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A SIMPLE PROOF OF POPOVICIU'S INEQUALITY

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Abstract. T. Popoviciu [5] has proved in 1965 the following inequality relating the values of a convex function $f : I \to \mathbb{R}$ at the weighted arithmetic means of the subfamilies of a given family of points $x_1, ..., x_n \in I$:

$$\sum_{1 \le i_1 < \cdots < i_p \le n} (\lambda_{i_1} + \cdots + \lambda_{i_p}) f\left(\tfrac{\lambda_{i_1} x_{i_1} + \cdots + \lambda_{i_p} x_{i_p}}{\lambda_{i_1} + \cdots + \lambda_{i_p}}\right)$$
$$\le \tbinom{n-2}{p-2} \left[\tfrac{n-p}{p-1} \sum_{i=1}^{n} \lambda_i f(x_i) + \left(\sum_{i=1}^{n} \lambda_i\right) f\left(\tfrac{\lambda_1 x_1 + \cdots + \lambda_n x_n}{\lambda_1 + \cdots + \lambda_n}\right)\right].$$

Here $n \ge 3$, $p \in \{2, ..., n-1\}$ and $\lambda_1, ..., \lambda_n$ are positive numbers (representing weights). The aim of this paper is to give a simple argument based on mathematical induction and a majorization lemma.

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T. Popoviciu [5] has proved in 1965 the following inequality relating the values of a convex function $f : I \to \mathbb{R}$ at the weighted arithmetic means of the different subfamilies of a given family of points $x_1, ..., x_n \in I$:

$$\sum_{1 \le i_1 < \cdots < i_p \le n} (\lambda_{i_1} + \cdots + \lambda_{i_p}) f\left(\tfrac{\lambda_{i_1} x_{i_1} + \cdots + \lambda_{i_p} x_{i_p}}{\lambda_{i_1} + \cdots + \lambda_{i_p}}\right)$$
$$\le \tbinom{n-2}{p-2} \left[\tfrac{n-p}{p-1} \sum_{i=1}^{n} \lambda_i f(x_i) + \left(\sum_{i=1}^{n} \lambda_i\right) f\left(\tfrac{\lambda_1 x_1 + \cdots + \lambda_n x_n}{\lambda_1 + \cdots + \lambda_n}\right)\right].$$

Here $n \ge 3$, $p \in \{2, ..., n-1\}$ and $\lambda_1, ..., \lambda_n$ are positive numbers (representing weights); I is a nonempty interval.

The inequality above (denoted $(P_{n,p})$ in what follows) is nontrivial even in the case of triplets (that is, when $n = 3$ and $p = 2$). Several alternative approaches of $(P_{3,2})$ are discussed in the recent book of C. P. Niculescu and L.-E. Persson [2]. See [4] and [3] for additional information.

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Theoretically, Popoviciu's inequality is a refinement of Jensen's inequality since it yields

$$f\left(\frac{\sum\limits_{i=1}^{n}\lambda_i x_i}{\sum\limits_{i=1}^{n}\lambda_i}\right) \leq \frac{1}{\binom{n-1}{p-1}\left(\sum\limits_{i=1}^{n}\lambda_i\right)} \sum_{1\leq i_1<\cdots<i_p\leq n}\left(\lambda_{i_1}+\cdots+\lambda_{i_p}\right) f\left(\frac{\lambda_{i_1}x_{i_1}+\cdots+\lambda_{i_p}x_{i_p}}{\lambda_{i_1}+\cdots+\lambda_{i_p}}\right)$$
$$\leq \frac{n-p}{n-1}\frac{\sum\limits_{i=1}^{n}\lambda_i f(x_i)}{\sum\limits_{i=1}^{n}\lambda_i} + \frac{p-1}{n-1} f\left(\frac{\sum\limits_{i=1}^{n}\lambda_i x_i}{\sum\limits_{i=1}^{n}\lambda_i}\right) \leq \frac{\sum\limits_{i=1}^{n}\lambda_i f(x_i)}{\sum\limits_{i=1}^{n}\lambda_i}.$$

The aim of the present paper is to offer a simple argument of $(P_{n,p})$ based on mathematical induction and the following variant of the majorization inequality:

Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be a convex function. If $x_1,...,x_n\in[a,b]$ and a convex combination $\sum\limits_{k=1}^{n}\mu_k x_k$ of these points equals a convex combination $\lambda_1 a+\lambda_2 b$ of the endpoints, then*

$$\sum_{k=1}^{n}\mu_k f(x_k)\leq \lambda_1 f(a)+\lambda_2 f(b).$$

Proof. This can be established easily by using the barycentric coordinates (in our case the fact that every point $x_k\in[a,b]$ can be expressed uniquely as a convex combination of a and b).

