REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 38 (2009) no. 1, pp. 11–23
ictp.acad.ro/jnaat

BERNSTEIN-TYPE OPERATORS ON TRIANGLES

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Abstract. The aim of the paper is to construct some univariate Bernstein-type operators on triangle, their product and Boolean sum, which interpolate a given function on the edges respectively at the vertices of triangle. Using the modulus of continuity and the Peano's theorem the remainders of corresponding approximation formulas are studied.

MSC 2000. 41A35, 41A36, 41A80.
Keywords. Bernstein operator, product operator, Boolean sum operator, modulus of continuity, error evaluation.

1. INTRODUCTION

Beginning with the paper by R. E. Barnhill, G. Birkhoff and W. J. Gordon [2], the blending interpolation operators on triangle were largely studied [3, 4, 5, 7, 9, 10, 16, 17, 21, 22, 24]. We also mention the applications of such interpolation schemes to computer aided geometric design and to finite element analysis [1, 3, 11, 13, 14, 15, 18, 19, 20, 33].

The aim of this paper is to construct Bernstein-type operators wich also interpolate the value of a given function on the border of triangle. Using modulus of continuity respectively Peano's theorem the remainders of the corresponding approximation formulas are as well studied. The accuracy of approximation is also illustrated by graphics of given functions and of the suitable Bernstein-type approximation.

By affine invariance it is sufficient to consider only the standard triangle $T_h = \left\{(x,y) \in \mathbb{R}^2 \mid x \geqslant 0, y \geqslant 0, x+y \leqslant h\right\}$, for $h > 0$.

2. UNIVARIATE OPERATORS

Let f be a real-valued function defined on T_h. Through the point $(x,y) \in T_h$, one considers the parallel lines to the coordinate axes which intersect the edges Γ_i, $i = 1,2,3$, of the triangle at the points $(0,y)$ and $(h-y,y)$ respectively $(x,0)$ and $(x,h-x)$ (Figure 1).

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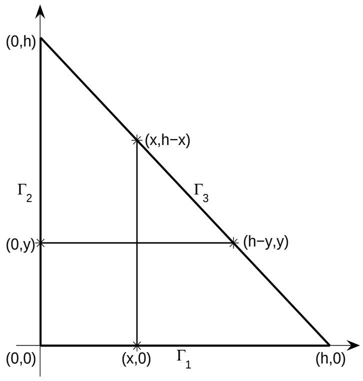

Fig. 1. Standard triangle.

Let $\Delta_m^x = \left\{ i\frac{h-y}{m},\ i = \overline{0,m} \right\}$ and $\Delta_n^y = \left\{ j\frac{h-x}{n},\ j = \overline{0,n} \right\}$ be uniform partitions of the intervals $[0, h-y]$ and $[0, h-x]$ respectively.

One considers the Bernstein-type operators B_m^x and B_n^y defined by

$$\left(B_m^x f\right)(x,y) = \sum_{i=0}^{m} p_{m,i}(x,y)\, f\left(i\frac{h-y}{m}, y\right),$$

where

$$p_{m,i}(x,y) = \binom{m}{i} x^i (h-x-y)^{m-i} \Big/ (h-y)^m, \quad 0 \leqslant x+y \leqslant h,$$

respectively

$$\left(B_n^y f\right)(x,y) = \sum_{j=0}^{n} q_{n,j}(x,y)\, f\left(x, j\frac{h-x}{n}\right),$$

with

$$q_{n,j}(x,y) = \binom{n}{j} y^j (h-x-y)^{n-j} \Big/ (h-x)^n, \quad 0 \leqslant x+y \leqslant h.$$

Theorem 1. *If f is a real-valued function defined on T_h then:*

(i) $B_m^x f = f$ *on* $\Gamma_2 \cup \Gamma_3$;

(ii) $\left(B_m^x e_{i0}\right)(x,y) = x^i,\ i = 0,1\ \left(\operatorname{dex}\left(B_m^x\right) = 1\right),$

$$\left(B_m^x e_{20}\right)(x,y) = x^2 + \frac{x(h-x-y)}{m},$$

$$\left(B_m^x e_{ij}\right)(x,y) = \begin{cases} y^j x^i, & i = 0,1,\ j \in \mathbb{N}; \\ y^j \left(x^2 + \frac{x(h-x-y)}{m}\right), & i = 2,\ j \in \mathbb{N}; \end{cases}$$

where $e_{ij}(x,y) = x^i y^j$ *and* $\operatorname{dex}\left(B_m^x\right)$ *is the degree of exactness of the operator* B_m^x.

