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# APPROXIMATION BY COMPLEX BERNSTEIN-KANTOROVICH AND STANCU-KANTOROVICH POLYNOMIALS AND THEIR ITERATES IN COMPACT DISKS*

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**Abstract.** In this paper, Voronovskaja-type results with quantitative upper estimates and the exact orders in simultaneous approximation by some complex Kantorovich-type polynomials and their iterates in compact disks in $\mathbb{C}$ are obtained.

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## 1. INTRODUCTION AND AUXILIARY RESULTS

The complex Bernstein polynomials, the complex Bernstein-Stancu polynomials depending on two parameters $0 \le \alpha \le \beta$ and the complex Bernstein-Stancu polynomials depending on one parameter $0 \le \gamma$, are defined by the same formulas as in the case of real variable, by

$$B_n(f)(z) = \sum_{k=0}^{n} p_{n,k}(z) f(k/n), \text{ see e.g. [9]},$$

$$S_n^{(\alpha,\beta)}(f)(z) = \sum_{k=0}^{n} p_{n,k}(z) f[(k+\alpha)/(n+\beta)], \text{ see [14]},$$

$$S_n^{<\gamma>}(f)(z) = \sum_{k=0}^{n} p_{n,k}^{<\gamma>}(z) f(k/n), \text{ see [13]},$$

respectively, where $z \in \mathbb{C}$, $p_{n,k}(z) = \binom{n}{k} z^k (1-z)^{n-k}$ and

$$p_{n,k}^{<\gamma>}(z) = \binom{n}{k} \frac{z(z+\gamma)...(z+(k-1)\gamma)(1-z)(1-z+\gamma)...(1-z+(n-k-1)\gamma)}{(1+\gamma)(1+2\gamma)...(1+(n-1)\gamma)}.$$

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In the very recent book [2] and the papers [3], [4], [5], results on simultaneous approximation and of Voronovskaja-type, with quantitative estimates in compact disks, for the above defined complex Bernstein-type polynomials and their iterates were obtained.

The main aim of this paper is to extend these kind of results to the following Kantorovich variants of these polynomials, defined by

$$K_n(f)(z) = (n+1)\sum_{k=0}^{n} p_{n,k}(z)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\mathrm{d}t, \text{ see [8]},$$

and

$$K_n^{(\alpha,\beta)}(f)(z) = (n+1+\beta)\sum_{k=0}^{n} p_{n,k}(z)\int_{(k+\alpha)/(n+1+\beta)}^{(k+1+\alpha)/(n+1+\beta)} f(t)\mathrm{d}t, \text{ see [1]}.$$

For our purpose, we need the following known results.

Theorem 1.1. *Let $f : \mathbb{D}_R \to \mathbb{C}$ be analytic in $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ with $R > 1$, i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$. Suppose $1 \le r < r_1 < R$. Then for all $|z| \le r$ and $n, p \in \mathbb{N}$, we have:*

(i) (a) *(see [2, pp. 264, Theorem 3.4.1 (v)] or [4 , Theorem 2.1, the case $\alpha = \beta = 0$])*

$$|B_n^{(p)}(f)(z) - f^{(p)}(z)| \le \frac{M_{2,r_1}(f)p!r_1}{n(r_1-r)^{p+1}},$$

*where $0 < M_{2,r_1}(f) = 2\sum_{j=2}^{\infty} j(j-1)|c_j|r_1^j < \infty$;*

(b) *(see [3, Theorem 2.1 (ii)])*

$$\left|B_n(f)(z) - f(z) - \frac{z(1-z)}{2n}f''(z)\right| \le \frac{5(1+r)^2}{2n} \cdot \frac{M_r(f)}{n},$$

*where $M_r(f) = \sum_{k=3}^{\infty} |c_k|k(k-1)(k-2)^2 r^{k-2} < \infty$;*

(ii) (a) *(see [4, Theorem 2.1])*

$$|[S_n^{(\alpha,\beta)}(f)]^{(p)}(z) - f^{(p)}(z)| \le \frac{M_{2,r_1}^{(\beta)}(f)p!r_1}{(n+\beta)(r_1-r)^{p+1}},$$