A second argument is based on the geometric meaning of convexity. Denoting by $A(x)$ the affine function joining $(a,f(a))$ with $(b,f(b))$, we have

$$\begin{aligned}\sum_{k=1}^{n}\mu_k f(x_k) &\leq \sum_{k=1}^{n}\mu_k A(x_k) = A\left(\sum_{k=1}^{n}\mu_k x_k\right)\\ &= A(\lambda_1 a+\lambda_2 b) = \lambda_1 A(a)+\lambda_2 A(b)\\ &= \lambda_1 f(a)+\lambda_2 f(b).\end{aligned}$$

□

It is worth to mention that Lemma 1 still works (with obvious changes) within the framework of convex functions on simplices.

We pass now to the proof of Popoviciu's inequality, by considering first the case where $n\in\mathbb{N}$, $n\geq 3$ and $p=n-1$:

$$(P_{n,n-1})\quad \sum_{1\leq i\leq n}\lambda_i f(x_i)+(n-2)\left(\sum_{1\leq i\leq n}\lambda_i\right) f\left(\frac{\sum\limits_{1\leq i\leq n}\lambda_i x_i}{\sum\limits_{1\leq i\leq n}\lambda_i}\right)$$
$$\geq \sum_{1\leq j\leq n}\left(\sum_{1\leq i\leq n,\, i\neq j}\lambda_i\right) f\left(\frac{\sum\limits_{1\leq i\leq n,\, i\neq j}\lambda_i x_i}{\sum\limits_{1\leq i\leq n,\, i\neq j}\lambda_i}\right).$$

Clearly, we may assume

$$x_1 \le x_2 \le \cdots \le x_n.$$

Choose $k \in \{1, ..., n-1\}$ such that

$$x_k \le \frac{\sum_{i=1}^{n} \lambda_i\, x_i}{\sum_{i=1}^{n} \lambda_i} \le x_{k+1}$$

and put

$$y_j = \frac{\sum_{1\le i\le n,\, i\ne j} \lambda_i\, x_i}{\sum_{1\le i\le n,\, i\ne j} \lambda_i}$$

for $j = 1, ..., n$. Then it is clear that

$$\frac{\sum_{i=1}^{n} \lambda_i\, x_i}{\sum_{i=1}^{n} \lambda_i} \le y_j \le \frac{\sum_{i=k+1}^{n} \lambda_i\, x_i}{\sum_{i=k+1}^{n} \lambda_i} \tag{1}$$

for all $j \in \{1, ..., k\}$.

We have

$$\begin{aligned}
\frac{\sum_{j=1}^{k}\left(\sum_{1\le i\le n,\, i\ne j} \lambda_i\right) y_j}{\sum_{j=1}^{k}\left(\sum_{1\le i\le n,\, i\ne j} \lambda_i\right)} &= \frac{\sum_{j=1}^{k}\left(\sum_{1\le i\le n,\, i\ne j} \lambda_i\right) \frac{\sum_{1\le i\le n,\, i\ne j} \lambda_i\, x_i}{\sum_{1\le i\le n,\, i\ne j} \lambda_i}}{\sum_{j=1}^{k}\left(\sum_{1\le i\le n,\, i\ne j} \lambda_i\right)} \\
&= \frac{(k-1) \sum_{i=1}^{n} \lambda_i\, x_i + \sum_{i=k+1}^{n} \lambda_i\, x_i}{(k-1) \sum_{i=1}^{n} \lambda_i + \sum_{i=k+1}^{n} \lambda_i} \\
&= \frac{(k-1)\left(\sum_{i=1}^{n} \lambda_i\right) \frac{\sum_{i=1}^{n} \lambda_i\, x_i}{\sum_{i=1}^{n} \lambda_i} + \left(\sum_{i=k+1}^{n} \lambda_i\right) \frac{\sum_{i=k+1}^{n} \lambda_i\, x_i}{\sum_{i=k+1}^{n} \lambda_i}}{(k-1) \sum_{i=1}^{n} \lambda_i + \sum_{i=k+1}^{n} \lambda_i}
\end{aligned}$$

so that by (1) and Lemma 1 we infer the inequality

$$\begin{aligned}
&\sum_{j=1}^{k}\left(\sum_{1\le i\le n,\, i\ne j} \lambda_i\right) f\left(\frac{\sum_{1\le i\le n,\, i\ne j} \lambda_i x_i}{\sum_{1\le i\le n,\, i\ne j} \lambda_i}\right) \le \\
&\le (k-1)\left(\sum_{i=1}^{n} \lambda_i\right) f\left(\frac{\sum_{i=1}^{n} \lambda_i x_i}{\sum_{i=1}^{n} \lambda_i}\right) + \left(\sum_{i=k+1}^{n} \lambda_i\right) f\left(\frac{\sum_{i=k+1}^{n} \lambda_i x_i}{\sum_{i=k+1}^{n} \lambda_i}\right).
\end{aligned}$$