Proof. The interpolation property (i) follows from the relations

$$p_{m,i}(0,y) = \begin{cases} 1, & \text{for } i = 0, \\ 0, & \text{for } i > 0, \end{cases}$$

and

$$p_{m,i}(h-y,y) = \begin{cases} 1, & \text{for } i = m, \\ 0, & \text{for } i < m. \end{cases}$$

Regarding the properties (ii), we have

$$\begin{aligned} (B_m^x e_{00})(x,y) &= \sum_{i=0}^{m} \tbinom{m}{i} \left(\tfrac{x}{h-y}\right)^i \left(1 - \tfrac{x}{h-y}\right)^{m-i} \\ &= \left(\tfrac{x}{h-y} + 1 - \tfrac{x}{h-y}\right)^m = 1; \end{aligned}$$

$$\begin{aligned} (B_m^x e_{10})(x,y) &= \sum_{i=0}^{m} \tbinom{m}{i} \left(\tfrac{x}{h-y}\right)^i \left(1 - \tfrac{x}{h-y}\right)^{m-i} i\tfrac{h-y}{m} \\ &= (h-y)\sum_{i=1}^{m} \tbinom{m-1}{i-1} \left(\tfrac{x}{h-y}\right)^i \left(1 - \tfrac{x}{h-y}\right)^{m-i} \\ &= x\sum_{i=0}^{m-1} \tbinom{m-1}{i} \left(\tfrac{x}{h-y}\right)^i \left(1 - \tfrac{x}{h-y}\right)^{m-1-i} \\ &= x\left(\tfrac{x}{h-y} + 1 - \tfrac{x}{h-y}\right)^{m-1} = x; \end{aligned}$$

$$\begin{aligned} (B_m^x e_{20})(x,y) &= \sum_{i=0}^{m} \tbinom{m}{i} \left(\tfrac{x}{h-y}\right)^i \left(1 - \tfrac{x}{h-y}\right)^{m-i} i^2 \left(\tfrac{h-y}{m}\right)^2 \\ &= \left(\tfrac{h-y}{m}\right)^2 \sum_{i=2}^{m} \tbinom{m}{i} i(i-1) \left(\tfrac{x}{h-y}\right)^i \left(1 - \tfrac{x}{h-y}\right)^{m-i} + \tfrac{x(h-y)}{m} \\ &= \tfrac{m-1}{m}x^2 \sum_{i=2}^{m} \tbinom{m-2}{i-2} \left(\tfrac{x}{h-y}\right)^{i-2} \left(1 - \tfrac{x}{h-y}\right)^{m-i} + \tfrac{x(h-y)}{m} \\ &= \tfrac{m-1}{m}x^2 + \tfrac{x(h-y)}{m} = x^2 + \tfrac{x(h-x-y)}{m}; \end{aligned}$$

$$(B_m^x e_{ij})(x,y) = y^j (B_m^x e_{i0})(x,y), \ i = 0,1,2, \ j \in \mathbb{N}.$$

$\square$

Remark 1. In the same way it is proved that:

$$B_n^y f = f \quad \text{on } \Gamma_1 \cup \Gamma_3;$$
$$\left(B_n^y e_{0j}\right)(x,y) = y^j, \; j = 0,1;$$
$$\left(B_n^y e_{02}\right)(x,y) = y^2 + \tfrac{y(h-x-y)}{n};$$
$$\left(B_n^y e_{ij}\right)(x,y) = \begin{cases} x^i y^j, & j = 0,1, \; i \in \mathbb{N}; \\ x^i \left(y^2 + \tfrac{y(h-x-y)}{n}\right), & j = 2, \; i \in \mathbb{N}. \end{cases}$$