*where $0 < M_{2,r_1}^{(\beta)}(f) = 2\sum_{j=2}^{\infty} j(j-1)|c_j|r_1^j + 2\beta r\sum_{j=1}^{\infty} j|c_j|r^{j-1} < \infty$;*

(b) *(see [4, proof of the Theorem 2.2])*

$$\left|S_n^{(\alpha,\beta)}(f)(z) - f(z) + \frac{\beta z-\alpha}{n+\beta}f'(z) - \frac{nz(1-z)}{2(n+\beta)^2}f''(z)\right| = O\left[\frac{1}{(n+\beta)^2}\right],$$

*where the positive constant in $O(1/(n+\beta)^2)$ depends on $f, r, \alpha$ and $\beta$, but is independent of $n$ and $z$;*

(c) *(see* [4, *Theorem* 3.2]*) Denoting the* $m$*th iterate by* ${}^{m}S_n^{(\alpha,\beta)}(f)(z)$*, we have*

$$|{}^{m}S_n^{(\alpha,\beta)}(f)(z) - f(z)| \leq \frac{2m}{n+\beta}\sum_{k=1}^{\infty}|c_k|\cdot|\beta k + k(k-1)|r^k;$$

(iii) *(see* [5, *Theorem* 2.1]*)*

$$|[S_n^{<\gamma>}(f)]^{(p)}(z) - f^{(p)}(z)| \leq \frac{M_{2,r_1,n}^{<\gamma>}(f)p!r_1}{(r_1-r)^{p+1}},$$

*where*

$$0 < M_{2,r_1,n}^{<\gamma>}(f) = \frac{2}{n}\sum_{j=2}^{\infty}j(j-1)|c_j|r_1^j + \frac{\gamma(r_1+1)}{6r_1}\sum_{j=2}^{\infty}j(j-1)(2j-1)|c_j|r_1^j < \infty.$$

(iv) *(see* [6, *Theorem* 3.1]*) If* $f$ *is not a polynomial of degree* $\leq \max\{1, p-1\}$*, then we have*

$$\|B_n^{(p)}(f) - f^{(p)}\|_r \sim \frac{1}{n},$$

*where* $||f||_r = \sup\{|f(z)|; |z| \leq r\}$ *and the constants in the equivalence depend only on* $f$*,* $r$ *and* $p$*.*

(v) *(see* [7, *Theorem* 3.1]*) Let* $0 \leq \alpha \leq \beta$ *with* $\alpha + \beta > 0$*. If* $f$ *is not a polynomial of degree* $\leq p-1$ *then we have*

$$\|[S_n^{(\alpha,\beta)}(f)]^{(p)} - f^{(p)}\|_r \sim \frac{1}{n+\beta},$$

*where the constants in the equivalence depend only on* $f$*,* $\alpha$*,* $\beta$*,* $r$ *and* $p$*.*

Remark 1.2. The Voronovskaja-type result in [4, Theorem 2.2] holds for $|z| \leq 1$. The proof of the above point (ii) (b), is immediate by replacing in the proof of Theorem 2.2 in [4] the condition $|z| \leq 1$ by $|z| \leq r$. □

## 2. COMPLEX BERNSTEIN-KANTOROVICH POLYNOMIALS

For our purpose also will be useful the next classical result.

Theorem 2.1. *(see e.g.* [9, *pp.* 30]*) Denoting* $F(z) = \int_0^z f(t)\mathrm{d}t$*, we have the relationship*

$$K_n(f)(z) = B'_{n+1}(F)(z), z \in \mathbb{C}.$$

Now, as a consequence of Theorem 2.1 and Theorem 1.1, (iv), we immediately get the following.