Or, by Jensen's inequality,

$$\sum_{j=1}^{k}\left(\sum_{1\le i\le n,\, i\ne j}\lambda_i\right) f\left(\frac{\sum\limits_{1\le i\le n,\, i\ne j}\lambda_i x_i}{\sum\limits_{1\le i\le n,\, i\ne j}\lambda_i}\right) \le$$

$$\le (k-1)\left(\sum_{i=1}^{n}\lambda_i\right) f\left(\frac{\sum\limits_{i=1}^{n}\lambda_i x_i}{\sum\limits_{i=1}^{n}\lambda_i}\right) + \sum_{i=k+1}^{n}\lambda_i f(x_i)$$

and

$$\sum_{j=k+1}^{n}\left(\sum_{1\le i\le n,\, i\ne j}\lambda_i\right) f\left(\frac{\sum\limits_{1\le i\le n,\, i\ne j}\lambda_i x_i}{\sum\limits_{1\le i\le n,\, i\ne j}\lambda_i}\right) \le$$

$$\le (n-k-1)\left(\sum_{i=1}^{n}\lambda_i\right) f\left(\frac{\sum\limits_{i=1}^{n}\lambda_i x_i}{\sum\limits_{i=1}^{n}\lambda_i}\right) + \sum_{i=1}^{k}\lambda_i f(x_i)$$

whence we may conclude $(P_{n,n-1})$.

Consider now the case where $n\in\mathbb{N}$, $p\ge 3$. We will prove that

$$(P_{n,p}) \Rightarrow (P_{n,p-1})$$

that is, if Popoviciu's inequality works for families of n weighted points by grouping them into subfamilies of size $p\in\{3,...,n-1\}$ then it also works by grouping them into subfamilies of size $p-1$.

By Lemma 1,

$$\lambda_{i_1} f(x_{i_1}) + \cdots + \lambda_{i_p} f(x_{i_p}) + (k-2)(\lambda_{i_1}+\cdots+\lambda_{i_p})\, f\left(\tfrac{\lambda_{i_1}x_{i_1}+\cdots+\lambda_{i_p}x_{i_p}}{\lambda_{i_1}+\cdots+\lambda_{i_p}}\right) \ge$$

$$\ge \sum_{j=1}^{p}\left(\sum_{1\le k\le p,\, k\ne j}\lambda_{i_k}\right) f\left(\frac{\sum\limits_{1\le k\le p,\, k\ne j}\lambda_{i_k}x_{i_k}}{\sum\limits_{1\le k\le p,\, k\ne j}\lambda_{i_k}}\right)$$

whence

$$\sum_{1\le i_1<\cdots<i_p\le n}(\lambda_{i_1}+\cdots+\lambda_{i_p})\, f\left(\tfrac{\lambda_{i_1}x_{i_1}+\cdots+\lambda_{i_p}x_{i_p}}{\lambda_{i_1}+\cdots+\lambda_{i_p}}\right) \ge \tfrac{1}{p-2}\Bigg(-\tbinom{n-1}{p-1}\sum_{i=1}^{n}\lambda_i f(x_i)$$

$$+(n-p+1)\sum_{1\le i_1<\cdots<i_{p-1}\le n}(\lambda_{i_1}+\cdots+\lambda_{i_{p-1}})\, f\left(\tfrac{\lambda_{i_1}x_{i_1}+\cdots+\lambda_{i_{p-1}}x_{i_{p-1}}}{\lambda_{i_1}+\cdots+\lambda_{i_{p-1}}}\right)\Bigg).$$

By our hypothesis we get

$$\binom{n-2}{p-1}\left(\frac{n-p}{p-1}\sum_{i=1}^{n}\lambda_i f(x_i)+\left(\sum_{i=1}^{n}\lambda_i\right)f\left(\frac{\sum\limits_{i=1}^{n}\lambda_i x_i}{\sum\limits_{i=1}^{n}\lambda_i}\right)\right)\geq$$

$$\geq\frac{1}{p-2}\left(-\binom{n-1}{p-1}\sum_{i=1}^{n}\lambda_i f(x_i)+(n-p+1)\right.$$

