REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 37 (2008) no. 2, pp. 217–220
ictp.acad.ro/jnaat

SOLVING INVERSE PROBLEMS VIA WEAK CONTRACTIVE MAPS

ŞTEFAN M. ŞOLTUZ

Abstract. We prove a "collage" theorem for weak contractive maps and we use it for inverse problems.

MSC 2000. 65J22, 47N40.
Keywords. Weak contractive maps, inverse problems.

1. INTRODUCTION

Let X be a real Banach space, $T : X \to X$ be an operator.

DEFINITION 1. *The operator T is weak contractive, if there exist a constant $q \in [0, 1)$ and $L > 0$, such that*

$$0 < q + L < 1 \tag{1}$$

and for each $x, y \in X$,

$$\|Tx - Ty\| \leq q \|x - y\| + L \|y - Tx\| . \tag{2}$$

Let $F(T)$ denote the fixed point set with respect to X for the map T. Suppose that $x^* \in F(T)$. A typical inverse problem is the following:

PROBLEM 2. For given $\varepsilon > 0$ and a "target" $\bar{x}$, find a weak contractive map T_ε such that $\left\|\bar{x} - x^*_{T_\varepsilon}\right\| < \varepsilon$, where $x^*_{T_\varepsilon} = T_\varepsilon\left(x^*_{T_\varepsilon}\right)$ is the fixed point of the weak contractive mapping T_ε.

Randomly selecting various in weak contractive maps class, finding their fixed points and computing the distance from our target is an extremely tedious procedure. Consider now the following problem which we shall fit in our framework and which is very useful for practitioners, see [3]. Recently, Kunze et all., see [3], [4], [5], have considered a class of inverse problems for ordinary differential equations and provided a mathematical basis for solving them within the framework of Banach spaces and contractions. We shall consider the same framework of Banach spaces and the larger class of weak contractive maps. Such maps were considered in [1] and [2].

*"Tiberiu Popoviciu" Institute of Numerical Analysis, P.O. Box. 68-1, Cluj-Napoca, Romania, e-mail: smsoltuz@gmail.com.

PROBLEM 3. Let $\bar{x} \in X$ be a target and let $\delta > 0$ be given. Find T_δ a weak contractive map such that $\|\bar{x} - T_\delta \bar{x}\| < \delta$.

In other words, instead of searching for weak contractive maps whose fixed points lie close to target $\bar{x}$, we search for weak contractive maps that send $\bar{x}$ close to itself.

2. MAIN RESULTS

We shall give a lower bound to approximation error in terms of the collage error.

PROPOSITION 4. *Let X be a real Banach space and T a weak contractive map with contraction factor $q \in (0,1)$, $L > 0$, and fixed point $x^* \in X$. Then for any $x \in X$,*

$$\frac{1}{1+q+L} \|x - Tx\| \leq \|x^* - x\| .$$

Proof. If $x = x^*$, the above inequality holds. If $x \neq x^*, \forall x \in X$, then one obtains

$$\begin{aligned} \|Tx - x\| &\leq \|x^* - x\| + \|x^* - Tx\| \\ &= \|x^* - x\| + \|Tx^* - Tx\| \\ &\leq \|x^* - x\| + q\|x^* - x\| + L\|x - Tx^*\| \\ &\leq (1 + q + L)\|x^* - x\| . \end{aligned}$$

From which one gets the conclusion. □

THEOREM 5. *(**Collage theorem for weak contractive maps**) Let X be a real Banach space and T a weak contractive map with contraction factor $q \in (0,1)$, $L > 0$, satisfying (1), and fixed point $x^* \in X$. Then for any $x \in X$,*

$$\|x^* - x\| \leq \frac{1}{1-(q+L)} \|x - Tx\| .$$

Proof. The weak contractive condition assures that the fixed point x^* is unique. If $x = x^*$, the above inequality holds. If $x \neq x^*, \forall x \in X$, then one obtains

$$\begin{aligned} \|x^* - x\| &\leq \|Tx^* - Tx\| + \|Tx - x\| \\ &\leq q\|x^* - x\| + L\|x - Tx^*\| + \|Tx - x\| \\ &= (q + L)\|x^* - x\| + \|Tx - x\| . \end{aligned}$$

From which one gets the conclusion by using (1). □

REMARK 6. To summarize, under appropriate conditions, we have the following bounds

$$\frac{1}{1+q+L} \|x - Tx\| \leq \|x^* - x\| \leq \frac{1}{1-(q+L)} \|x - Tx\| .$$

□

The above “Collage Theorem” allows us to reformulate the inverse Problem 2 in the particular and more convenient Problem 3.