□

Now, let us consider the approximation formula

$$f = B_m^x f + R_m^x f.$$

Theorem 2. *If* $f(\cdot, y) \in C[0, h-y]$ *then*

$$\left|\left(R_m^x f\right)(x,y)\right| \leqslant \left(1 + \tfrac{h}{2\delta\sqrt{m}}\right)\omega\left(f(\cdot,y);\delta\right), \quad y \in [0,h], \tag{1}$$

where $\omega(f(\cdot,y);\delta)$ *is the modulus of continuity of the function* f *with regard to the variable* x.

Moreover, if $\delta = \frac{1}{\sqrt{m}}$ *then*

$$\left|\left(R_m^x f\right)(x,y)\right| \leqslant \left(1 + \tfrac{h}{2}\right)\omega\left(f(\cdot,y);\tfrac{1}{\sqrt{m}}\right), \quad y \in [0,h]. \tag{2}$$

Proof. We have

$$\left|\left(R_m^x f\right)(x,y)\right| \leqslant \sum_{i=0}^{m} p_{m,i}(x,y)\left|f(x,y) - f\left(i\tfrac{h-y}{m}, y\right)\right|.$$

As,

$$\left|f(x,y) - f\left(i\tfrac{h-y}{m}, y\right)\right| \leqslant \left(\tfrac{1}{\delta}\left|x - i\tfrac{h-y}{m}\right| + 1\right)\omega\left(f(\cdot,y);\delta\right)$$

one obtains

$$\begin{aligned}\left|\left(R_m^x f\right)(x,y)\right| &\leqslant \sum_{i=0}^{m} p_{m,i}(x,y)\left(\tfrac{1}{\delta}\left|x - i\tfrac{h-y}{m}\right| + 1\right)\omega\left(f(\cdot,y);\delta\right) \\ &\leqslant \left[1 + \tfrac{1}{\delta}\left(\sum_{i=0}^{m} p_{m,i}(x,y)\left(x - i\tfrac{h-y}{m}\right)^2\right)^{1/2}\right]\omega\left(f(\cdot,y);\delta\right) \\ &= \left[1 + \tfrac{1}{\delta}\sqrt{\tfrac{x(h-x-y)}{m}}\right]\omega\left(f(\cdot,y);\delta\right).\end{aligned}$$

Since

$$\max_{T_h}\left[x(h-x-y)\right] = \tfrac{h^2}{4}, \tag{3}$$

it follows that

$$\left|\left(R_m^x f\right)(x,y)\right| \leqslant \left(1 + \tfrac{h}{2\delta\sqrt{m}}\right)\omega\left(f(\cdot,y);\delta\right).$$

For $\delta = 1/\sqrt{m}$, one obtains (2). $\square$

THEOREM 3. *If* $f(\cdot, y) \in C^2[0,h]$ *then*

$$\left(R_m^x f\right)(x,y) = -\tfrac{x(h-x-y)}{2m} f^{(2,0)}(\xi, y), \quad \xi \in [0, h-y] \tag{4}$$

and

$$\left|\left(R_m^x f\right)(x,y)\right| \leqslant \tfrac{h^2}{8m} M_{20} f, \quad (x,y) \in T_h, \tag{5}$$

where

$$M_{ij} f = \max_{T_h} \left| f^{(i,j)}(x,y) \right|.$$

Proof. As, $\operatorname{dex}(B_m^x) = 1$, by Peano's theorem, one obtains

$$\left(R_m^x f\right)(x,y) = \int_0^{h-y} K_{20}(x,y;s) f^{(2,0)}(s,y)\,\mathrm{d}s,$$

where the kernel

$$K_{20}(x,y;s) := R_m^x\left[(x-s)_+\right] = (x-s)_+ - \sum_{i=0}^{m} p_{m,i}(x,y)\left(i\tfrac{h-y}{m} - s\right)_+$$

does not change the sign $(K_{20}(x,y;s) \leqslant 0,\ x \in [0,h-y])$.

