)$ be the linear space of all Bochner measurable functions $f : \mathbb{R} \to X$ with $\int_{\mathbb{R}} ||f(s)||^p \, \mathrm{d}s < \infty$, which is a Banach space with respect to the norm

$$||f||_p := \left(\int_{\mathbb{R}} ||f(s)||^p \, \mathrm{d}s\right)^{1/p}.$$

Let $C_b(\mathbb{R}, X)$ be the linear space of all functions $f : \mathbb{R} \to X$, which are continuous and bounded, and let $C_0(\mathbb{R}, X) := \{f \in C_b(\mathbb{R}, X) : \lim\limits_{t\to\pm\infty} f(t) = 0\}$. If $C_{b0} := \{f \in C_b(\mathbb{R}, X) : \lim\limits_{t\to\infty} f(t) = 0\}$ and $C_{0b} := \{f \in C_b(\mathbb{R}, X) : \lim\limits_{t\to-\infty} f(t) = 0\}$, then $C_b(\mathbb{R}, X), C_0(\mathbb{R}, X), C_{b0}(\mathbb{R}, X)$ and $C_{0b}(\mathbb{R}, X)$ are Banach spaces with respect to the norm $|||f||| := \sup\limits_{t\in\mathbb{R}} ||f(t)||$. Let $C_c(\mathbb{R}, X)$ denote the space of all continuous functions $f : \mathbb{R} \to X$ with compact support.

Let $\mathcal{U} = \{U(t, s)\}_{t\geq s}$ be an evolution family on X. We denote by $\mathcal{V}_{\mathcal{U}}$ the linear space of all functions $v : \mathbb{R} \to X$ with the property that for every $t_0 \in \mathbb{R}$, $v(t_0) \in X_1(t_0) \cup X_3(t_0)$. We associate with $\mathcal{U}$ the integral equation given by the variation of constants formula:

$$(E_{\mathcal{U}}) \qquad f(t) = U(t, s)f(s) + \int_s^t U(t, \tau)v(\tau)\mathrm{d}\tau, \quad \forall t \geq s,$$

where $f \in C_b(\mathbb{R}, X)$ and $v \in C_c(\mathbb{R}, X)$.

DEFINITION 7. *Let $p \in (1, \infty)$. The pair $(C_b(\mathbb{R}, X), C_c(\mathbb{R}, X))$ is said to be uniformly p-admissible for $\mathcal{U}$ if there is $L > 0$ such that the following properties hold:*

(i) *for every $v \in C_c(\mathbb{R}, X)$ there exist $f \in C_{b0}(\mathbb{R}, X)$ and $g \in C_{0b}(\mathbb{R}, X)$ such that the pairs (f, v) and (g, v) satisfy the equation $(E_{\mathcal{U}})$;*

(ii) *if $v \in C_c(\mathbb{R}, X)$ and $f \in C_{b0}(\mathbb{R}, X) \cup C_{0b}(\mathbb{R}, X)$ are such that the pair (f, v) satisfies the equation $(E_{\mathcal{U}})$, then $|||f||| \leq L \ \max\{||v||_1, ||v||_p\}$;*

(iii) *if $v \in C_c(\mathbb{R}, X) \cap \mathcal{V}_{\mathcal{U}}$ and $f \in C_0(\mathbb{R}, X)$ are such that the pair (f, v) satisfies the equation $(E_{\mathcal{U}})$, then $|||f||| \leq L \ ||v||_p$.*

The main result of this section is:

THEOREM 8. *Let $p \in (1, \infty)$. An evolution family $\mathcal{U} = \{U(t, s)\}_{t\geq s}$ is uniformly exponentially trichotomic if and only if the pair $(C_b(\mathbb{R}, X), C_c(\mathbb{R}, X))$ is uniformly p-admissible for $\mathcal{U}$.*

Proof. Necessity follows from Theorem 3.10 in [8].

Sufficiency. From Theorem 3.8 in [8] it follows that there are three families of projections $\{P_k(t)\}_{t\in\mathbb{R}}$, $k \in \{1, 2, 3\}$ and two constants $K, \nu > 0$ such that conditions (i), (ii), (iv)–(viii) from Definition 2 are satisfied. Moreover, $\text{Im } P_k(t_0) = X_k(t_0)$, for all $t_0 \in \mathbb{R}$ and $k \in \{1, 2, 3\}$.

