REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 37 (2008) no. 2, pp. 133–142
ictp.acad.ro/jnaat

START ITERATION FOR EZQUERRO-HERNÁNDEZ METHOD

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Abstract. We present a numerical method for solving nonlinear equation systems, namely Ezquerro-Hernández method with a rate of convergence equal to 4. The main result of this article is an algorithm that determines a start iteration for the method within its quadruple convergence sphere.

MSC 2000. 65H10.
Keywords. Ezquerro-Hernández method, start iteration from sphere of cvadruplu convergence.

1. EZQUERRO–HERNANDEZ METHOD

X and Y are two real or complex Banach spaces. $F : D \subset X \to Y$ is a nonlinear operator, 3 times Fréchet differentiable over the convex and open set $D_0 \subset D$. Lets assume that there is a linear operator $F'(x^{(0)})^{-1} \in \mathfrak{L}_{\mathfrak{M}}(Y, X)$ for $x^{(0)} \in D_0$. Let $F(x) = 0$, then $\Gamma(x) = F'(x)^{-1}$ is the nonlinear operator, $\mathcal{N}(x) = \Gamma(x)F(x)$ is the Newton operator and the sequences $\left\{x^{(k)}\right\}$ and $\left\{y^{(k)}\right\}$ are defined by:

$$(1)\qquad \begin{cases} y & = & x - \mathcal{N}(x), \\ \mathcal{H}(x,y) & = & \Gamma(x)\left[F'\left(\frac{1}{3}x + \frac{2}{3}y\right) - F'(x)\right], \\ \widehat{x} & = & y - \frac{3}{4}\mathcal{H}(x,y)\left[I - \frac{3}{2}\mathcal{H}(x,y)\right](y-x), \end{cases}$$

where x and y denote the current iteration and $\widehat{x}$ denotes the next iteration. The method given by (1) was proposed by Ezquerro and Hernández in [3]. This method has an *R-order* of convergence equal to 4.

2. PRELIMINARY LEMMAS AND THEOREMS

Let consider the following conditions to be fulfilled:

(c_1) $\|\Gamma_0\| = \left\|\Gamma\left(x^{(0)}\right)\right\| \leq \beta$,
(c_2) $\left\|\mathcal{N}\left(x^{(0)}\right)\right\| \leq \eta$,
(c_3) $\|F''(x)\| \leq M_2$, for $\forall\ x \in D_0$,

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(c_4) $\left\|F'''(x)\right\| \leq M_3$, for $\forall\ x \in D_0$,

(c_5) $\left\|F'''(x) - F'''(y)\right\| \leq L\left\|x-y\right\|$, for $\forall\ x, y \in D_0$, with $L \geq 0$.

Let denote $A = \beta\eta M_2$, $B = \beta\eta^2 M_3$ and $E = \beta\eta^3 L$. $\{a_k\}$, $\{b_k\}$, $\{c_k\}$ and $\{d_k\}$, are defined as sequences with the initial values

$$a_0 = c_0 = 1, \quad b_0 = 2A/3, \quad d_0 = A(1+A)/2,$$

given by:

$$\text{(2)} \qquad \begin{cases} a_{k+1} &= \frac{a_k}{1-Aa_k(c_k+d_k)}, \\ c_{k+1} &= \frac{32}{2187} \cdot \frac{27\left(4+\beta_k^2\right)A^3a_k^2+18ABa_k+17E}{\beta_k^4 b_k^4} a_{k+1}d_k^4, \\ b_{k+1} &= \frac{2A}{3}a_{k+1}c_{k+1}, \\ d_{k+1} &= \frac{3}{4}\beta_{k+1}b_{k+1}c_{k+1}, \end{cases}$$

for any $k \in \mathbb{N}^*$. In the formulas above we denoted $1 + 3b_k/2$ by β_k, for any $k \in \mathbb{N}^*$.

Lemma 1. *The following inequalities are true for any* $k \in \mathbb{N}^*$.

(I_k) $\left\|\Gamma_k\right\| = \left\|\Gamma\left(x^{(k)}\right)\right\| \leq \beta a_k$,

(II_k) $\left\|y^{(k)} - x^{(k)}\right\| = \left\|\mathcal{N}\left(x^{(k)}\right)\right\| \leq \eta c_k$,

(III_k) $\left\|\mathcal{H}\left(x^{(k)}, y^{(k)}\right)\right\| \leq b_k$,

(IV_k) $\left\|x^{(k+1)} - y^{(k)}\right\| \leq \eta d_k$,

(V_k) $\left\|x^{(k+1)} - x^{(k)}\right\| \leq \eta(c_k + d_k)$.