By mean value theorem, it follows that

$$\left(R_m^x f\right)(x,y) = f^{(2,0)}(\xi,y) \int_0^{h-y} K_{20}(x,y;s)\,\mathrm{d}s, \quad \xi \in [0,h-y],$$

i.e.

$$\left(R_m^x f\right)(x,y) = -\tfrac{x(h-x-y)}{2m} f^{(2,0)}(\xi,y).$$

Now, using (3) it is also obtained (5). $\square$

REMARK 2. From (4) it follows that

- if $f(\cdot,y)$ is a concave function then $\left(R_m^x f\right)(x,y) \geqslant 0$, i.e. $\left(B_m^x f\right)(x,y) \leqslant f(x,y)$,
- if $f(\cdot,y)$ is a convex function then $\left(R_m^x f\right)(x,y) \leqslant 0$, i.e. $\left(B_m^x f\right)(x,y) \geqslant f(x,y)$,

for $x \in [0,h-y]$ and $y \in [0,h]$. $\square$

REMARK 3. For the remainder $R_n^y f$ of the approximation formula

$$f = B_n^y f + R_n^y f,$$

we also have:

A. if $f \in C[0,h-x]$ then

$$\left|\left(R_n^y f\right)(x,y)\right| \leqslant \left(1 + \tfrac{h}{2\delta\sqrt{n}}\right)\omega\left(f(x,\cdot);\delta\right), \quad x \in [0,h], \tag{6}$$

and

$$\left|\left(R_n^y f\right)(x,y)\right| \leqslant \left(1 + \tfrac{h}{2}\right)\omega\left(f(x,\cdot);\tfrac{1}{\sqrt{n}}\right), \quad x \in [0,h], \tag{7}$$

respectively

B. if $f \in C^2[0,h]$ then

$$\left(R_n^y f\right)(x,y) = -\tfrac{y(h-x-y)}{2n} f^{(0,2)}(x,\eta)\,,\ \eta \in [0, h-x] \tag{8}$$

and

$$\left|\left(R_n^y f\right)(x,y)\right| \leqslant \tfrac{h^2}{8n} M_{02} f, \quad (x,y) \in T_h. \tag{9}$$

$\square$

3. PRODUCT OPERATORS

Let $P_{mn} = B_m^x B_n^y$ respectively $Q_{nm} = B_n^y B_m^x$ be the products of operators B_m^x and B_n^y.

We have

$$(P_{mn} f)(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} p_{m,i}(x,y)\, q_{n,j}\left(i\tfrac{h-y}{m}, y\right) f\left(i\tfrac{h-y}{m}, j\tfrac{(m-i)h+iy}{mn}\right).$$

Remark 4. The nodes of the operator P_{mn} are as in the Figure 2, for $i = \overline{0,m}$, $j = \overline{0,n}$, and $y \in [0,h]$. $\square$

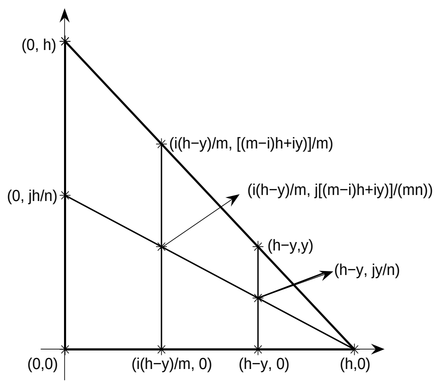

Fig. 2. Nodes of operator P_{mn}.

Theorem 4. *The operator P_{mn} satisfies the following relations:*

(i) $(P_{mn} f)(x,0) = (B_m^x f)(x,0)\,,$

(ii) $(P_{mn} f)(0,y) = (B_n^y f)(0,y)\,,$

(iii) $(P_{mn} f)(x,h-x) = f(x,h-x)\,, \quad x,y \in [0,h]\,.$

The proofs follow by a straightforward computation.