Corollary 2.2. *Let* $f : \mathbb{D}_R \to \mathbb{C}$ *be analytic in* $\mathbb{D}_R$ *with* $R > 1$ *and* $1 \leq r < R$*.*

(i) *If* $f$ *is not a polynomial of degree* $\leq 0$ *then for all* $n \in \mathbb{N}$ *we have*

$$\|K_n(f) - f\|_r \sim \frac{1}{n},$$

*where the constants in the equivalence depend only on* $f$ *and* $r$*.*

(ii) *If $f$ is not a polynomial of degree $\leq \max\{1, p-1\}$ then for all $p, n \in \mathbb{N}$ we have*

$$\|K_n^{(p)}(f) - f^{(p)}\|_r \sim \frac{1}{n},$$

*with the constants in the equivalence depending only on $f$, $r$ and $p$.*

*Proof.* We combine Theorem 2.1, (i) with Theorem 1.1, (iv).

(i) We get

$$\|K_n(f) - f\|_r = \|B'_{n+1}(F) - F'\|_r \sim \frac{1}{n+1},$$

if $F$ is not a polynomial of degree $\leq \max\{1, 1\} = 1$, which ends the proof.

(ii) We obtain

$$\|K_n^{(p)}(f) - f^{(p)}\|_r = \|B_{n+1}^{(p+1)}(F) - F^{(p+1)}\|_r \sim \frac{1}{n+1},$$

if $F$ is not a polynomial of degree $\leq \max\{1, p\} = p$, which ends the proof.

□

Upper estimates with explicit constants in Voronovskaja's theorem and in approximation by $K_n(f)$ can be derived as follows.

THEOREM 2.3. *Let $f : \mathbb{D}_R \to \mathbb{C}$ be analytic in $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ with $R > 1$, i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$. Suppose $1 \leq r < r_1 < R$. Then for all $|z| \leq r$ and $n, p \in \mathbb{N}$, we have:*

(i)

$$|K_n^{(p)}(f)(z) - f^{(p)}(z)| \leq \frac{C_{2,r_1}(f)(p+1)!r_1}{(n+1)(r_1-r)^{p+2}},$$

*where* $0 < C_{2,r_1}(f) = 2\sum_{j=2}^{\infty}(j-1)|c_{j-1}|r_1^j < \infty$ *;*

(ii)

$$\left|K_n(f)(z) - f(z) - \frac{1-2z}{2(n+1)} \cdot f'(z) - \frac{z(1-z)}{2(n+1)} \cdot f''(z)\right| \leq \frac{r_1 C_{r_1,n+1}(f)}{(r_1-r)^2},$$

*where*

$$C_{r_1,n}(f) = \frac{5(1+r_1)^2}{2n} \cdot \frac{\sum_{k=3}^{\infty} |c_{k-1}|(k-1)(k-2)^2 r_1^{k-2}}{n}.$$

*Proof.* (i) Combining Theorem 2.1 with Theorem 1.1, (i) (a), we obtain

$$|K_n^{(p)}(f)(z) - f^{(p)}(z)| = |B_{n+1}^{(p+1)}(F)(z) - F^{(p+1)}(z)| \leq \frac{M_{2,r_1}(F)(p+1)!r_1}{(n+1)(r_1-r)^{p+2}},$$

where $0 < M_{2,r_1}(F) = 2\sum_{j=2}^{\infty} j(j-1)|C_j|r_1^j < \infty$ and $F(z) = \sum_{k=0}^{\infty} C_k z^k$, $z \in \mathbb{D}_R$. But we also get

$$F(z) = \int_0^z \left[\sum_{k=0}^{\infty} c_k t^k\right] \mathrm{d}t = \sum_{k=0}^{\infty} \frac{c_k}{k+1} z^{k+1} = \sum_{k=1}^{\infty} \frac{c_{k-1}}{k} z^k,$$

which implies $C_k = \frac{c_{k-1}}{k}$ and $C_{2,r_1}(f) = 2\sum\limits_{j=2}^{\infty}(j-1)|c_{j-1}|r_1^j$.