$$\left.\sum_{1\leq i_1<\cdots<i_{p-1}\leq n}(\lambda_{i_1}+\cdots+\lambda_{i_{p-1}})\,f\left(\frac{\lambda_{i_1}x_{i_1}+\cdots+\lambda_{i_{p-1}}x_{i_{p-1}}}{\lambda_{i_1}+\cdots+\lambda_{i_{p-1}}}\right)\right),$$

that is,

$$\left(\binom{n-2}{p-2}\frac{n-p}{p-1}+\binom{n-1}{p-1}\frac{1}{p-2}\right)\sum_{i=1}^{n}\lambda_i f(x_i)+\binom{n-2}{p-2}\left(\sum_{i=1}^{n}\lambda_i\right)f\left(\frac{\sum\limits_{i=1}^{n}\lambda_i x_i}{\sum\limits_{i=1}^{n}\lambda_i}\right)$$

$$\geq\frac{n-p+1}{p-2}\sum_{1\leq i_1<\cdots<i_{p-1}\leq n}(\lambda_{i_1}+\cdots+\lambda_{i_{p-1}})\,f\left(\frac{\lambda_{i_1}x_{i_1}+\cdots+\lambda_{i_{p-1}}x_{i_{p-1}}}{\lambda_{i_1}+\cdots+\lambda_{i_{p-1}}}\right).$$

Since

$$\binom{n-2}{p-2}\frac{n-p}{p-1}+\binom{n-1}{p-1}\frac{1}{p-2}=\binom{n-2}{p-2}\frac{n-p+1}{p-2}$$

and

$$\binom{n-2}{p-2}=\binom{n-2}{p-3}\frac{n-p+1}{p-2}$$

we can restate the last inequality as follows:

$$\binom{n-2}{(p-1)-2}\frac{n-(p-1)}{(p-1)-1}\sum_{i=1}^{n}\lambda_i f(x_i)+\left(\sum_{i=1}^{n}\lambda_i\right)f\left(\frac{\sum\limits_{i=1}^{n}\lambda_i x_i}{\sum\limits_{i=1}^{n}\lambda_i}\right)$$

$$\geq\sum_{1\leq i_1<\cdots<i_{p-1}\leq n}(\lambda_{i_1}+\cdots+\lambda_{i_{p-1}})\,f\left(\frac{\lambda_{i_1}x_{i_1}+\cdots+\lambda_{i_{p-1}}x_{i_{p-1}}}{\lambda_{i_1}+\cdots+\lambda_{i_{p-1}}}\right),$$

which proves to be precisely $(P_{n,p-1})$.

The proof of Popoviciu's inequality is now complete.

Remark 2. The induction step is not necessary in deriving the unweighted case of the inequalities $(P_{n,2})$:

$$(\mathrm{nP}_{n,2})\quad (n-2)\frac{f(x_1)+\cdots+f(x_n)}{n}+f\left(\frac{x_1+\cdots+x_n}{n}\right)\geq\frac{2}{n}\sum_{1\leq j<k\leq n}f\left(\frac{x_j+x_k}{2}\right)$$

for all $x_1,...,x_n$ in the domain of f.

In fact, assuming that

$$x_1\leq\cdots\leq x_n$$

we will consider first the case where

$$\frac{x_1+x_n}{2} \le \frac{x_1+\cdots+x_n}{n}.$$

Then, by Lemma 1 we get

$$(M)\quad \frac{1}{n}\left(f(x_1)+f\left(\frac{x_1+x_2}{2}\right)+\cdots+f\left(\frac{x_1+x_n}{2}\right)\right) \le \frac{1}{2}\left(f(x_1)+f\left(\frac{x_1+\cdots+x_n}{n}\right)\right)$$

while from Jensen's inequality we infer that

$$(J)\qquad \frac{2}{n}\sum_{2\le j<k\le n} f\left(\frac{x_j+x_k}{2}\right) \le \frac{2}{n}\frac{n-2}{2}\sum_{i=2}^{n} f(x_i).$$

Summing up (M) and (J) we get $(\mathrm{nP}_{n,2})$. The case where

$$\frac{x_1+x_n}{2} \ge \frac{x_1+\cdots+x_n}{n}$$

can be treated in a similar way (changing the role of the indices 1 and n in (M)). □

At first glance Popoviciu's inequality is a one real variable result. This impression is strongly supported by the existence of counterexamples even in the two real variables context. For example, think at an upsidedown regular triangular pyramid (viewed as the graph of a convex function). Besides, all known arguments of $(P_{n,p})$ make use of the ordering of $\mathbb{R}$.

However, as Professor Constantin P. Niculescu called to our attention, it is possible to develop a higher dimensional theory of convexity based on $(P_{3,2})$. This makes the objective of our joint paper [1].

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