REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 37 (2008) no. 2, pp. 143–149

ELEMENTARY SPLINE FUNCTIONS

GHEORGHE COMAN* and TEODORA CĂTINAŞ*

Abstract. The aim of this paper is to define the elementary spline functions in analogy with the definition of the polynomial spline functions, given by I. J. Schonberg. Also, it is described a method for constructing the elementary spline functions. Finally, some examples are given.

MSC 2000. 65D07.
Keywords. Elementary spline function, Cauchy problem, error estimation.

First, we present the I. J. Schoenberg's definition of polynomial spline function given in [7].

Definition 1. [7] *Let x_+^{n-1} denote the truncated power function defined as x^{n-1} if $x \geq 0$ and 0 if $x < 0$ $(n = 1, 2, ...)$. Let ξ_ν $(\nu = 1, ..., k)$ be a given finite sequence of increasing abscissae. By a* spline function *of degree $n-1$ we mean a function of the form*

$$S_{n-1,k}(x) = P_{n-1}(x) + \sum_{\nu=1}^{k} C_\nu (x - \xi_\nu)_+^{n-1},$$

where P_{n-1} is a polynomial of degree $\leq n-1$. Equivalently, this function may be defined by separate polynomials of degree $\leq n-1$ in each of the $k+1$ intervals $(-\infty, \xi_1)$, (ξ_2, ξ_3), ..., (ξ_k, ∞), such that the composite function has $n-2$ continuous derivatives for all real x.

Remark 2. As it is mentioned by I. J. Schoenberg, in [9], the polynomial spline functions were already used for approximation of functions by T. Popoviciu in [4]. In particular, he showed that a continuous non-concave function of order n in a finite interval $[a, b]$ is the uniform limit of elementary functions of order n that are also non-concave of order n in $[a, b]$ [4]. $\square$

The elementary spline functions will be introduced here as an approximate solution of a Cauchy problem regarding linear differential equations, using the method described in [3].

† "Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, Str. M. Kogălniceanu Nr. 1, 400084 Cluj-Napoca, Romania, e-mail: {`ghcoman, tcatinas`}@`math.ubbcluj.ro`.

One considers the Cauchy problem

$$y^{(n)} + a_1 y^{(n-1)} + ... + a_n y = 0 \tag{1}$$

and

$$\begin{cases} y(x_0) = y_0 \\ y'(x_0) = y_0' \\ ... \\ y^{(n-1)}(x_0) = y_0^{(n-1)}, \end{cases} \tag{2}$$

with $a_1, ..., a_n \in C[a,b]$.

We set $x_k \in [a,b]$, $k = 0, ..., m$ as points of a partition of the interval $[a,b]$, namely $a = x_0 < x_1 < ... < x_m = b$.

A method for approximating the solution y of the Cauchy problem (1)–(2), on the interval $[a,b]$, is that of attaching, on each subinterval $[x_{k-1}, x_k]$, $k = 1, ..., m$, a Cauchy problem regarding a linear differential equation with constant coefficients and to approximate the solution y, on the corresponding subinterval, with the solution of the attached problem.

More precisely, on each subinterval $[x_{k-1}, x_k]$, $k = 1, ..., m$, one considers the Cauchy problem:

$$y_k^{(n)} + \alpha_{k1} y_k^{(n-1)} + ... + \alpha_{kn} y_k = 0, \qquad \alpha_{ki} \in \mathbb{R},\ i = 1, ..., n; \tag{3}$$

$$\begin{cases} y_k(x_{k-1}) = y_{k-1}(x_{k-1}) \\ y_k'(x_{k-1}) = y_{k-1}'(x_{k-1}) \\ ... \\ y_k^{(n-1)}(x_{k-1}) = y_{k-1}^{(n-1)}(x_{k-1}), \qquad \text{for } k = 1, ..., m, \end{cases} \tag{4}$$

with

$$\begin{cases} y_0(x_0) = y_0 \\ y_0'(x_0) = y_0' \\ ... \\ y_0^{(n-1)}(x_0) = y_0^{(n-1)}. \end{cases}$$

The constants $\alpha_{ki} \in \mathbb{R}$, $i = 1, ..., n$; $k = 1, ..., m$ may be chosen in several ways. For example, they may be chosen as

$$\alpha_{ki} = a_i(\beta_k), \qquad \beta_k \in [x_{k-1}, x_k],\ (\text{e.g., } \beta_k = \tfrac{x_{k-1}+x_k}{2});$$
$$\alpha_{ki} = \min_{x \in [x_{k-1}, x_k]} |a_i(x)|;$$
$$\alpha_{ki} = \max_{x \in [x_{k-1}, x_k]} |a_i(x)|.$$

In this way, the solving of the Cauchy problem (1)–(2) is reduced to solving m Cauchy problems (3)–(4) regarding linear differential equations with constant coefficients.

