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ON THE ACCELERATION OF THE CONVERGENCE OF CERTAIN ITERATIVE PROCEEDINGS (I)

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Abstract. The research elaborated in this paper has its origin in the study of the convergence of the sequences generated through the use of certain methods derived from the well-known Newton-Kantorovich method for the simultaneous approximation of the solution of an equation in a linear normed space, and of the inverse of the Fréchet differential at this solution. An important place is given in the paper to the notion of convergence order of an approximant sequence of the solution of an equation. Considering given an approximant sequence which verifies certain conditions expressed through the inequalities (25), we will build another approximant sequence through the relations (22), with ameliorated convergence order. We will analyse certain special cases and, in the same time, we will determine optimal methods from the point of view of the convergence order.

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1. INTRODUCTION

Let us consider the normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ having θ_X and θ_Y as zero elements.

We will note by $(X, Y)^*$ the set of the linear and continous mappings defined on X with values in Y; we know that this set is a linear normed space with the norm:

$$\|\cdot\| : (X, Y)^* \to \mathbb{R};\ \|U\| = \sup \{ \|U(x)\|_Y \,|\, x \in X, \|x\|_X = 1 \},$$

the supremum of the definition being of course finite.

It is also known that if $(Y, \|\cdot\|_Y)$ is a Banach space, then $((X, Y)^*, \|\cdot\|)$ is a Banach space as well.

We will consider now a set $D \subseteq X$, a nonlinear function $f : D \to Y$ and, using this function, the nonlinear equation:

$$f(x) = \theta_Y. \tag{1}$$

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Our aim is to study the existence of a solution of this equation, namely an element $\overline{x}$ from D so that for $x = \overline{x}$ the equality (1) is verified and also to approximate this solution by a sequence $(x_n)_{n\geq 0} \subseteq D$, with elements obtained by a recurrence formula.

If we suppose that the function $f : D \to Y$ admits the Fréchet differential $f'(x) \in (X, Y)^*$ at every point $x \in D$, the sequence $(x_n)_{n\geq 0}$ can be generated from the well-known iterative method known as the Newton-Kantorovich iterative method. In this case, for any $n \geq 0$ the equality

$$f'(x_n)(x_{n+1} - x_n) + f(x_n) = \theta_Y \tag{2}$$

is verified.

If the function $f : D \to Y$ and the initial element $x_0 \in D$ verify certain conditions, we can prove that for any $n \geq 0$ there exists the mapping $[f'(x_n)]^{-1} \in (Y, X)^*$ and in this way the recurrence relation (2) will be written for any $n \geq 0$ as

$$x_{n+1} = x_n - \left[f'(x_n)\right]^{-1} f(x_n) \tag{3}$$

(see [9] and [10]).

The mapping $[f'(x_n)]^{-1} \in (Y, X)^*$ will be found by solving the linear equation (2) and this operation must be accomplished for every $n \geq 0$, that is, for every iteration step needed in order to obtain an element $x_n \in D$ corresponding to the criterion of error imposed to the approximant of the solution of equation (1).

We can surpass this difficulty using an additional sequence. Thus, besides the solution $\overline{x} \in D$ that is approximated by the sequence $(x_n)_{n\geq 0} \subseteq D$, the linear mapping $[f'(x)]^{-1} \in (Y, X)^*$ will be approximated at the same time by an additional sequence $(A_n)_{n\geq 0} \subseteq (Y, X)^*$.

The manner of obtaining the additional sequence is that of the approximation of a certain linear mapping's inverse. So if we consider a mapping $U \in (X, Y)^*$ and another mapping $A_0 \in (Y, X)^*$ so that $\|\mathbf{I}_Y - UA_0\| < 1$ ($\mathbf{I}_Y$ represents the identical mapping from Y) then there exists the inverse to the right of the mapping U, denoted by $U_d^{-1} \in (Y, X)^*$ and this mapping will be obtained as the limit of the sequence $(A_n)_{n\geq 0} \subseteq (Y, X)^*$ generated by the recurrence sequence:

$$A_{n+1} = A_n \sum_{k=0}^{r} (\mathbf{I}_Y - UA_n)^k \tag{4}$$

starting from the mapping A_0, where $r \in \mathbb{N}$ is given.