Proof. The proofs can be followed in [1]. □

According to lemma 1, if the following conditions are fulfilled:

- $x^{(k)},\ y^{(k)} \in D_0$,
- $Aa_k(c_k + d_k) < 1$,
- the sequence $\{c_k + d_k\}$ is a Cauchy sequence,

then the sequence defined by (1) is convergent.

The polynomial

$$P(s) = 1 - \frac{3s}{2}\left(1 + \frac{3s}{4}\left(1 + \frac{3s}{2}\right)\right) = \frac{-27s^3-18s^2-24+16}{16}$$

has a single real and positive root that we denote by σ,

$$\sigma = \frac{2}{9}\left[\frac{7-3\sqrt{6}}{5}\sqrt[3]{\left[5\left(7+3\sqrt{6}\right)\right]^2} + \sqrt[3]{5\left(7+3\sqrt{6}\right)} - 1\right].$$

The approximative value of this root is $\sigma \approx 0.433752794292925\ldots$.

The real functions $f : [0, \sigma) \to \mathbb{R}$ and $h : (1, \infty) \to \mathbb{R}$ are defined as:

$$\text{(3)} \qquad f(s) = \frac{27}{16} \cdot \frac{s^4}{P(s)^2}, \quad h(t) = \frac{18ABt+17E}{108A^3t^2},$$

where $A, B, E > 0$, and $g : [0,\sigma) \times (1,\infty) \to \mathbb{R}$ is defined as

$$g(s,t) = f(s)\left(1 + \tfrac{1}{4}\left(1 + \tfrac{3s}{2}\right)^2 + h(t)\right). \tag{4}$$

LEMMA 2. *For the functions f and g defined by* (3) *and* (4) *the following proprieties stand:*

(i) *f is increasing for $s \in [0,\sigma)$ and $f(0) = 0$,*
(ii) *f' is increasing for $s \in [0,\sigma)$ and $f'(0) = 0$,*
(iii) *$g(s,t) < g(s,1)$ for $t > 1$,*
(iv) *$g_1(s) = g(s,1)$ is increasing for $s \in [0,\sigma)$, and $g_1(0) = 0$,*
(v) *$g_1'(s)$ is increasing for $s \in [0,\sigma)$ and $g_1'(0) = 0$.*

Proof. The proofs can be followed in [1]. □

Lets consider the polynomial

$$R(\tau) = 27(\tau - 1)(2\tau - 1)(\tau^2 + \tau + 2)(\tau^2 + 2\tau + 4). \tag{5}$$

Using the sequences $\{a_k\}$, $\{b_k\}$, $\{c_k\}$ and $\{d_k\}$ we can state that

$$b_{k+1} = g(b_k, a_k) \quad \text{for } \forall\ k \in \mathbb{N}^*. \tag{6}$$

The following theorem proves the four proprieties of the sequences $\{a_k\}$, $\{b_k\}$, $\{c_k\}$ and $\{d_k\}$.

THEOREM 3. *Let consider the following constants*

$$A \in \left(0, \tfrac{1}{2}\right), \quad B \in \left(0, \tfrac{R(A)-17E}{18A}\right), \quad E \in \left(0, \tfrac{R(A)}{17}\right), \tag{7}$$

where R is the polynomial defined by (5)*. Then the following inequalities are true:*

(i_k) *$b_{k+1} < b_k$ for any $k \in \mathbb{N}^*$,*
(ii_k) *$Aa_k(c_k + d_k) < 1$ for any $k \in \mathbb{N}^*$,*
(iii_k) *$a_k \geq 1$ for any $k \in \mathbb{N}^*$,*
(iv_k) *$a_k < a_{k+1}$ for any $k \in \mathbb{N}^*$.*

Proof. The proofs can be followed in [1]. □

THEOREM 4. *In the conditions* (7) *there exists $r \in [0,\infty)$ so that*

$$r = \sum_{k=0}^{\infty}(c_k + d_k). \tag{8}$$

Proof. The proofs can be followed in [1]. □

3. THE CONVERGENCE OF EZQUERRO-HERNÁNDEZ METHOD

THEOREM 5. *We consider the nonlinear operator $F : D \subset X \to Y$, that is 3 times Fréchet differentiable over the convex and open set $D_0 \subset D$, where X and Y are Banach spaces. We assume that the following conditions are fulfilled.*