Let us consider the characteristic equation of (3):

$$r_k^n + \alpha_{k1} r_k^{(n-1)} + ... + \alpha_{kn} = 0. \tag{5}$$

The solution y_k of the problem (3)–(4) depends on the nature of the solution of the characteristic equation (5).

As a consequence of this statement is that a fundamental system of solutions $y_{k1},...,y_{kn}$ of the equation (3) depends on the nature of the roots of the equation (5). Thus, if the characteristic equation (5) has:

- n real and distinct roots $r_{k1},...,r_{kn}$, then

$$y_{k1}(x) = \mathrm{e}^{r_{k1}x}, ..., y_{kn}(x) = \mathrm{e}^{r_{kn}x}.$$

- p multiple roots $r_{k1},...,r_{kp}$, with multiplicity orders, respectively, $\mu_{k1},...,\mu_{kp}$, (with $\mu_{k1}+...+\mu_{kp}=n$) then

$$y_{k1}(x) = \mathrm{e}^{r_{k1}x}; y_{k2}(x) = x\mathrm{e}^{r_{k1}x}; ...; y_{k,\mu_{k1}}(x) = x^{\mu_{k1}-1}\mathrm{e}^{r_{k1}x};$$
$$y_{k,\mu_{k1}+1}(x) = \mathrm{e}^{r_{k2}x}; y_{k,\mu_{k1}+2}(x) = x\mathrm{e}^{r_{k2}x}; ...; y_{k,\mu_{k1}+\mu_{k2}}(x) = x^{\mu_{k2}-1}\mathrm{e}^{r_{k2}x};$$
$$...$$
$$y_{k,n-\mu_{kp}}(x) = \mathrm{e}^{r_{kp}x}; y_{k,n-\mu_{kp}+1}(x) = x\mathrm{e}^{r_{kp}x}; ...; y_{k,n}(x) = x^{\mu_{kp}-1}\mathrm{e}^{r_{kp}x}.$$

- complex roots, for example, $r = u \pm \mathrm{i}v$, then the functions $\mathrm{e}^{ux}\cos vx$ and $\mathrm{e}^{ux}\sin vx$ are solutions of the equation (5).
- multiple complex roots, for example, $r = u+\mathrm{i}v$ with multiplicity order μ, then the functions

$$\begin{aligned}&\mathrm{e}^{ux}\cos vx,\ \mathrm{e}^{ux}\sin vx\\ &x\mathrm{e}^{ux}\cos vx,\ x\mathrm{e}^{ux}\sin vx\\ &...\\ &x^{\mu-1}\mathrm{e}^{ux}\cos vx,\ x^{\mu-1}\mathrm{e}^{ux}\sin vx\end{aligned}$$

  are solutions of the equation (5).

If $y_{k1},...,y_{kn}$ is a fundamental system of solutions, generated by the roots of the characteristic equation (5), then

$$y_k = c_1y_{k1} + ... + c_ny_{kn}, \qquad c_i \in \mathbb{R},\ i = 1,...,n,$$

is the solution of the Cauchy problem (3)–(4), with the constants $c_1,...,c_n$ obtained by the conditions (4).

Remark 3. The function $\bar{y} \in C^{n-1}[a,b]$ defined by $\bar{y}|_{[x_{k-1},x_k]} = y_k$, for all $k = 1,...,m$, is considered as an approximation of the solution y of the given problem (1)–(2), i.e., $y \approx \bar{y}$ on $[a,b]$, where $\bar{y}|_{[x_{k-1},x_k]}$ is the restriction of the function $\bar{y}$ to the interval $[x_{k-1},x_k]$. □

Remark 4. As y_k, $k = 1,...,m$ are elementary functions (polynomials, exponential or trigonometric functions), $\bar{y}$ is a piecewise elementary function. □

Similarly to the definition of the polynomial spline function, as a piecewise polynomial function, given by I. J. Schonberg, we have the following definition.