(ii) Replacing in Theorem 1.1, (i) (b), $n$ by $n+1$, $r$ by $r_1$ and $f$ by $F$, for all $|z| \le r_1$ and $n \in \mathbb{N}$, we obtain

$$\left|B_{n+1}(F)(z) - F(z) - \tfrac{z(1-z)}{2(n+1)}F''(z)\right| \le \tfrac{5(1+r_1)^2}{2(n+1)} \cdot \tfrac{M_{r_1}(F)}{n+1},$$

where

$$\begin{aligned} M_{r_1}(F) &= \sum_{k=3}^{\infty} |C_k| k(k-1)(k-2)^2 r_1^{k-2} = \\ &= \sum_{k=3}^{\infty} |c_{k-1}|(k-1)(k-2)^2 r_1^{k-2} := A_{r_1}(f). \end{aligned}$$

Here again we wrote $F(z) = \sum\limits_{k=0}^{\infty} C_k z^k$, for all $z \in \mathbb{D}_R$.

Now, denoting $C_{r_1,n}(f) = \frac{5(1+r_1)^2}{2n} \cdot \frac{A_{r_1}(f)}{n}$, by $\Gamma$ the circle of radius $r_1 > r$ and center 0, and $E_n(F)(z) = B_{n+1}(F)(z) - F(z) - \frac{z(1-z)}{2(n+1)}F''(z)$, since for any $|z| \le r$ and $v \in \Gamma$, we have $|v - z| \ge r_1 - r$, by the Cauchy's formula it follows that for all $|z| \le r$ and $n \in \mathbb{N}$, we obtain

$$|E_n'(F)(z)| = \tfrac{1}{2\pi}\left|\int_{\Gamma} \tfrac{E_n(f)(z)}{(v-z)^2}\mathrm{d}v\right| \le C_{r_1,n+1}(f)\tfrac{1}{2\pi}\tfrac{2\pi r_1}{(r_1-1)^2} = C_{r_1,n+1}(f)\cdot\tfrac{r_1}{(r_1-r)^2}.$$

But by Theorem 2.1 we obtain

$$E_n'(F)(z) = K_n(f)(z) - f(z) - \tfrac{1-2z}{2(n+1)} \cdot f'(z) - \tfrac{z(1-z)}{2(n+1)} \cdot f''(z),$$

which proves the theorem. □

## 3. COMPLEX STANCU-KANTOROVICH POLYNOMIALS DEPENDING ON TWO PARAMETERS

For our purpose will be useful the next result.

THEOREM 3.1. *Denoting $F(z) = \int_0^z f(t)\mathrm{d}t$, we have the relationship*

$$K_n^{(\alpha,\beta)}(f)(z) = \tfrac{n+1+\beta}{n+1}[S_{n+1}^{(\alpha,\beta)}(F)]'(z), z \in \mathbb{C}.$$

*Proof.* The theorem is immediate by the following formula

$$\begin{aligned} [S_{n+1}^{(\alpha,\beta)}(F)]'(z) &= (n+1+\beta)\sum_{k=0}^{n} p_{n,k}(z)\left[F\left(\tfrac{k+\alpha+1}{n+\beta+1}\right) - F\left(\tfrac{k+\alpha}{n+1+\beta}\right)\right] \\ &\quad - \beta\sum_{k=0}^{n} p_{n,k}(z)\left[F\left(\tfrac{k+\alpha+1}{n+\beta+1}\right) - F\left(\tfrac{k+\alpha}{n+1+\beta}\right)\right] \\ &= K_n^{(\alpha,\beta)}(f)(z) - \tfrac{\beta}{n+1+\beta}K_n^{(\alpha,\beta)}(f)(z). \end{aligned}$$

□

As a consequence of Theorem 3.1 and Theorem 1.1, (v), we also get the following.