(a) *Local conditions:*
  (a_1) *There is* $\Gamma_0 = \Gamma\left(x^{(0)}\right) = F'\left(x^{(0)}\right)^{-1}$, $x^{(0)} \in D_0$,
  (a_2) $\|\Gamma_0\| \le \beta$,
  (a_3) $\left\|\Gamma_0 F\left(x^{(0)}\right)\right\| \le \eta$,
  (a_4) $S\left(x^{(0)}, r\right) \subset D_0$, *where* r *is given by* (8),
(b) *Global conditions:*
  (b_1) $\|F''(x)\| \le M_2$, $\forall x \in D_0$,
  (b_2) $\|F'''(x)\| \le M_3$, $\forall x \in D_0$,
  (b_3) $\|F'''(x) - F'''(y)\| \le L\|x-y\|$, $\forall x, y \in D_0$,
(c) *Restrictions:*
  (c_1) $A \in (0, \frac{1}{2})$,
  (c_2) $B \in (0, \frac{R(A)-17E}{18A})$,
  (c_3) $E \in (0, \frac{R(A)}{17})$,
  where $A = M_2\beta\eta$, $B = M_3\beta\eta^2$, $E = L\beta\eta^3$ *and* R *is the polynomial defined by the formula* (5).

Then the sequence $\left\{x^{(k)}\right\}$, *given by the recurrent formula* (1), *is well defined, and* $\left\{x^{(k)}\right\}$, $\left\{y^{(k)}\right\} \subset S\left(x^{(0)}, r\eta\right)$ *for any* $k \in \mathbb{N}^*$. *The sequence* $\left\{x^{(k)}\right\}$ *converges to* $x^\star$, *the solution for the equation* $F(x) = 0$ *and* $x^\star \in \overline{S\left(x^{(0)}, r\eta\right)}$. *This solution, namely* $x^\star$ *is unique in* $S\left(x^{(0)}, 2/(M_2\beta - r\eta)\right) \cap D_0$. *The estimated error is given by the formula*

$$\left\|x^\star - x^{(k)}\right\| \le \eta \sum_{j=k}^{\infty}(c_k + d_k) \le \tfrac{3(2+A+A^2)}{4A} \cdot \tfrac{b_1}{\sqrt[3]{\gamma}} \sum_{j=k}^{\infty} \left(\sqrt[3]{\gamma}\right)^{4^{j-1}}, \tag{9}$$

where $\gamma = b_2/b_1$.

Proof. The proofs can be followed in [1]. □

Remark 1. The *R-order* of convergence of *Ezquerro-Hernández* method can be computed by the inequality (9) and is equal to 4.

Remark 2. The advantage of *Ezquerro-Hernández* method is that the *R-order* of convergence 4 is obtained without the the use of the 2^{nd} and 3^{rd} derivative. The calculus of the 2^{nd} and 3^{rd} derivative is needed to check the convergence conditions of theorem 5, for the initial iteration $x^{(0)}$.

4. IMPLEMENTATION OF EZQUERRO-HERNÁNDEZ METHOD

In order the present the programs for *Ezquerro-Hernández* method lets consider a nonlinear equation in $\mathbb{R}^2$

$$\begin{cases} x_1^2 - 2\cos(3x_1) - x_2 - k_1 &= 0 \\ x_1 - x_2^3 + 3\cos(2.5x_2) + k_2 &= 0, \end{cases} \tag{10}$$

where $k_1 = 8.822260523769353$ and $k_2 = 4.149013443610321$.

EXAMPLE 6. Mathcad dedicated variable ORIGIN has the value 1.

$$ORIGIN := 1.$$

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a nonlinear function

$$F(x) := \begin{pmatrix} x_1^2 - 2\cos(3x_1) - x_2 - k_1 \\ x_1 - x_2^3 + 3\cos\left(\frac{5}{2}x_2\right) + k_2 \end{pmatrix}$$

for the nonlinear equation $F(x) = 0$. The $1st$ order derivative of F is

$$F'(x) := \begin{pmatrix} 2x_1 + 6\sin(3x_1) & -1 \\ 1 & -3x_2^2 - \frac{15}{2}\sin\left(\frac{5}{2}x_2\right) \end{pmatrix}.$$