Definition 5. *The function $\bar{y}$, defined above, is called* an elementary spline function.

Because $\bar{y}$ was constructed as an approximation of the solution of the n-th order differential equation (1), Definition 5 can be completed by the following definition.

Definition 6. *The function $\bar{y}$ is an elementary spline function of order n.*

In the case of an uniform partition of the interval, for the approximation error we have the following estimation.

Proposition 7. *If $x_k = a + kh$, $k = 0, ..., m$, with $h = (b-a)/m$, is an uniform partition of the interval $[a, b]$ then the following estimation holds:*

$$\|y - \bar{y}\| \leq \varepsilon h \mathrm{e}^{Lh} \tfrac{1-\mathrm{e}^{(b-a)L}}{1-\mathrm{e}^{hL}},$$

where

$$L = \max\{1 + \|a_1\|, \|a_2\|, ..., \|a_n\|\}$$
$$\varepsilon = \max_{1\leq k\leq m}\{\|a_1 - \alpha_{k1}\|, ..., \|a_n - \alpha_{kn}\|\} \cdot \|y\|$$

Example 8. Suppose that $x_k = a + kh$, $k = 0, ..., m$; $h = (b-a)/m$ is a uniform partition of the interval $[a, b]$ and $\beta_k = (x_{k-1} + x_k)/2$, $k = 1, ..., m$.

We consider the following Cauchy problem:

$$\begin{cases} y'' + (2x-1)y = 0, & x \in [0,1] \\ y(0) = 1 \\ y'(0) = 0. \end{cases} \tag{6}$$

□

Case 1. Consider $x_k = kh$, $k = 0, ..., 4$, with $h = 1/4$. On the interval $[0, \frac{1}{4}]$ we have

$$\begin{cases} y_1'' - \frac{3}{4}y_1 = 0, \\ y_1(0) = 1 \\ y_1'(0) = 0. \end{cases}$$

As the characteristic equation in this case has two real and distinct roots, $r_{1,2} = \pm\frac{\sqrt{3}}{2}$, it follows that

$$y_1(x) = c_1 \mathrm{e}^{-\frac{\sqrt{3}}{2}x} + c_2 \mathrm{e}^{\frac{\sqrt{3}}{2}x},$$

and after some computation we get the constants

$$c_1 = c_2 = \tfrac{1}{2}.$$

On the interval $[\frac{1}{4}, \frac{1}{2}]$ we obtain

$$\begin{cases} y_2'' - \frac{1}{4}y_2 = 0, \\ y_2(\frac{1}{4}) = y_1(\frac{1}{4}) \\ y_2'(\frac{1}{4}) = y_1'(\frac{1}{4}). \end{cases}$$

and it follows

$$y_2(x) = c_1' \mathrm{e}^{-\frac{1}{2}x} + c_2' \mathrm{e}^{\frac{1}{2}x},$$

with

$$c_1' = \tfrac{1+\sqrt{3}}{4} \mathrm{e}^{\frac{(1-\sqrt{3})}{8}} + \tfrac{1-\sqrt{3}}{4} \mathrm{e}^{\frac{(1+\sqrt{3})}{8}}$$
$$c_2' = \tfrac{1-\sqrt{3}}{4} \mathrm{e}^{-\frac{(1+\sqrt{3})}{8}} + \tfrac{1+\sqrt{3}}{4} \mathrm{e}^{\frac{(\sqrt{3}-1)}{8}}.$$

In the same way we get

$$y_3(x) = c_1'' \cos \tfrac{x}{2} + c_2'' \sin \tfrac{x}{2}, \qquad x \in [\tfrac{1}{2}, \tfrac{3}{4}],$$

with

$$c_2'' = (-c_1' \mathrm{e}^{-\frac{1}{4}} + c_2' \mathrm{e}^{\frac{1}{4}}) \cos \tfrac{1}{4} + (c_1' \mathrm{e}^{-\frac{1}{4}} + c_2' \mathrm{e}^{\frac{1}{4}}) \sin \tfrac{1}{4}$$
$$c_1'' = (-c_1' \mathrm{e}^{-\frac{1}{4}} + c_2' \mathrm{e}^{\frac{1}{4}} - c_2'' \sin \tfrac{1}{4}) \Big/ \cos \tfrac{1}{4}$$