It remains to prove that $\sup\limits_{t\in\mathbb{R}} ||P_k(t)|| < \infty$, for every $k \in \{1, 2, 3\}$. Let $L > 0$ be given by Definition 7 and let $M, \omega > 0$ be given by Definition 1.

Step 1 We prove that there is $\gamma > 0$ such that

$$||U(t,t_0)x|| \le \gamma \ ||x||, \quad \forall x \in \text{Im } P_1(t_0) + \text{Im } P_2(t_0), \ \forall t \ge t_0. \tag{3.1}$$

Let $\alpha : \mathbb{R} \to [0, 2]$ be a continuous function with supp $\alpha \subset (0, 1)$ and $\int_0^1 \alpha(\tau)\ \mathrm{d}\tau = 1$. Let $t_0 \in \mathbb{R}$ and let $x \in \text{Im } P_1(t_0) + \text{Im } P_2(t_0)$. We consider the functions

$$v : \mathbb{R} \to X, \quad v(t) = \alpha(t - t_0)U(t,t_0)x$$

$$f : \mathbb{R} \to X, \quad f(t) = \int_{-\infty}^{t} \alpha(\tau - t_0)\ \mathrm{d}\tau\ U(t,t_0)x.$$

We have that $v \in C_c(\mathbb{R}, X)$. Observing that $f(t) = U(t,t_0)x$, for $t \ge t_0 + 1$, from $x \in \text{Im } P_1(t_0) + \text{Im } P_2(t_0)$, we obtain that

$$\sup_{t\ge t_0+1} ||f(t)|| < \infty.$$

Since $f(t) = 0$, for $t \le t_0$ we deduce that $f \in C_{0b}(\mathbb{R}, X)$. In addition, it is easily checked that the pair (f, v) satisfies the equation $(E_{\mathcal{U}})$, so $|||f||| \le L \max\{||v||_1, ||v||_p\}$. This implies that

$$||U(t,t_0)x|| = ||f(t)|| \le 2LM\mathrm{e}^{\omega}||x||, \quad \forall t \ge t_0 + 1. \tag{3.2}$$

Setting $\gamma = \max\{2LM\mathrm{e}^{\omega}, M\mathrm{e}^{\omega}\}$, from (3.2) we conclude that (3.1) holds.

Step 2 We prove that $\sup\limits_{t\in\mathbb{R}} ||P_3(t)|| < \infty$. Indeed, for every $t \in \mathbb{R}$ we set $P(t) := P_1(t) + P_2(t)$.

Let $\gamma > 0$ be given by Step 1 and let $T > 0$ be such that $\mathrm{e}^{\nu T} > K\gamma$.

Let $t_0 \in \mathbb{R}$ and let $x \in X$ with $P(t_0)x \ne 0$ and $P_3(t_0)x \ne 0$. Then

$$\begin{aligned} M\mathrm{e}^{\omega T} \left|\left| \tfrac{P(t_0)x}{||P(t_0)x||} + \tfrac{P_3(t_0)x}{||P_3(t_0)x||} \right|\right| &\ge \left|\left| U(t_0+T,t_0)\ \left(\tfrac{P(t_0)x}{||P(t_0)x||} + \tfrac{P_3(t_0)x}{||P_3(t_0)x||} \right) \right|\right| \\ &\ge \left(\tfrac{1}{K}\ \mathrm{e}^{\nu T} - \gamma \right). \end{aligned}$$

Setting $\alpha = (\mathrm{e}^{\nu T} - \gamma K)/(KM\mathrm{e}^{\omega T})$ we obtain that

$$\left|\left| \tfrac{P(t_0)x}{||P(t_0)x||} + \tfrac{P_3(t_0)x}{||P_3(t_0)x||} \right|\right| \ge \alpha. \tag{3.3}$$