The variable **d** that represents the space dimension, the identity matrix and the operators Γ and $\mathcal{N}$ ar considered as

$$d := 2 \quad I := indentity(d) \quad \Gamma(x) := F'(x)^{-1} \quad \mathcal{N}(x) := \Gamma(x) \cdot F(x),$$

where the Mathcad function **indentity(d)** was was used to generate the identity matrix of order **d**. The function H and Q form (1) are defined in Mathcad:

$$H(x, y) := \Gamma(x) \cdot \left[F'\left(x + \tfrac{2}{3}(y - x)\right) - F'(x)\right]$$

and

$$Q(x, y) := \tfrac{3}{4} \cdot H(x, y) \cdot \left[I - \tfrac{3}{2}H(x, y)\right].$$

Now the *Ezquerro-Hernández* method operator can be defined:

$$EH(x) := [I - Q(x, x - \mathcal{N}(x))] \cdot \mathcal{N}(x) \tag{11}$$

With this preparation done the program that applies the *Ezquerro-Hernández* method can be easily written.

PROGRAM 7. The *Ezquerro-Hernández* method program.

$$EzHe(x, \varepsilon) := \left| \begin{array}{l} z \leftarrow x^{\mathrm{T}} \\ while\ |EH(x)| \geq \varepsilon \\ \quad \left| \begin{array}{l} x \leftarrow x - EH(x) \\ z \leftarrow stack(z, x^{\mathrm{T}}) \end{array} \right. \\ return\ z \end{array} \right.$$

Let the initial vector be

$$x := \begin{pmatrix} 2.3 & 6 \end{pmatrix}^{\mathrm{T}}$$

and $\varepsilon := 10^{-15}$, the program *EzHe* outputs the 8 iterations for *Ezquerro-Hernández* method.

$$EzHe(x, 10^{-15}) = \begin{pmatrix} 2.3 & 6 \\ 3.39029632223816170 & 3.1088000642353517 \\ 0.44114493152486567 & 2.2109495464523180 \\ 7.80798684999809320 & 2.5526294911507867 \\ 3.12767020347179870 & 2.1135931271578080 \\ 2.99759733296023570 & 2.0015643192458255 \\ 2.99999999991826140 & 2.0000000000800786 \\ 2.99999999999999960 & 1.9999999999999996 \end{pmatrix}.$$

5. THE CHOICE OF THE INITIAL ITERATION

One can see that the random choice of the initial iteration, in general, can not ensure a convergence with and order equal to 4 for the *Ezquerro-Hernández* method starting with the first step. In order to determine a start iteration that ensures a convergence of order 4, form the first step, we use the local conditions (a_1)–(a_3), the global conditions (b_1)–(b_3) and the restriction (c_1)–(c_3) from the convergence theorem 5.

We choose a convex domain $D_0 \subset D = \mathbb{R}^2$ that contains a solution $x^\star$ for the equation $F(x) = 0$. For the equation presented in the example (10) we choose a disk centered in $x^{(0)} = (c_1\ c_2)^{\mathrm{T}}$ with a radius r,

$$D_0 = \left\{x \mid \left|x - x^{(0)}\right| \le r,\ x \in \mathbb{R}^2\right\},$$

where $c_1 = 2.91$, $c_2 = 1.88$ and $r = 0.4$. We compute the topological degree of function F relative to the regular polygon, [5], [6], with m wedged subscribed within the circle. The Mathcad programs that compute the topological degree are:

Program 8. Program **P** generates **m** angles of the regular polygon subscribed within $c = (c_1\ c_2)^{\mathrm{T}}$ with a radius of r (the last angle $m+1$ is the same as the first one).

$$P(c,r,m) := \left| \begin{array}{l} for\ k \in 1..m+1 \\ \quad \left| \begin{array}{l} t_k \leftarrow \dfrac{2(k-1)\pi}{m} \\ P_{1,k} \leftarrow c_1 + r \cdot \cos(t_k) \\ P_{2,k} \leftarrow c_2 + r \cdot \sin(t_k) \end{array} \right. \\ return\ P \end{array} \right.$$

Program 9. Subprogram **q** calculates the matrix determinant.

$$q(F,x,y)) := \left| \begin{pmatrix} sign\,(F(x)_1) & sign\,(F(y)_1) \\ sign\,(F(x)_2) & sign\,(F(y)_2) \end{pmatrix} \right|$$

Program 10. Subprogram **Q**.