and

$$y_4(x) = c_1''' \cos \tfrac{\sqrt{3}}{2} x + c_2''' \sin \tfrac{\sqrt{3}}{2} x, \qquad x \in [\tfrac{3}{4}, 1]$$

with

$$c_2''' = \tfrac{1}{\sqrt{3}} \Big(c_1'' (\sqrt{3} \sin \tfrac{3\sqrt{3}}{8} \cos \tfrac{3}{8} - \sin \tfrac{3}{8} \cos \tfrac{3\sqrt{3}}{8}) + c_2'' (\sqrt{3} \sin \tfrac{3\sqrt{3}}{8} \sin \tfrac{3}{8} + \cos \tfrac{3}{8} \cos \tfrac{3\sqrt{3}}{8}) \Big)$$
$$c_1''' = (c_1'' \cos \tfrac{3}{8} + c_2'' \sin \tfrac{3}{8} - c_2''' \sin \tfrac{3\sqrt{3}}{8}) \Big/ \cos \tfrac{3\sqrt{3}}{8}.$$

Remark 9. The solution of the problem (6), $\bar{y}$, has the following properties: $\bar{y} \in C^1[0,1]$ and $\bar{y}|_{[0,\frac{1}{2}]}$ is an exponential function, respectively, $\bar{y}|_{[\frac{1}{2},1]}$ is a trigonometric function.

For the error we have

$$\|y - \bar{y}\| \le \tfrac{7}{16} \mathrm{e}^{\frac{1}{4}} \frac{1-\mathrm{e}}{1-\mathrm{e}^{\frac{1}{4}}} \|y\| .$$

□

In Figure 1 we plot the graph of the solution $\bar{y}$, for Case 1.

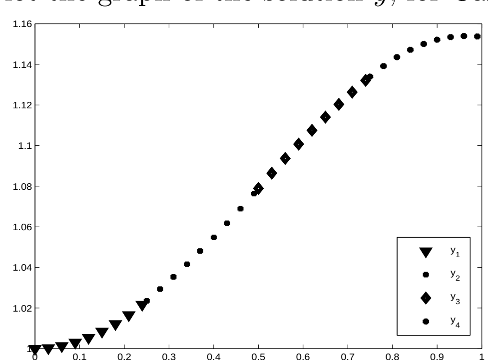

Figure 1: Graph of $\bar{y}$.

Case 2. Consider $x_k = kh$, $k = 0, ..., 3$, with $h = 1/3$. On interval $[0, \frac{1}{3}]$ we have

$$\begin{cases} y_1'' - y_1 = 0, & x \in [0, \frac{1}{3}] \\ y_1(0) = 1 \\ y_1'(0) = 0. \end{cases}$$

Hence, we get

$$y_1(x) = c_1 \mathrm{e}^{-\frac{\sqrt{3}}{2}x} + c_2 \mathrm{e}^{\frac{\sqrt{3}}{2}x},$$

with the constants

$$c_1 = c_2 = \tfrac{1}{2}.$$

If $x \in [\frac{1}{3}, \frac{2}{3}]$ we obtain

$$\begin{cases} y_2'' = 0, & x \in [\frac{1}{3}, \frac{2}{3}] \\ y_2(\frac{1}{3}) = y_1(\frac{1}{3}) \\ y_2'(\frac{1}{3}) = y_1'(\frac{1}{3}). \end{cases}$$

It follows that

$$y_2(x) = c_1' + c_2' x,$$

and after some computation we get the constants

$$\begin{aligned} c_1' &= (\tfrac{1}{2} + \tfrac{\sqrt{3}}{12})\mathrm{e}^{-\frac{\sqrt{3}}{6}} + (\tfrac{1}{2} - \tfrac{\sqrt{3}}{12})\mathrm{e}^{\frac{\sqrt{3}}{6}} \\ c_2' &= \tfrac{\sqrt{3}}{4}(\mathrm{e}^{\frac{\sqrt{3}}{6}} - \mathrm{e}^{-\frac{\sqrt{3}}{6}}). \end{aligned}$$