One can verify the fact that for any $n \geq 0$ the relations

$$(5)\qquad \begin{cases} \mathbf{I}_Y - UA_{n+1} = (\mathbf{I}_Y - UA_n)^{r+1}, \\ \mathbf{I}_Y - UA_n = (\mathbf{I}_Y - UA_0)^{(r+1)^n}, \\ \|A_{n+1} - A_n\| \leq \|A_0\| \frac{d^{(r+1)^n}}{1-d^{(r+1)^n}} \exp \sum_{k=1}^{r} \frac{d^k}{1-d^{kr}}, \end{cases}$$

are true. In the above $d = \|\mathbf{I}_Y - UA_0\|$ and for any $u \in \mathbb{R}$ we have $\exp(u) = e^u$. Again, for any $n \geq 0$ and $p \in \mathbb{N}$ the following inequality is true

$$(6)\qquad \|A_{n+p} - A_n\| \leq \|A_0\| \frac{d^{(r+1)^n}}{\left[1-d^{(r+1)^n}\right]^2} \exp \sum_{k=1}^{r} \frac{d^k}{1-d^{kr}}.$$

If $(X, \|\cdot\|_X)$ is a Banach space, we deduce the convergence of the sequence $(A_n)_{n\geq 0}$ to the mapping $U_d^{-1} \in (Y,X)^*$ for which we have the estimates

$$(7)\qquad \begin{cases} \left\|U_d^{-1}\right\| \leq \frac{\|A_0\|}{1-d^{(r+1)^n}} \exp \sum_{k=1}^{r} \frac{d^k}{1-d^{kr}}, \\ \left\|U_d^{-1} - A_n\right\| \leq \|A_0\| \frac{d^{(r+1)^n}}{\left[1-d^{(r+1)^n}\right]^2} \exp \sum_{k=1}^{r} \frac{d^k}{1-d^{kr}}. \end{cases}$$

On account of the aforementioned results the following definition is justified

Definition 1. *For a given mapping $U \in (X,Y)^*$ and a number $r \in \mathbb{N}$, the mapping:*

$$(8)\qquad \mathbf{S}_U^{(r+1)} : (Y,X)^* \to (Y,X)^*, \quad \mathbf{S}_U^{(r+1)}(A) = A \sum_{k=0}^{r} (\mathbf{I}_Y - UA)^k$$

is called a mapping of approximation with the order $r+1$ of the inverse to right of U.

Remark 2. If $A_0 \in (Y,X)^*$ is given, using a certain approximation mapping we can build a sequence $(A_n)_{n\geq 0}$ that will be in fact the sequence of the successive approximations generated through the mapping (8), namely for any $n \geq 0$ we have the equality $A_{n+1} = \mathbf{S}_U^{(r+1)}(A_n)$, a relation of recurrence that goes back to the relation (4) and if $\|\mathbf{I}_Y - UA_0\| < 1$ there exists $U_d^{-1} = \lim_{n\to\infty} A_n \in (Y,X)^*$. □

We return to the iterative method of Newton-Kantorovich for the equation (1).

Definition 3. *If the function $f : D \to Y$ admits the Fréchet differential at every point $x \in D$ and this differential is invertible, then the function:*

$$(9)\qquad Q : D \to X;\ Q(x) = x - \left[f'(x)\right]^{-1} f(x)$$

is called the iterative operator of Newton attached to the function $f : D \to Y$.

It is clear that for the existence of a certain iterative operator of Newton, in addition to the differentiability at every point $x \in D$ of the mapping $f : D \to Y$, it is necessary to suppose the invertibility of the mapping $f'(x) \in (X, Y)^*$ at every point $x \in D$ as well.

So the iterative method of Newton-Kantorovich is in fact the method of successive approximations generated by the mapping Q, the recurrence relation (3) will be written in the form $x_{n+1} = Q(x_n)$.

In order to avoid this last difficulty we will use the operator

$$\mathbf{R} : D \times (Y, X)^* \to X,\ \mathbf{R}(x, A) = x - \mathbf{S}^{(r+1)}_{f'(x)}(A) f(x) \tag{10}$$

with $r \in \mathbb{N} \cup \{0\}$.