$$Q(F,c,r,m) := \left| \begin{array}{l} N \leftarrow P(c,r,m) \\ Q \leftarrow \begin{pmatrix} q\left(F,c,N^{\langle 1\rangle}\right) \\ q\left(F,c,N^{\langle 2\rangle}\right) \\ q\left(F,N^{\langle 1\rangle},N^{\langle 2\rangle}\right) \end{pmatrix} \\ for\ k \in 2..m \\ \quad Q \leftarrow augment\left[Q, \begin{pmatrix} q\left(F,c,N^{\langle k\rangle}\right) \\ q\left(F,c,N^{\langle k+1\rangle}\right) \\ q\left(F,N^{\langle k\rangle},N^{\langle k+1\rangle}\right) \end{pmatrix}\right] \\ return\ Q \end{array} \right.$$

Program 11. Program **gradt** that calculates the topological degree for function **F** relative to the regular polygon with the angles on the circle $c = (c_1\ c_2)^{\mathrm{T}}$ with a radius **r**.

$$gradt(F,c,r,m) := \left| \begin{array}{l} N \leftarrow P(c,r,m) \\ for\ k \in 1..m \\ \quad s_k \leftarrow sign\left[\left|\begin{pmatrix} c_1 & c_2 & 1 \\ N_{1,k} & N_{2,k} & 1 \\ N_{1,k+1} & N_{2,k+1} & 1 \end{pmatrix}\right|\right] \\ Q \leftarrow Q(F,c,r,m) \\ return\ \dfrac{\left|\displaystyle\sum_{k=1}^{m} s_k \sum_{j=1}^{d+1} (-1)^j Q_{j,k}\right|}{2^d d!} \end{array} \right.$$

Since the topological degree of function $\boldsymbol{F}$ relative to the regular polygon $\boldsymbol{P}$ is

$$gradt(F, 2.91, 1.88, 0.4, 31) = 1,$$

namely different from 0, it implies that the function $\boldsymbol{F}$ has at least one root within the polygon $\boldsymbol{P}$, [2], [4]. Therefore the disk centered in **c** with a radius **r** also contains at least one root of function $\boldsymbol{F}$.

We verify if there is Γ_0. That means that we have to compute $\Gamma_0 = \Gamma\left(x^{(0)}\right)$,

$$\Gamma_0 = \begin{pmatrix} 0.10707668676926453 & -0.03449884967970596 \\ 0.03449884967970596 & -0.33330336776140457 \end{pmatrix}.$$

The equation of a circle of radius r centered on $(c_1\ c_2)^{\mathrm{T}}$ is

$$C(x) := (x_1 - c_1)^2 + (x_2 - c_2)^2 - r^2.$$

Form the following nonlinear programming problems we can determine the constants M_2, M_3 and the Lipschitz constant $\boldsymbol{L}$:

(1) let $F_1''(x)$ and $F_2''(x)$ be the matrices that constitute the $2nd$ order tensor the represents the $2nd$ derivative of function F. These matrices are extracted from the matrix $F'(x)$, namely the $1st$ order derivative of the nonlinear function F.

$$F_1''(x) := \begin{pmatrix} 2 + 18\cos(3x_1) & 0 \\ 0 & 0 \end{pmatrix},$$

$$F_2''(x) := \begin{pmatrix} 0 & 0 \\ 0 & -6x_2 - \frac{75}{4}\cos\left(\frac{5}{2}x_2\right) \end{pmatrix}.$$

The objective function

$$nF''(x) := \max\left[norme(F_1''(x)), norme(F_2''(x))\right].$$

We solve the nonlinear programming problem, where the initial value is: $x := x^{(0)}$,

$$Given\ C(x) \leq 0 \quad gradt(F, c, r, m) \neq 0 \quad \xi := maximize(nF'', x),$$

then, using the optimum value ξ, we obtain the constant

$$M_2 := F''(\xi), \quad \text{namely} \quad M_2 = 29.33101477081403.$$

(2) The $3rd$ order derivative of function F is a $3rd$ order tensor that is composed by 4 matrices:

$$F_1'''(x) := \begin{pmatrix} -54\sin(3x_1) & 0 \\ 0 & 0 \end{pmatrix}, \quad F_4'''(x) := \begin{pmatrix} 0 & 0 \\ 0 & \frac{375}{8}\sin\left(\frac{5}{2}x_2\right) - 6 \end{pmatrix}$$

and

$$F_2'''(x) = F_3'''(x) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The process that provided the matrices $F_1''(x)$ and $F_2''(x)$ from $F'(x)$, is similarly used to obtain $F_1'''(x)$ and $F_2'''(x)$ from $F_1''(x)$ and to obtain from $F_2''(x)$ the matrices $F_3'''(x)$ and $F_4'''(x)$. The objective function

$$nF'''(x) := \max\left[norme(F_1'''(x)), norme(F_2'''(x)), norme(F_3'''(x)), norme(F_4'''(x))\right].$$