COROLLARY 3.2. *Let $f : \mathbb{D}_R \to \mathbb{C}$ be analytic in $\mathbb{D}_R$ with $R > 1$, $1 \le r < R$ and $0 \le \alpha \le \beta$, $\alpha + \beta > 0$.*

(i) *If $f$ is not identical 0, then for all $n \in \mathbb{N}$ we have*

$$\|K_n^{(\alpha,\beta)}(f) - f\|_r \sim \tfrac{1}{n+\beta},$$

*where the constants in the equivalence depend only on $f$, $r$, $\alpha$ and $\beta$.*

(ii) *If $f$ is not a polynomial of degree $\le p-1$ then for all $p, n \in \mathbb{N}$ we have*

$$\|[K_n^{(\alpha,\beta)}(f)]^{(p)} - f^{(p)}\|_r \sim \tfrac{1}{n+\beta},$$

*with the constants in the equivalence depending only on $f$, $r$, $\alpha$, $\beta$ and $p$.*

*Proof.* We combine Theorem 3.1 with Theorem 1.1, (v).

(i) We get

$$\|K_n^{(\alpha,\beta)}(f) - f\|_r = \|[S_{n+1}^{(\alpha,\beta)}(F)]' - F'\|_r \sim \tfrac{1}{n+\beta},$$

if $F$ is not a polynomial of degree $\le 0$, which ends the proof.

(ii) We obtain

$$\|[K_n^{(\alpha,\beta)}(f)]^{(p)} - f^{(p)}\|_r = \|[S_{n+1}^{(\alpha,\beta)}(F)]^{(p+1)} - F^{(p+1)}\|_r \sim \tfrac{1}{n+\beta},$$

if $F$ is not a polynomial of degree $\le p$, which ends the proof. $\square$

Upper estimates with explicit constants in Voronovskaja's theorem and in approximation by $K_n^{(\alpha,\beta)}(f)(z)$ polynomials can be derived as follows.

THEOREM 3.3. *Let $f : \mathbb{D}_R \to \mathbb{C}$ be analytic in $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ with $R > 1$, i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$. Suppose $1 \le r < r_1 < R$. Then for all $|z| \le r$ and $n, p \in \mathbb{N}$, we have:*

(i)

$$|[K_n^{(\alpha,\beta)}(f)]^{(p)}(z) - f^{(p)}(z)| \le \tfrac{C_{2,r_1}^{(\beta)}(f)(p+1)!r_1}{(n+1)(r_1-r)^{p+2}} + \tfrac{\beta}{n+1}||f||_r,$$

*where* $0 < C_{2,r_1}^{(\beta)}(f) = 2\sum_{j=2}^{\infty}(j-1)|c_{j-1}|r_1^j + 2\beta\sum_{j=1}^{\infty}|c_{j-1}|r_1^j < \infty$ ;

(ii)

$$\left|K_n^{(\alpha,\beta)}(f)(z) - f(z) + \left(\tfrac{\beta z-\alpha}{n+1} - \tfrac{1-2z}{2(n+\beta+1)}\right)f'(z) - \tfrac{z(1-z)}{2(n+\beta+1)}f''(z)\right| \le$$
$$\le \tfrac{C(f,r_1,\alpha,\beta)}{(n+1)(n+\beta+1)} \cdot \tfrac{r_1}{(r_1-r)^2},$$

*where $C(f, r_1, \alpha, \beta)$ is a positive constant depending only on $f$, $r_1$, $\alpha$ and $\beta$.*

*Proof.* (i) Combining Theorem 3.1 with Theorem 1.1, (ii) (a), for all $|z| \le r$ we obtain