The property (i) or (ii) imply that $(P_{mn} f)(0,0) = f(0,0)$.

Remark 5. The product operator P_{mn} interpolates the function f at the vertex $(0,0)$ and on the hypothenuse $x+y=h$ of the triangle T_h. $\square$

The product operator Q_{nm}, given by

$$(Q_{nm}f)(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}p_{m,i}\left(x,j\tfrac{h-x}{n}\right)q_{n,j}(x,y)\,f\left(i\tfrac{(n-j)h+jx}{mn},j\tfrac{h-x}{n}\right)$$

has the nodes as in Figure 3, for $i=\overline{0,m}$, $j=\overline{0,n}$, $x\in[0,h]$, and the

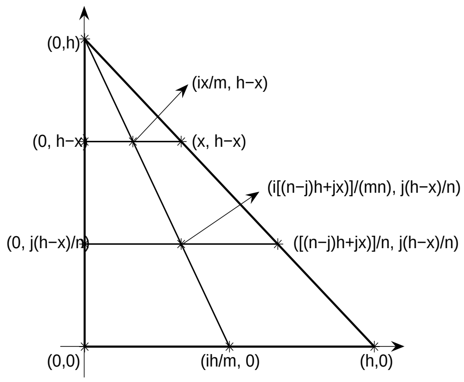

Fig. 3. Nodes of operator Q_{nm}.

properties

(i') $(Q_{nm}f)(x,0)=(B_m^x f)(x,0)$,
(ii') $(Q_{nm}f)(0,y)=(B_n^y f)(0,y)$,
(iii') $(Q_{nm}f)(h-y,y)=f(h-y,y)$, $\quad x,y\in[0,h]$.

Let us consider the approximation formula

$$f=P_{mn}f+R_{mn}^P f.$$

THEOREM 5. *If $f\in C(T_h)$ then*

$$\left|\left(R_{mn}^P f\right)(x,y)\right|\leqslant(1+h)\,\omega\left(f;\tfrac{1}{\sqrt{m}},\tfrac{1}{\sqrt{n}}\right),\quad (x,y)\in T_h. \tag{10}$$

Proof. We have

$$\begin{aligned}\left|\left(R_{mn}^P f\right)(x,y)\right|\leqslant\Bigg[&\tfrac{1}{\delta_1}\sum_{i=0}^{m}\sum_{j=0}^{n}p_{m,i}(x,y)\,q_{n,j}\left(i\tfrac{h-y}{m},y\right)\left|x-i\tfrac{h-y}{m}\right|\\&+\tfrac{1}{\delta_2}\sum_{i=0}^{m}\sum_{j=0}^{n}p_{m,i}(x,y)\,q_{n,j}\left(i\tfrac{h-y}{m},y\right)\left|y-j\tfrac{(m-i)h+iy}{mn}\right|\\&+\sum_{i=0}^{m}\sum_{j=0}^{n}p_{m,i}(x,y)\,q_{n,j}\left(i\tfrac{h-y}{m},y\right)\Bigg]\omega(f;\delta_1,\delta_2).\end{aligned}$$

After some transformations, one obtains

$$\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}\left(x,y\right) q_{n,j}\left(i\tfrac{h-y}{m},y\right)\left|x-i\tfrac{h-y}{m}\right| \leqslant \sqrt{\tfrac{x(h-x-y)}{m}}$$
$$\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}\left(x,y\right) q_{n,j}\left(i\tfrac{h-y}{m},y\right)\left|y-j\tfrac{(m-i)h+iy}{mn}\right| \leqslant \sqrt{\tfrac{y(h-x-y)}{m}},$$

while

$$\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}\left(x,y\right) q_{n,j}\left(i\tfrac{h-y}{m},y\right) = 1.$$