In order to approximate the solution of the equation (1) we will use a sequence $(x_n)_{n\geq 0} \subseteq D$ where for every $n \geq 0$ we have

$$\begin{cases} x_{n+1} = \mathbf{R}(x_n, A_n), \\ A_{n+1} = \mathbf{S}^{(q+1)}_{f'(x_{n+1})}(A_n), \end{cases} \tag{11}$$

namely

$$\begin{cases} x_{n+1} = x_n - \mathbf{S}^{(r+1)}_{f'(x_n)}(A_n) f(x_n), \\ A_{n+1} = \mathbf{S}^{(q+1)}_{f'(x_{n+1})}(A_n), \end{cases} \tag{12}$$

or, even in more detail,

$$\begin{cases} x_{n+1} = x_n - A_n \left[\sum\limits_{k=0}^{r} (\mathbf{I}_Y - f'(x_n) A_n)^k\right] f(x_n), \\ A_{n+1} = A_n \sum\limits_{k=0}^{q} (\mathbf{I}_Y - f'(x_{n+1}) A_n)^k, \end{cases} \tag{13}$$

with $x_0 \in D$ and $A_0 \in (Y, X)^*$, which are the initial elements of the proceeding and are arbitrarily chosen. Also, in relations (11)–(13), we use the numbers $r \in \mathbb{N} \cup \{0\}$ and $q \in \mathbb{N}$, in order to generate a certain unconstant sequence $(A_n)_{n\geq 0}$, the use of a certain number $q \geq 1$ is necessary.

It has been shown [1], [2] that the order of the convergence speed of the approximant sequence $(x_n)_{n\geq 0} \subseteq D$ to a solution of the equation $f(x) = \theta_Y$ is 2, that is, there exists $K, L > 0$ so that for any $n \geq 0$ we have the inequalities

$$\begin{cases} \|f(x_{n+1})\|_Y \leq K \|f(x_n)\|_Y^2, \\ \|x_{n+1} - x_n\|_X \leq L \|f(x_n)\|_Y. \end{cases} \tag{14}$$

This order is not dependent on the natural numbers $r \in \mathbb{N} \cup \{0\}$ and $q \in \mathbb{N}$.

For this reason special interest is given to the case of $r = 0,\ q = 1$, for the simplicity of the calculations needed. In this case the relations (13) will be written under the form

$$\begin{cases} x_{n+1} = x_n - A_n f(x_n), \\ A_{n+1} = A_n (2\mathbf{I}_Y - f'(x_{n+1}) A_n), \end{cases} \tag{15}$$

a method proposed by S. Ul'm [13], and also to the case of $r = 0$, $q = 2$ in which the choice of the initial elements $x_0 \in D$ and $A_0 \in (Y, X)^*$ is most favorable in order to ensure the convergence [1].

For the detailed study of these methods one can refer to the papers [3], [5], [6], [7] and [8].

Method (13) can be extended. In order to this it is necessary to use an operator $Q : D \to X$ with $Q(D) \subseteq D$. With the aid of this operator we build a sequence $(x_n)_{n\geq 0} \subseteq D$ starting from an arbitrary element $x_0 \in D$ and for a certain $n \geq 0$, $x_n \in D$ being known, we will determine the element $x_{n+1} \in D$ that verifies the equality

$$f'(x_n)(x_{n+1} - x_n) + f(Q(x_n)) = 0. \tag{16}$$

If for any $n \geq 0$ there exists the mapping $[f'(x_n)]^{-1} \in (Y, X)^*$, we can say that the sequence $(x_n)_{n\geq 0}$ is generated starting from $x_0 \in D$ through the recurrence relation

$$x_{n+1} = Q(x_n) - \left[f'(x_n)\right]^{-1} f(Q(x_n)). \tag{17}$$

Such a method is known under the name of the iterative method of the Traub type.

Obviously, if $Q = \mathbf{I}_X$ the iterative method generated from (17) goes back to the method of Newton-Kantorovich (2)-(3).