THEOREM 7. *If Problem 3 has a solution, then Problem 2 has a solution too.*

Proof. Let $\varepsilon > 0$ and $\bar{x} \in X$ be given. For $\delta := (1-(q+L))\,\varepsilon$, let T_δ a weak contractive map be such that $\|\bar{x} - T_\delta \bar{x}\| < \delta$. If $x^*_{T_\delta}$ is the fixed point of the weak contractive mapping T_δ, then, by Theorem 5,

$$\left\|\bar{x} - x^*_{T_\delta}\right\| \le \tfrac{1}{1-(q+L)} \left\|\bar{x} - T_\delta \bar{x}\right\| \le \tfrac{1}{1-(q+L)}\delta = \varepsilon.$$

□

Note that shrinking the distance between two operators, one of them from weak contractive map reduces the distance between their fixed points.

PROPOSITION 8. *Let X be a real Banach space and T_1 weak contractive map with contraction factor $q_1 \in (0,1)$ and $T_2 : X \to X$ a map such that $x_1^*, x_2^* \in X$ are distinct fixed points for T_1 and T_2. Then*

$$\|x_1^* - x_2^*\| \le \tfrac{1}{1-(q_1+L_1)} \sup_{x\in X} \|T_1 x - T_2 x\|\,.$$

Proof. One obtains

$$\begin{aligned}\|x_1^* - x_2^*\| &= \|T_1 x_1^* - T_2 x_2^*\| \le \|T_1 x_1^* - T_1 x_2^*\| + \|T_1 x_2^* - T_2 x_2^*\| \\ &\le q\,\|x_1^* - x_2^*\| + L_1\,\|x_2^* - T_1 x_1^*\| + \sup_{x\in X}\|T_1 x - T_2 x\|\,,\end{aligned}$$

from which we get the conclusion. □

THEOREM 9. *Let X be a real Banach space, $T : X \to X$, $\bar{x} = T\bar{x}$ and suppose there exists T_1 a weak contractive map such that $\sup_{x\in X}\|T_1 x - Tx\| \le \varepsilon$. Then*

$$\|\bar{x} - T_1\bar{x}\| \le \tfrac{1+(q+L)}{1-(q+L)}\varepsilon.$$

Proof. Let $x^* = T_1 x^*$, and by use of Proposition 8 we obtain

$$\|\bar{x} - x^*\| \le \tfrac{1}{1-(q+L)} \left(\sup_{x\in X}\|T_1 x - Tx\|\right).$$

Thus,

$$\begin{aligned}\|\bar{x} - T_1\bar{x}\| &\le \|\bar{x} - x^*\| + \|x^* - T_1\bar{x}\| \\ &\le \|\bar{x} - x^*\| + \|T_1 x^* - T_1\bar{x}\| \\ &\le \|\bar{x} - x^*\| + q\,\|\bar{x} - x^*\| + L\,\|\bar{x} - T_1 x^*\| \\ &= (1+(q+L))\,\|\bar{x} - x^*\| \\ &\le \tfrac{1+(q+L)}{1-(q+L)}\left(\sup_{x\in X}\|T_1 x - Tx\|\right) \le \tfrac{1+(q+L)}{1-(q+L)}\varepsilon.\end{aligned}$$

□

REFERENCES

[1] Berinde, V., *Approximating fixed points of weak contractions using Picard iteratioon*, Nonlinear Analysis Forum, **9**, pp. 43–53, 2004.

[2] Berinde, M. and Berinde, V., *On a general class of multi-valued weakly Picard mappings,* J. Math. Anal. Appl., **326**, pp. 772–782, 2007.

[3] Kunze, H.E. and Vrscay, E.R., *Solving inverse problems for ordinary differential equations using the Picard contraction mapping*, Inverse Problems, **15**, pp. 745–770, 1999.

[4] Kunze, H.E. and Gomes, S., *Solving an inverse problem for Urison-type integral equations using Banach's fixed point theorem*, Inverse Problems, **19**, pp. 411–418, 2003.

[5] Kunze, H.E., Hicken, J.E. and Vrscay, E.R., *Inverse problems for ODEs using contraction maps and suboptimality for the 'collage method'*, Inverse Problems, **20**, pp. 977–991, 2004.

Received by the editors: July 5, 2008.