It follows,

$$\left|\left(R_{mn}^{P}f\right)(x,y)\right| \leqslant \left(\tfrac{1}{\delta_1}\sqrt{\tfrac{x(h-x-y)}{m}}+\tfrac{1}{\delta_2}\sqrt{\tfrac{y(h-x-y)}{m}}+1\right)\omega\left(f;\delta_1,\delta_2\right).$$

Taking into account that

$$\tfrac{x(h-x-y)}{m} \leqslant \tfrac{h^2}{4m}, \quad \tfrac{y(h-x-y)}{n} \leqslant \tfrac{h^2}{4n},$$

one obtains

$$\left|\left(R_{mn}^{P}f\right)(x,y)\right| \leqslant \left(\tfrac{1}{\delta_1}\tfrac{h}{2\sqrt{m}}+\tfrac{1}{\delta_2}\tfrac{h}{2\sqrt{n}}+1\right)\omega\left(f;\delta_1,\delta_2\right),$$

hence

$$\left|\left(R_{mn}^{P}f\right)(x,y)\right| \leqslant (1+h)\,\omega\left(f;\tfrac{1}{\sqrt{m}},\tfrac{1}{\sqrt{n}}\right).$$

□

4. BOOLEAN SUM OPERATORS

Let

$$S_{mn} := B_m^x \oplus B_n^y = B_m^x + B_n^y - B_m^x B_n^y,$$
$$T_{nm} := B_n^y \oplus B_m^x = B_n^y + B_m^x - B_n^y B_m^x,$$

be the Boolean sums of the Bernstein-type operators B_m^x and B_n^y.

THEOREM 6. *If f is a real-valued function defined on T_h then*

$$S_{mn}f\Big|_{\partial T_h} = f\Big|_{\partial T_h}.$$

Proof. We have

$$S_{mn}f = \left(B_m^x + B_n^y - B_m^x B_n^y\right) f.$$

The interpolation properties of B_m^x, B_n^y and the properties (i)–(iii) of the operator P_{mn} imply that

$$\begin{aligned}(S_{mn}f)(x,0) &= (B_m^x f)(x,0) + f(x,0) - (B_m^x f)(x,0) = f(x,0),\\ (S_{mn}f)(0,y) &= f(0,y) - (B_n^y f)(0,y) + (B_n^y f)(0,y) = f(0,y),\\ (S_{mn}f)(x,h-x) &= f(x,h-x) + f(x,h-x) - f(x,h-x) = f(x,h-x),\end{aligned}$$

for all $x, y \in [0,h]$. □

Let $R_{mn}^S f$ be the remainder of the Boolean sum approximation formula

$$f = S_{mn}f + R_{mn}^S f.$$

THEOREM 7. *If $f \in C(T_h)$ then*

$$\begin{aligned}\left|\left(R_{mn}^S f\right)(x,y)\right| \leqslant{} & \left(1+\tfrac{h}{2}\right)\omega\left(f(\cdot,y);\tfrac{1}{\sqrt{m}}\right) + \left(1+\tfrac{h}{2}\right)\omega\left(f(x,\cdot);\tfrac{1}{\sqrt{n}}\right)\\ & + (1+h)\,\omega\left(f;\tfrac{1}{\sqrt{m}},\tfrac{1}{\sqrt{n}}\right), \quad (x,y)\in T_h.\end{aligned}$$

Proof. From the equality

$$f - S_{mn}f = f - B_m^x f + f - B_n^y f - (f - P_{mn}f)$$

one obtains

$$\left|\left(R_{mn}^S f\right)(x,y)\right| \leqslant \left|\left(R_m^x f\right)(x,y)\right| + \left|\left(R_n^y f\right)(x,y)\right| + \left|\left(R_{mn}^P f\right)(x,y)\right|.$$