We apply to the methods of the Traub type the same modification that was used in the cases of the methods of the Newton-Kantorovich type. Thus we will obtain the pair of sequences $(x_n)_{n\geq 0} \subseteq D$ and $(A_n)_{n\geq 0} \subseteq (Y, X)^*$ generated from the recurrence relations

$$\begin{cases} x_{n+1} = Q(x_n) - A_n \left[\sum\limits_{k=0}^{r} (\mathbf{I}_Y - f'(x_n) A_n)^k\right] f(Q(x_n)), \\ A_{n+1} = A_n \left[\sum\limits_{k=0}^{q} (\mathbf{I}_Y - f'(x_{n+1}) A_n)^k\right]. \end{cases} \tag{18}$$

It is known that if the operator $Q : D \to X$ has the order p in connection with the function $f : D \to Y$, namely there exist the numbers $K, L > 0$ so that for any $x \in D$ the following relations are true

$$\begin{cases} \|f(Q(x_n))\|_Y \leq K \|f(x_n)\|_Y^p, \\ \|Q(x) - x\|_X \leq L \|f(x_n)\|_Y, \end{cases} \tag{19}$$

then the order of the convergence speed of the sequence $(x_n)_{n\geq 0} \subseteq D$ generated through the relations (16) and (18) is $p + 1$ (see [4], [6], [8], [11] and [12]).

Furthermore, as we have shown in [7], given a sequence $(y_n)_{n\geq 0} \subseteq D$ with a **convergence order** $p \geq 1$ **in connection with the function** $f : D \to Y$

and the sequence $(x_n)_{n\geq 0}$, meaning there exists the numbers $K_0, L_0 > 0$ so that, for any $n \geq 0$ the following inequalities are true

$$\begin{cases} \|f(y_n)\|_Y \leq K_0 \|f(x_n)\|_Y^p, \\ \|y_n - x_n\|_X \leq L_0 \|f(x_n)\|_Y, \end{cases} \tag{20}$$

then the sequence $(x_n)_{n\geq 0} \subseteq D$ generated by the relation:

$$x_{n+1} = y_n - [f'(x_n)]^{-1} f(y_n), \tag{21}$$

or through the relations:

$$\begin{cases} x_{n+1} = y_n - A_n \left[\sum_{k=0}^{r} (\mathbf{I}_Y - f'(x_n) A_n)^k \right] f(y_n), \\ A_{n+1} = A_n \sum_{k=0}^{q} (\mathbf{I}_Y - f'(x_{n+1}) A_n)^k \end{cases} \tag{22}$$

has the convergence order $p+1$ in connection with the same function $f : D \to Y$.

In the paper [3] we have considered a mapping $Q : D \times (Y, X)^* \to X$ with the property that for any $A \in (Y, X)^*$ the inclusion $Q(D, A) \subseteq D$ takes place. Using this mapping we will build an iterative method for the approximation of the solution of the equation (1), with the sequences $(x_n)_{n\geq 0}$ and $(A_n)_{n\geq 0}$ that are obtained form the recurrence relations

$$\begin{cases} x_{n+1} = Q(x_n, A_n) - A_n \left[\sum_{k=0}^{r} (\mathbf{I}_Y - f'(x_n) A_n)^k \right] f(Q(x_n, A_n)), \\ A_{n+1} = A_n \sum_{k=0}^{q} (\mathbf{I}_Y - f'(x_{n+1}) A_n)^k, \end{cases} \tag{23}$$

starting from the elements $x_0 \in D$ and $A_0 \in (Y, X)^*$ arbitrarily chosen.

In order to ensure the convergence of the approximation sequence $(x_n)_{n\geq 0} \subseteq D$ generated through the relations (23), we will replace the relations (19) with others more adequate to the mapping $Q : D \times (Y, X)^* \to X$. Thus we will suppose that the numbers $m \in \mathbb{N}$ and $M > 0$, and, for any $i = \overline{1, m}$, the numbers $p_i, q_i, K_i > 0$ with $p_i + q_i > 1$ exist so that for any $x \in D$ and any $A \in (Y, X)^*$ the following inequalities are true:

$$\begin{cases} \|f(Q(x, A))\|_Y \leq \sum_{i=1}^{m} K_i \|f(x)\|_Y^{p_i} \cdot \|\mathbf{I}_Y - f'(x) A\|^{q_i}, \\ \|Q(x, A) - x\|_X \leq M \|f(x)\|_Y. \end{cases} \tag{24}$$

In this paper we will study the more general case of an iterative proceeding in which a sequence $(y_n)_{n\geq 0} \subseteq D$ is given and a sequence $(x_n)_{n\geq 0} \subseteq D$ is built using the recurrence relation (22).