For $x \in [\frac{2}{3}, 1]$ we have

$$\begin{cases} y_3'' + \frac{2}{3} y_3 = 0, & x \in [\frac{2}{3}, 1] \\ y_3(\frac{2}{3}) = y_2(\frac{2}{3}) \\ y_3'(\frac{2}{3}) = y_2'(\frac{2}{3}). \end{cases}$$

Hence, we get

$$y_3(x) = c_1'' \cos\sqrt{\tfrac{2}{3}}x + c_2'' \sin\sqrt{\tfrac{2}{3}}x,$$

with

$$\begin{aligned} c_2'' &= c_1' \sin\tfrac{2}{3}\sqrt{\tfrac{2}{3}} + c_2' \left(\tfrac{2}{3}\sin\tfrac{2}{3}\sqrt{\tfrac{2}{3}} + \sqrt{\tfrac{3}{2}}\cos\tfrac{2}{3}\sqrt{\tfrac{2}{3}} \right) \\ c_1'' &= \left(c_1' + \tfrac{2}{3}c_2' - c_2'' \sin\tfrac{2}{3}\sqrt{\tfrac{2}{3}}\right) \Big/ \cos\tfrac{2}{3}\sqrt{\tfrac{2}{3}}. \end{aligned}$$

Remark 10. In this case, the solution of the problem (6), $\bar{y}$, has the following properties: $\bar{y} \in C^1[0,1]$ and $\bar{y}|_{[0,\frac{1}{3}]}$ is an exponential function, $\bar{y}|_{[\frac{1}{3},\frac{2}{3}]}$ is a polynomial function and $\bar{y}|_{[\frac{2}{3},1]}$ is a trigonometric function.

For the error we have

$$\|y - \bar{y}\| \le \tfrac{5}{9}\mathrm{e}^{\frac{1}{3}} \frac{1-\mathrm{e}}{1-\mathrm{e}^{\frac{1}{3}}} \|y\|.$$

□

In Figure 2 we plot the graph of the solution $\bar{y}$, for Case 2.

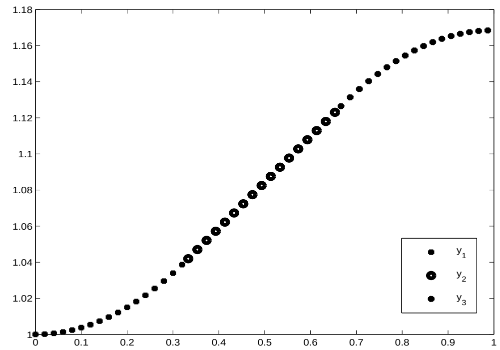

FIGURE 2: Graph of $\bar{y}$.

REFERENCES

[1] COMAN, GH., *Numerical Analysis,* Ed. Libris, Cluj-Napoca, 1995 (in Romanian).

[2] IONESCU, D. V., *Applications of the succesive approximation method to the numerical integration of differential equations,* Studii şi cercetări matematice, Cluj-Napoca, **11**, pp. 273–286, 1960 (in Romanian).

[3] PAVEL, G., *The approximative solution of the Cauchy problem for a linear differential equation of order n*, Studia Universitatis Babeş-Bolyai, Mathematica, no. 2, pp. 49–53, 1978 (in Romanian).

[4] POPOVICIU, T., *Notes sur les fonctions convexes d'ordre superieur*, Bull. Math. de la Societe Roumaine des Science, **43**, pp. 95–141, 1941.

[5] RUS, I. A., *Differential equations, integral equations and dynamical systems,* Ed. House Transilvania Press, Cluj-Napoca, 1996.

[6] SCHONBERG, I. J., *Contributions to the problem of approximations of equidistant data by analytic functions, Quart. Appl. Math.,* **4**, no. 1, pp. 45–99, 1946.

[7] SCHONBERG, I. J., *Spline functions, convex curves and mechanical quadrature,* Bull. Amer. Math. Soc., **64**, no. 1, pp. 352–357, 1958.

[8] SCHONBERG, I. J., O*n trigonometric spline interpolation,* J. Mathematics and Mechanics, **13**, no. 5, pp. 795–825, 1964.

[9] SCHONBERG, I. J., *On spline interpolation at all integer points of the real axis,* Mathematica, **10** (**33**), no. 1, pp. 151–170, 1968.

Received by the editors: June 17, 2008.