Now, from (2), (7) and (10), the proof follows. □

REMARK 6. Analoguous relations can be obtained for the remainders of the product approximation formula

$$f = Q_{nm}f + R_{nm}^Q f = B_n^y B_m^x f + R_{nm}^Q f$$

and for the Boolean sum formula

$$f = T_{nm}f + R_{nm}^T f = \left(B_n^y \oplus B_m^x\right) f + R_{nm}^T f.$$

□

5. EXAMPLES

Finally, one considers two test functions, generally used in literature (see, e.g. [22])

$$\begin{aligned} f_1(x,y) &= \tfrac{1}{3}\exp\left[-\tfrac{81}{16}\left(\left(x-\tfrac{1}{2}\right)^2 + \left(y-\tfrac{1}{2}\right)^2\right)\right],\\ f_2(x,y) &= \tfrac{1}{6}\,\frac{\frac{5}{4}+\cos\frac{5}{4}y}{1+(3x-1)^2}.\end{aligned}$$

In FIGURE 4 we plot the graphs of f_1 and f_2.

FIGURE 5 and FIGURE 6 contain the graphs of $B_m^x f$, $B_n^y f$, $P_{mn}f$, $S_{mn}f$, with $h=1$, $m=5$, $n=6$, for f_1 and f_2 respectively.

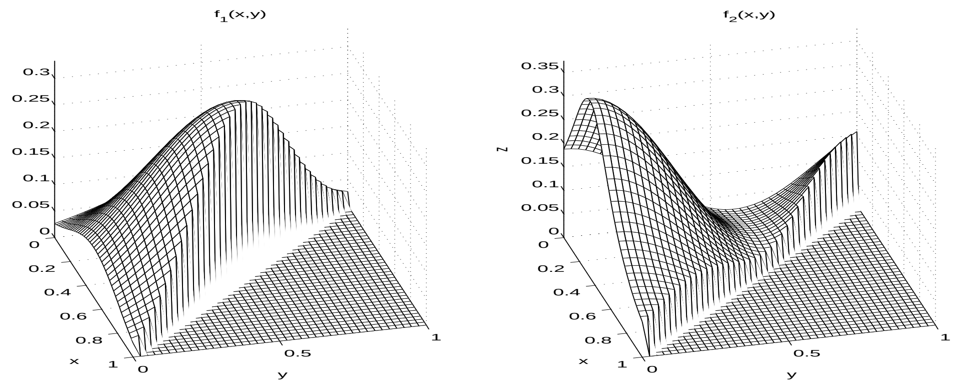

Fig. 4. Graphs of f_1 and f_2.