However, we will replace the inequalities (20). Supposing that the numbers $m \in \mathbb{N}$ and $L > 0$ exist, and for any $i = \overline{1, m}$ the numbers $p_i, q_i, K_i > 0$ exist with $p_i + q_i \geq 1$ so that for any $n \geq 0$ the following inequalities are true

$$\begin{cases} \|f(y_n)\|_Y \leq \sum\limits_{i=1}^{m} K_i \|f(x_n)\|_Y^{p_i} \cdot \|\mathbf{I}_Y - f'(x_n) A_n\|^{q_i}, \\ \|y_n - x_n\|_X \leq M \|f(x_n)\|_Y. \end{cases} \tag{25}$$

It is clear that if there exists the mapping $Q : D \times (Y, X)^* \to X$ so that for any $n \geq 0$ we have $y_n = Q(x_n, A_n)$, the proposed iterative proceeding goes back to the iterative proceeding generated through the relations (23), while the inequalities (24) are verified.

2. THE CONVERGENCE OF THE ITERATIVE PROCEEDING (??), (??)

In order to simplify the writing we will introduce the function

$$g : [0, +\infty[\times [0, +\infty[\to \mathbb{R}, \; g(u, v) = \sum_{i=1}^{m} K_i u^{p_i} v^{q_i}$$

where the number $m \in \mathbb{N}$ and for any $i = \overline{1, m}$ the elements $K_i, p_i, q_i > 0$ with $p_i + q_i > 1$ are the same for which the inequalities (25) are verified. It is evident that for any $u, v \geq 0$ it results $g(u, v) \geq 0$.

We will consider as well the function:

$$h : [0, +\infty[\to \mathbb{R}, \; h(t) = 1 + t + t^2 + \ldots + t^r = \tfrac{1 - t^{r+1}}{1 - t}.$$

Finally, for $x_0 \in D$ and $R > 0$, we will note by $\mathcal{S}(x_0, R)$ the closed ball with the center at the point x_0 and the radius R, so we have:

$$\mathcal{S}(x_0, R) = \{x \in \mathbb{R} / \|x - x_0\|_X \leq R\}.$$

We suppose that the following hypothesis is fulfilled:

Hypothesis $\mathbf{I_1}$. *The function $f : D \to Y$ admits a Fréchet differential at every point $x \in D$, this differential being $f'(x) \in (X, Y)^*$. The function $f' : D \to (X, Y)^*$ verifies the Lipschitz condition, namely there exists a constant $L > 0$ so that for any $x \in D$ the following inequality is true:*

$$\left\|f'(x) - f'(y)\right\| \leq L \|x - y\|_X. \tag{26}$$

In the same context we suppose that for any $x \in D$ the mapping $f'(x) \in \in (X, Y)^*$ is invertible, therefore the inverse mapping $[f'(x)]^{-1} \in (Y, X)^*$ exists. We also suppose the existence of a number $B > 0$ so that for any $x \in D$ we have the inequality:

$$\left\|[f'(x)]^{-1}\right\| \leq B. \tag{27}$$

The main result of this paper is the following:

THEOREM 4. *If the following hypotheses are true:*

i) *the linear normed space $(X, \|\cdot\|_X)$ is a Banach space and the function $f : D \to Y$ verifies the hypothesis $\mathbf{I}_1$.*

ii) *the numbers* $C_1, C_2 > 0,\ r \in \mathbb{N} \cup \{0\}$ *and* $q \in \mathbb{N}$ *exist and the following conditions are verified:*

$$(28)\quad \begin{cases} \frac{L}{2} B^2 (1+C_2)^2 g^2 (C_1, C_2) h(C_2) + \\ + g(C_1, C_2) \left[C_2^{r+1} + LMBC_1 (1+C_2) h(C_2)\right] \le C_1 \\ C_2 + LMBC_1 (1+C_2) + LB^2 (1+C_2)^2 g(C_1, C_2) h(C_2) \le C_2^{\frac{1}{q+1}}, \end{cases}$$

together with the inequality:

$$(29)\quad d = \max\left\{\frac{1}{C_1} \|f(x_0)\|_Y, \frac{1}{C_2} \left\|\mathbf{I}_Y - f'(x_0) A_0\right\|\right\} < 1$$