In addition,

$$\begin{aligned} &\left|\left| \tfrac{P(t_0)x}{||P(t_0)x||} + \tfrac{P_3(t_0)x}{||P_3(t_0)x||} \right|\right| = \\ &= \tfrac{1}{||P_3(t_0)x||} \left|\left| (I - P(t_0))x + \tfrac{||P_3(t_0)x||}{||P(t_0)x||}\ P(t_0)x \right|\right| \le \\ &\le \tfrac{1}{||P_3(t_0)x||}\ (||x|| + |\ ||P_3(t_0)x|| - ||P(t_0)x||\ |) \le \tfrac{2||x||}{||P_3(t_0)x||}. \end{aligned} \tag{3.4}$$

From (3.3) and (3.4) we have that

$$||P_3(t_0)x|| \le (2/\alpha)||x||.$$

If $P(t_0)x = 0$, then $x = P_3(t_0)x$. Then, taking $\alpha_3 = \max\{(2/\alpha), 1\}$ we obtain that

$$||P_3(t_0)x|| \leq \alpha_3 \; ||x||, \quad \forall x \in X, \forall t_0 \in \mathbb{R}.$$

Step 3 We prove that $\sup_{t\in\mathbb{R}} ||P_k(t)|| < \infty$, for $k \in \{1, 2\}$.

From Step 2 we have that $q := \sup_{t\in\mathbb{R}} ||P_1(t) + P_2(t)|| < \infty$.

Let $h > 0$ be such that $K^2 \mathrm{e}^{-\nu h} < 1$ and let $\delta := (1 - K^2\mathrm{e}^{-\nu h})/(KM\mathrm{e}^{\omega h})$.

Let $t_0 \in \mathbb{R}$. Let $x \in X$ with $P_1(t_0)x \neq 0$ and $P_2(t_0)x \neq 0$. From

$$\begin{aligned} &M\mathrm{e}^{\omega h} \left|\left| \tfrac{P_1(t_0)x}{||P_1(t_0)x||} + \tfrac{P_2(t_0)x}{||P_2(t_0)x||} \right|\right| \geq \\ &\geq \left|\left| U(t_0+h, t_0) \left(\tfrac{P_1(t_0)x}{||P_1(t_0)x||} + \tfrac{P_2(t_0)x}{||P_2(t_0)x||} \right) \right|\right| \geq \tfrac{1}{K} - K\mathrm{e}^{-\nu h} \end{aligned}$$

we obtain that

$$\left|\left| \tfrac{P_1(t_0)x}{||P_1(t_0)x||} + \tfrac{P_2(t_0)x}{||P_2(t_0)x||} \right|\right| \geq \delta. \tag{3.5}$$

In addition,

$$\begin{aligned} &\left|\left| \tfrac{P_1(t_0)x}{||P_1(t_0)x||} + \tfrac{P_2(t_0)x}{||P_2(t_0)x||} \right|\right| = \tfrac{1}{||P_1(t_0)x||} \; \left|\left| P_1(t_0)x + \tfrac{||P_1(t_0)x||}{||P_2(t_0)x||} \; P_2(t_0)x \right|\right| = \\ &= \tfrac{1}{||P_1(t_0)x||} \; \left|\left| (P_1(t_0) + P_2(t_0))x + \tfrac{||P_1(t_0)x|| - ||P_2(t_0)x||}{||P_2(t_0)x||} \; P_2(t_0)x \right|\right| \leq \\ &\leq \tfrac{2||(P_1(t_0)+P_2(t_0))x||}{||P_1(t_0)x||} \leq \tfrac{2q||x||}{||P_1(t_0)x||}. \end{aligned} \tag{3.6}$$

From (3.5) and (3.6) we deduce that

$$||P_1(t_0)x|| \leq (2q/\delta)||x||. \tag{3.7}$$

If $P_2(t_0)x = 0$, then $x \in \mathrm{Im}(P_1(t_0) + P_3(t_0))$. Then

$$||P_1(t_0)x|| = ||x - P_3(t_0)x|| \leq (1 + \alpha_3)||x||. \tag{3.8}$$

Setting $\alpha_1 := \max\{(2q/\delta), 1 + \alpha_3\}$ from (3.7) and (3.8) it follows that

$$||P_1(t_0)x|| \leq \alpha_1 \; ||x||, \quad \forall x \in X, \forall t_0 \in \mathbb{R}.$$

Since for every $t_0 \in \mathbb{R}$, $P_2(t_0) = I - (P_1(t_0) + P_3(t_0))$, the proof is complete. $\square$

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