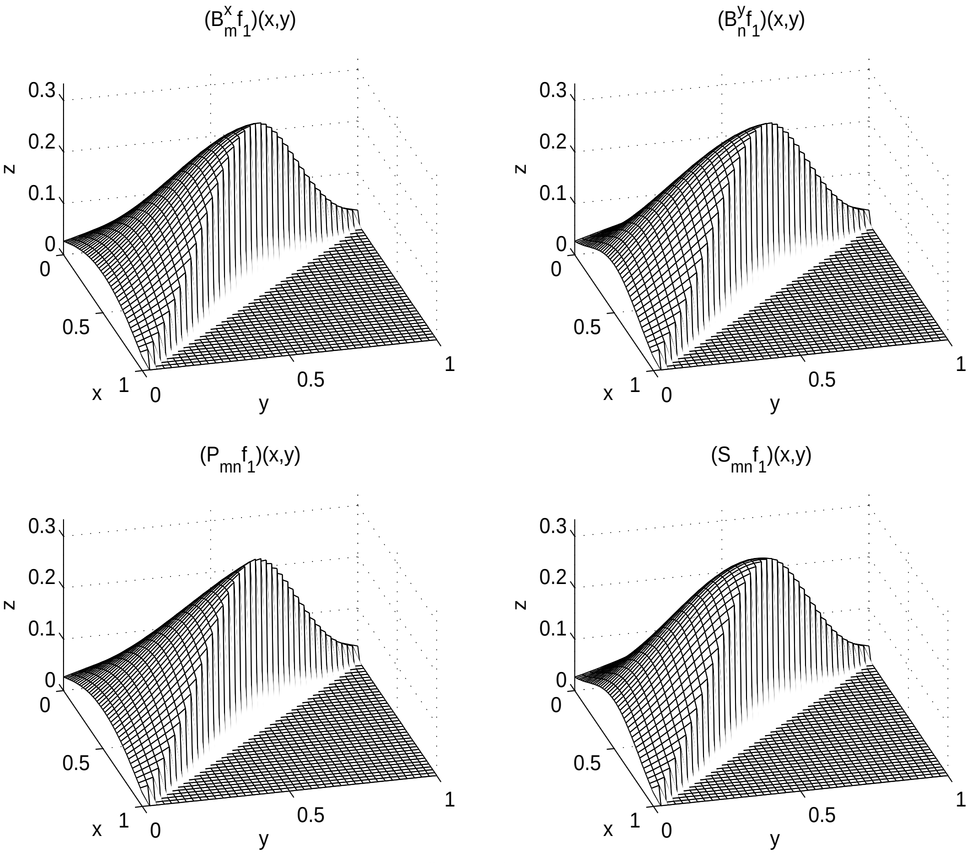

Fig. 5. Graphs of $B_m^x f_1$, $B_n^y f_1$, $P_{mn} f_1$, and $S_{mn} f_1$.

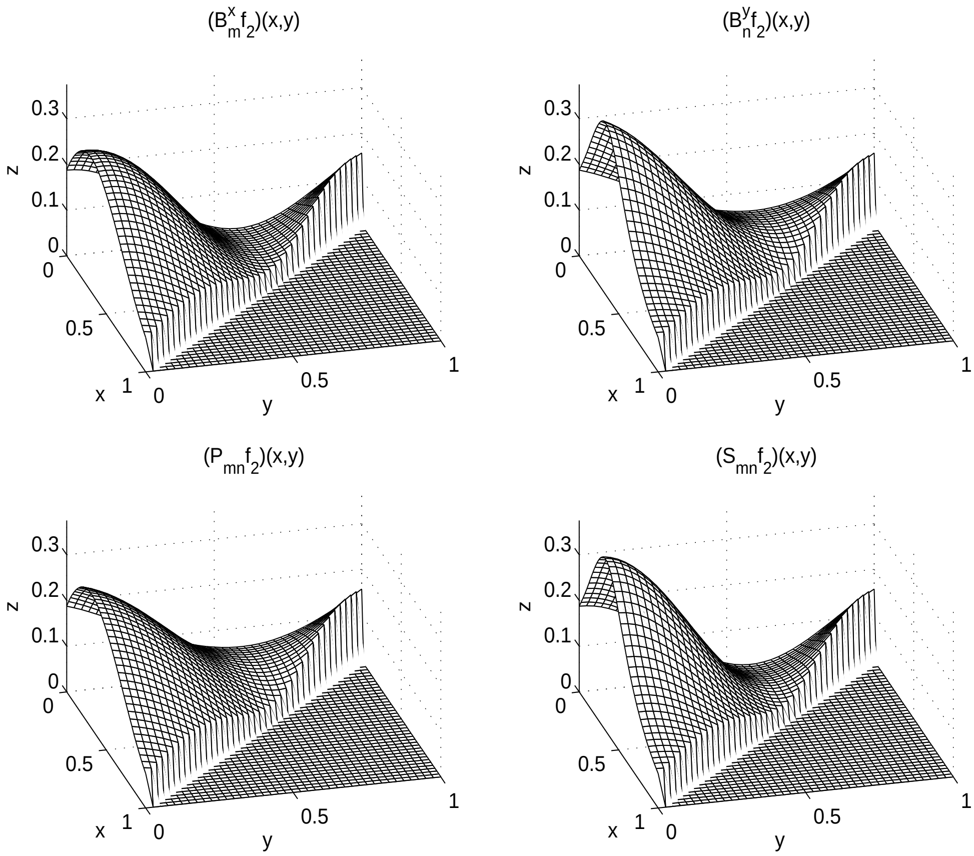

Fig. 6. Graphs of $B_m^x f_2$, $B_n^y f_2$, $P_{mn} f_2$, and $S_{mn} f_2$.

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Received by the editors: February 16, 2009.