and the inclusion relation $\mathcal{S}(x_0, R) \subseteq D$, *where:*

$$(30)\quad R = \left[2MC_1 + B(1+C_2) g(C_1, C_2) h(C_2)\right] \frac{d}{1 - d^{\alpha - 1}},$$

with the number α *has the value:*

$$\min\{2(p_1+q_1), ..., 2(p_m+q_m), p_1+q_1+1, ..., p_m+q_m+1, q+1\},$$

then:

j) *the sequences* $(x_n)_{n\ge0}, (y_n)_{n\ge0} \subseteq D$ *and* $(A_n)_{n\ge0} \subseteq (Y, X)^*$ *that for any* $n \ge 0$ *verify the relations* (22) *and the inequalities* (25) *and they are convergent;*

jj) *the equation* (1) *has a solution* $\overline{x} \in \mathcal{S}(x_0, R)$ *and* $\overline{x} = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$;

jjj) *the mapping* $\overline{A} = [f'(\overline{x})]^{-1} \in (Y, X)^*$ *exists and* $\overline{A} = \lim_{n\to\infty} A_n$;

jv) *for any* $n \ge 0$ *the following inequalities are true:*

$$(31)\quad \|x_{n+1} - x_n\|_X \le \left[MC_1 + B(1+C_2) g(C_1, C_2) h(C_2)\right] d^{\alpha^n},$$

$$(32)\quad \|A_{n+1} - A_n\| \le B(1+C_2) \frac{\beta d^{\alpha^n} - \left(\beta d^{\alpha^n}\right)^{q+1}}{1 - \beta d^{\alpha^n}},$$

where:

$$\beta = C_1 + BLMC_1(1+C_2) + B^2 L (1+C_2)^2 h(C_2) g(C_1, C_2),$$

$$(33)\quad \|\overline{x} - x_n\|_X \le \left[MC_1 + B(1+C_2) g(C_1, C_2) h(C_2)\right] \frac{d^{\alpha^n}}{1 - d^{\alpha^n(\alpha-1)}},$$

$$(34)\quad \left\|\overline{A} - A_n\right\| \le B(1+C_2) \sum_{k=1}^{q} \frac{\left(\beta d^{\alpha^n}\right)^k}{1 - d^{k\alpha^n(\alpha-1)}},$$

$$(35)\quad \begin{aligned} &\|\overline{x} - y_n\|_X \le \\ &\le \left[2MC_1 + B(1+C_2) g(C_1, C_2) h(C_2)\right] \frac{d^{\alpha^n}}{1 - d^{\alpha^n(\alpha-1)}}. \end{aligned}$$

Proof. Using the method of mathematical induction we will prove that the following assertions are true for any $n \ge 0$.

a) $x_n \in \mathcal{S}(x_0, R)$,
b) $\rho_n = \|f(x_n)\|_Y \leq C_1 d^{\alpha^n}$,
$\delta_n = \|I_Y - f'(x_n) A_n\| \leq C_2 d^{\alpha^n}$,
c) $\|A_n\| \leq B(1 + C_2)$,
d) $y_n \in \mathcal{S}(x_0, R)$.

Indeed, from the theorem's hypotheses it is clear that the assertions a)–c) are true for $n = 0$ with the specification for c) that:

$$\|A_0\| = \left\|[f'(x_0)]^{-1} + A_0 - [f'(x_0)]^{-1}\right\| \leq$$
$$\leq \left\|[f'(x_0)]^{-1}\right\| (1 + \|I_Y - f'(x_0) A_0\|) \leq B(1 + C_2).$$

For the assertion d) with $n = 0$ we have:

$$\|y_0 - x_0\|_X \leq M \|f(x_0)\|_Y \leq MC_1 d \leq R.$$

We suppose that the assertions a)–d) are true for values of indexes that do not go beyond a certain $n \geq 0$ and from this hypothesis we deduce that they are true for $n + 1$.