$$\begin{aligned}
&|[K_n^{(\alpha,\beta)}(f)]^{(p)}(z) - f^{(p)}(z)| = \\
&= \left|\tfrac{n+1+\beta}{n+1}[S_{n+1}^{(\alpha,\beta)}(F)]^{(p+1)}(z) - F^{(p+1)}(z)\right| \\
&\le \tfrac{n+1+\beta}{n+1}\left|[S_{n+1}^{(\alpha,\beta)}(F)]^{(p+1)}(z) - F^{(p+1)}(z)\right| + \tfrac{\beta}{n+1}|F^{(p+1)}(z)| \\
&\le \tfrac{n+1+\beta}{n+1}\cdot\tfrac{M_{2,r_1}^{(\beta)}(F)(p+1)!r_1}{(n+\beta+1)(r_1-r)^{p+2}} + \tfrac{\beta}{n+1}\cdot|f^{(p)}(z)| \\
&\le \tfrac{M_{2,r_1}^{(\beta)}(F)(p+1)!r_1}{(n+1)(r_1-r)^{p+2}} + \tfrac{\beta}{n+1}\cdot||f^{(p)}||_r,
\end{aligned}$$

and reasoning exactly as in the proof of Theorem 2.3, (i), we get

$$\begin{aligned}
M_{2,r_1}^{(\beta)}(F) &= 2\sum_{j=2}^{\infty} j(j-1)|C_j|r_1^j + 2\beta\sum_{j=1}^{\infty} j|C_j|r_1^j \\
&= 2\sum_{j=2}^{\infty}(j-1)|c_{j-1}|r_1^j + 2\beta\sum_{j=1}^{\infty}|c_{j-1}|r_1^j := C_{2,r_1}^{(\beta)}(f).
\end{aligned}$$

(ii) Replacing in Theorem 1.1, (ii) (b), $n$ by $n+1$, $r$ by $r_1$ and $f$ by $F$, for all $|z| \le r_1$ and $n \in \mathbb{N}$, we obtain

$$\left|S_{n+1}^{(\alpha,\beta)}(F)(z) - F(z) + \tfrac{\beta z-\alpha}{n+\beta+1}F'(z) - \tfrac{(n+1)z(1-z)}{2(n+\beta+1)^2}F''(z)\right| \le \tfrac{C(f,r_1,\alpha,\beta)}{(n+\beta+1)^2},$$

where the positive constant $C(f, r_1, \alpha, \beta)$ depends only on $f, r, \alpha$ and $\beta$. Let us denote

$$E_n(F)(z) = S_{n+1}^{(\alpha,\beta)}(F)(z) - F(z) + \tfrac{\beta z-\alpha}{n+\beta+1}F'(z) - \tfrac{(n+1)z(1-z)}{2(n+\beta+1)^2}F''(z).$$

If $\Gamma$ is the circle of radius $r_1 > r$ and center 0, and since for any $|z| \le r$ and $v \in \Gamma$, we have $|v - z| \ge r_1 - r$, by the Cauchy's formula it follows that for all $|z| \le r$ and $n \in \mathbb{N}$, we obtain as in the proof of Theorem 2.3, (ii)

$$|E_n'(F)(z)| \le C(f, r_1, \alpha, \beta)\cdot\tfrac{r_1}{(r_1-r)^2}\cdot\tfrac{1}{(n+\beta+1)^2}.$$

But by Theorem 3.1 we obtain

$$\begin{aligned}
E_n'(F)(z) &= \tfrac{n+1}{n+1+\beta}K_n^{(\alpha,\beta)}(f)(z) - f(z) + \tfrac{1}{n+\beta+1}[(\beta z-\alpha)f(z)]' \\
&\quad - \tfrac{n+1}{2(n+\beta+1)^2}[(z-z^2)f'(z)]' \\
&= \tfrac{n+1}{n+\beta+1}\cdot A,
\end{aligned}$$

where

$$A = K_n^{(\alpha,\beta)}(f)(z) - f(z) + f'(z)\left(\tfrac{\beta z-\alpha}{n+1} - \tfrac{1-2z}{2(n+\beta+1)}\right) - \tfrac{z(1-z)}{2(n+\beta+1)}f''(z),$$

which immediately proves the theorem. □

Concerning the $m$th iterates ${}^mK_n^{(\alpha,\beta)}(f)(z)$, we obtain the following result.