We solve the nonlinear programming problem with the initial value $x := x^{(0)}$,

$$Given\ C(x) \leq 0 \quad gradt(F, c, r, m) \neq 0 \quad \xi := maximize(nF''', x),$$

then using the optimum value ξ, we determine the constant

$$M_3 := nF'''(\xi), \quad \text{namely} \quad M_3 = 54.$$

(3) Let

$$\begin{aligned} n_1(u,v) &:= norme(F_1'''(u) - F_1'''(v)), \\ n_2(u,v) &:= norme(F_2'''(u) - F_2'''(v)), \\ n_3(u,v) &:= norme(F_3'''(u) - F_3'''(v)), \\ n_4(u,v) &:= norme(F_4'''(u) - F_4'''(v)), \end{aligned}$$

and the objective function

$$L(u,v) := \frac{\max\left[n_1(u,v), n_2(u,v), n_3(u,v), n_4(u,v)\right]}{|u-v|}.$$

We solve the nonlinear programming problem, where the initial values are $u = (P_{1,1}\ P_{2,1})^{\mathrm{T}}$ and $v := \left(P_{1,\lfloor m/2\rfloor}\ P_{2,\lfloor m/2\rfloor}\right)^{\mathrm{T}}$.

$$Given\ C(u) \le 0 \quad C(v) \le 0 \quad gradt(F,c,r,m) \ne 0$$

$$\begin{pmatrix} \mu \\ \omega \end{pmatrix} := maximize(L,u,v),$$

using the optimal values μ and ω we will obtain the Lipschitz constant

$$L := L(\mu,\omega), \quad \text{namely} \quad L = 161.98364830993293.$$

Let

$$\begin{aligned} \beta(x) &:= norme(\Gamma(x)), \\ \eta(x) &:= |\mathcal{N}(x)|, \\ A(x) &:= M_2 \cdot \beta(x) \cdot \eta(x), \\ B(x) &:= M_3 \cdot \beta(x) \cdot \eta(x)^2, \\ E(x) &:= L \cdot \beta(x) \cdot \eta(x)^3. \end{aligned}$$

Let R be a polynomial

$$R(\tau) := 27(\tau-1)(2\tau-1)(\tau^2+\tau+2)(\tau^2+2\tau+4)$$

and the objective function

$$\rho(x) := |EH(x)|.$$

where the function EH is defined by the formula (11). The initial value $x := x^{(0)}$ is used for the nonlinear programming problem

$$\begin{aligned} &Given \\ &\quad 0 < A(x) < \frac{1}{2} \\ &\quad 0 < B(x) < \frac{Q(A(x)) - 17E(x)}{18 \cdot A(x)} \\ &\quad 0 < D(x) < \frac{Q(A(x))}{17} \\ &\quad gradt(F,c,r,m) \ne 0 \\ &s := Maximize(\rho, x). \end{aligned}$$

The solution for the nonlinear programming problem is

$$s = \begin{pmatrix} 2.993115896952277 \\ 2.1055148260544714 \end{pmatrix},$$

The value for the objective function, in the optimum solution s, is the radius of the quadruple convergence disk $D_c = \{x \mid |x-s| \leq \rho(s)\ x \in \mathbb{R}^2\}$, a *Ezquerro-Hernández method*,

$$\rho(s) = 0.10419021465653865.$$

Let us verify the condition (a_4) from the convergence theorem 5. Since

$$\left|x^{(0)} - s\right| = 0.2403438975646355 < r - \rho(s) = 0.29580978534346136,$$

it implies that $D_c \subset D_0 \subset D$. The disk D_c contains the solution $x^\star$ for the equation $F(x) = 0$. The sequence $\left\{x^{(k)}\right\}$, with $x^{(0)} = s$, converges with a *R-order* of convergence equal to 4 starting with $x^\star$, and all the terms of the sequence are contained in D_c. This solution, namely s is the initial iteration that ensures an *R-order* of convergence equal to 4 starting with the first step.

$$EzHe(s, 10^{-15}) = \begin{pmatrix} 2.9900922931613527 & 2.1055148260544714 \\ 3.0001882531510615 & 2.0015649217352602 \\ 3.0000000000184297 & 2.0000000001656666 \\ 3 & 2 \end{pmatrix}.$$

ACKNOWLEDGEMENT. Dedicated to professor Ştefan Măruşter on the occasion of his $70th$ birthday.

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Received by the editors: May 8, 2008.