Let us look at them successively:

a) For any $i \in \{1, 2, ..., n\}$ we have:

$$\|x_{i+1} - x_i\|_X \leq \|y_i - x_i\|_X + \|A_i\| \cdot \|f(y_i)\|_Y \sum_{k=0}^{r} \left\|\mathbf{I}_Y - f'(x_i) A_i\right\|^k.$$

From the induction's hypothesis and the hypotheses of the theorem we deduce that

$$\|y_i - x_i\|_X \leq M \|f(x_i)\|_Y \leq MC_1 d^{\alpha^i},$$
$$\|A_i\| \leq B(1 + C_2)$$

and

$$\|f(y_i)\|_Y \leq \sum_{j=1}^{m} K_j \|f(x_i)\|_Y^{p_j} \cdot \left\|\mathbf{I}_Y - f'(x_i) A_i\right\|^{q_j} \leq$$
$$\leq \sum_{j=1}^{m} K_j C_1^{p_j} C_2^{q_j} d^{(p_j + q_j)\alpha^i}.$$

But for any $j \in \{1, 2, ..., m\}$ we have that $p_j + q_j \geq 1$, therefore for the same values of the natural number j we deduce that $(p_j + q_j)\alpha^i \geq \alpha^i$ and as $d < 1$ it result that

$$\|f(y_i)\|_Y \leq d^{\alpha^i} \sum_{j=1}^{m} K_j C_1^{p_j} C_2^{q_j} = g(C_1, C_2) d^{\alpha^i}.$$

Finally

$$\sum_{k=0}^{r}\left\|\mathbf{I}_Y - f'(x_i)A_i\right\|^k \le \sum_{k=0}^{r} C_2^k d^{k\alpha^i}.$$

As $k\alpha^i \ge 0$ and $d<1$ we deduce that $d^{k\alpha^i} \le 1$, therefore $\sum_{k=0}^{r} C_2^k d^{k\alpha^i} \le$ $\le \sum_{k=0}^{r} C_2^k = h(C_2)$.

In this way

$$\|x_{i+1}-x_i\|_X \le [MC_1 + B(1+C_2)g(C_1,C_2)h(C_2)]\,d^{\alpha^i},$$

from where

$$\text{(36)}\qquad \|x_{n+1}-x_0\|_X \le \sum_{i=0}^{n}\|x_{i+1}-x_i\|_X \le$$

$$\le [MC_1 + B(1+C_2)g(C_1,C_2)h(C_2)]\sum_{i=0}^{n} d^{\alpha^i} <$$

$$< [MC_1 + B(1+C_2)g(C_1,C_2)h(C_2)]\,\tfrac{d}{1-d^{\alpha-1}} < R,$$

therefore it is clear that $x_{n+1}\in\mathcal{S}(x_0,R)$.

b) It is clear that

$$\text{(37)}\qquad \begin{aligned}\|f(x_{n+1})\|_Y \le \|f(x_{n+1}) - f(y_n) - f'(y_n)(x_{n+1}-y_n)\|_Y +\\ + \|f(y_n)+f'(y_n)(x_{n+1}-y_n)\|_Y.\end{aligned}$$

Since the mapping $f' : D \to (Y,X)^*$ verifies the Lipschitz condition, it is known that for any $x,y\in D$ the following inequality is true:

$$\text{(38)}\qquad \left\|f(x)-f(y)-f'(y)(x-y)\right\|_Y \le \tfrac{L}{2}\|x-y\|_X^2.$$

Taking into account the fact that $x_{n+1}, y_n \in \mathcal{S}(x_0,R)$ we deduce that

$$\text{(39)}\qquad \left\|f(x_{n+1})-f(y_n)-f'(y_n)(x_{n+1}-y_n)\right\|_Y \le \tfrac{L}{2}\|x_{n+1}-y_n\|_X^2.$$

But

$$\|x_{n+1}-y_n\|_X \le \left\|S^{(r+1)}_{f'(x_n)}(A_n)\right\|\cdot\|f(y_n)\|_Y \le \|A_n\|\cdot$$

$$\cdot\left(\sum_{k=0}^{n}\left\|\mathbf{I}_Y - f'(x_n)A_n\right\|^k\right)\cdot\left(\sum_{j=1}^{m} K_j\|f(x_n)\|_Y^{p_j}\left\|I_Y - f'(x_n)A_n\right\|^{q_j}\right) \le$$

$$\le B(1+C_2)h(C_2)\sum_{j=1}^{m} K_j C_1^{p_j} C_2^{q_j} d^{(p_j+q_j)\alpha^n}.$$