THEOREM 3.4. *Let $f : \mathbb{D}_R \to \mathbb{C}$ be analytic in $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ with $R > 1$, i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$. Suppose $1 \le r < r_1 < R$. Then for all $|z| \le r$ and $n, p \in \mathbb{N}$, we have*

$$|[{}^m K_n^{(\alpha,\beta)}(f)]^{(p)}(z) - f^{(p)}(z)| \le \frac{2m}{n+1+\beta} \sum_{k=1}^{\infty} |c_{k-1}| \cdot |\beta + (k-1)| r^k \cdot \frac{(p+1)! r_1}{(r_1 - r)^{p+1}}.$$

*Proof.* First we easily observe that

$${}^m K_n^{(\alpha,\beta)}(f)(z) = \frac{\mathrm{d}}{\mathrm{d}z}[{}^m S_{n+1}^{(\alpha,\beta)}(F)](z),$$

where $F(z) = \int_0^z f(t)\mathrm{d}t = \sum_{k=0}^{\infty} C_k z^k$. Taking into account Theorem 1.1, (ii) (c), the Cauchy's theorem and reasoning exactly as in the proofs of Theorem 2.3, (i) and 3.3, (i), it follows

$$\begin{aligned}
|[{}^m K_n^{(\alpha,\beta)}(f)]^{(p)}(z) - f^{(p)}(z)| &= |[{}^m S_{n+1}^{(\alpha,\beta)}(F)]^{(p+1)}(z) - F^{(p+1)}(z)| \\
&\le \frac{2m}{n+1+\beta} \sum_{k=1}^{\infty} |C_k| \cdot |\beta k + k(k-1)| r^k \cdot \frac{(p+1)! r_1}{(r_1 - r)^{p+1}} \\
&= \frac{2m}{n+1+\beta} \sum_{k=1}^{\infty} |c_{k-1}| \cdot |\beta + (k-1)| r^k \cdot \frac{(p+1)! r_1}{(r_1 - r)^{p+1}},
\end{aligned}$$

which proves the theorem. □

REMARK 3.5. For $\beta = 0$ in Theorem 3.4 we get corresponding results for the iterates of classical complex Kantorovich polynomials. Note that in the real case, some asymptotic results for the iterates of Kantorovich polynomials were obtained in [10]. □

REMARK 3.6. If $\frac{m_n}{n} \to 0$ when $n \to \infty$, then by Theorem 3.4 it is immediate that

$$[{}^{m_n} K_n^{(\alpha,\beta)}(f)]^{(p)}(z) \to f^{(p)}(z),$$

uniformly with respect to $|z| \le 1$, for any $1 \le r < R$. □

REMARK 3.7. The Stancu-Kantorovich polynomials depending on the parameter $0 \le \gamma$ were introduced in [12] by

$$K_n^{<\gamma>}(f)(z) = (n+1) \sum_{k=0}^{n} p_{n,k}^{<\gamma>}(z) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\mathrm{d}t,$$

where

$$p_{n,k}^{<\gamma>}(z) = \binom{n}{k} \frac{z(z+\gamma)\ldots(z+(k-1)\gamma)(1-z)(1-z+\gamma)\ldots(1-z+(n-k-1)\gamma)}{(1+\gamma)(1+2\gamma)\ldots(1+(n-1)\gamma)}.$$

To prove analogous results for these polynomials too, we would need a similar connection between $[S_{n+1}^{<\gamma>}(F)]'(z)$ and $K_n^{<\gamma>}(f)(z)$, with those in Theorems 2.1 and 3.1. But this study is left as an open question. □

Remark 3.8. The complex Kantorovich polynomials of second order can be defined as in the case of real variable ([11]) by

$$Q_n(f)(z) = [B_{n+2}(H)(z)]'' , z \in \mathbb{C},$$

where $H(z) = \int_0^z F(u)\mathrm{d}u$, $F(u) = \int_0^u f(t)\mathrm{d}t$ and $B_{n+2}$ is the $(n+2)$-th Bernstein polynomial.

It is easy to see that similar approximation results with those for $K_n(f)(z)$ in Section 2 can be obtained for $Q_n(f)(z)$ too. □

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