From the definition of α we deduce that for any $j=\overline{1,m}$ we have that $p_j+q_j \ge \frac{\alpha}{2}$ and as $d<1$ we have

$$\text{(40)}\qquad \|x_{n+1}-y_n\|_X \le B(1+C_2)h(C_2)g(C_1,C_2)\,d^{\frac{1}{2}\alpha^{n+1}}.$$

From the inequalities (39) and (40) it results that

$$\text{(41)}\quad \begin{aligned}&\|f(x_{n+1})-f(y_n)-f'(y_n)(x_{n+1}-y_n)\|_Y\le\\&\le\tfrac{L}{2}B^2(1+C_2)^2h^2(C_2)\,g^2(C_1,C_2)\,d^{\alpha^{n+1}}.\end{aligned}$$

At the same time

$$\text{(42)}\quad \left\|f(y_n)+f'(y_n)(x_{n+1}-y_n)\right\|_Y\le\left\|\mathbf{I}_Y-f'(y_n)\,\mathbf{S}^{(r+1)}_{f'(x_n)}(A_n)\right\|\cdot\|f(y_n)\|.$$

Here

$$\text{(43)}\quad \left\|\mathbf{I}_Y-f'(x_n)\,\mathbf{S}^{(r+1)}_{f'(x_n)}(A_n)\right\|\le\left\|\mathbf{I}_Y-f'(x_n)A_n\right\|^{r+1}\le C_2^{r+1}d^{(r+1)\alpha^n}$$

and from $x_n,y_n\in\mathcal{S}(x_0,R)\subseteq D$ we deduce that

$$\left\|f'(x_n)-f'(y_n)\right\|\le L\|x_n-y_n\|_X\le LM\|f(x_n)\|_Y\le LMC_1d^{\alpha^n},$$

and finally

$$\begin{aligned}\left\|\mathbf{S}^{(r+1)}_{f'(x_n)}(A_n)\right\|&\le\|A_n\|\sum_{k=0}^{r}\left\|\mathbf{I}_Y-f'(x_n)A_n\right\|^k\le B(1+C_2)\sum_{k=0}^{r}C_2^kd^{k\alpha^n}\le\\&\le B(1+C_2)\,h(C_2).\end{aligned}$$

Taking into account the fact that $r\ge0$, from the inequality (43) we deduce that

(44)

$$\left\|\mathbf{I}_Y-f'(y_n)\,\mathbf{S}^{(r+1)}_{f'(x_n)}(A_n)\right\|\le C_2^{r+1}d^{(r+1)\alpha^n}+LMBC_1(1+C_2)\,h(C_2)\,d^{\alpha^n}.$$

The last expression is not great as $d^{\alpha^n}\left[C_2^{r+1}+LMBC_1(1+C_2)\,h(C_2)\right]$. From (42) and (2) we deduce that

$$\text{(45)}\quad \begin{aligned}&\|f(y_n)+f'(y_n)(x_{n+1}-y_n)\|_Y\le\\&\le d^{\alpha^n}\left[C_2^{r+1}+LMBC_1(1+C_2)\,h(C_2)\right]\cdot\\&\cdot\sum_{j=1}^{m}K_j\|f(x_n)\|_Y^{p_j}\|I_Y-f'(x_n)A_n\|^{q_j},\end{aligned}$$

and so

$$\text{(46)}\quad \begin{aligned}&\|f(y_n)+f'(y_n)(x_{n+1}-y_n)\|_Y\le\\&\le\left[C_2^{r+1}+LMBC_1(1+C_2)\,h(C_2)\right]\sum_{j=1}^{m}K_jC_1^{p_j}C_2^{q_j}d^{(p_j+q_j+1)\alpha^n}.\end{aligned}$$

As for any $j=\overline{1,m}$ we have $p_j+q_j+1\ge\alpha$, therefore $(p_j+q_j+1)\,\alpha^n\ge\alpha^{n+1}$ and $d^{(p_j+q_j+1)\alpha^n}\le d^{\alpha^{n+1}}$ and so:

$$\text{(47)}\quad \begin{aligned}&\|f(y_n)+f'(y_n)(x_{n+1}-y_n)\|_Y\le\\&\le\left[C_2^{r+1}+LMBC_1(1+C_2)\,h(C_2)\right]g(C_1,C_2)\,d^{\alpha^{n+1}}.\end{aligned}$$

From the relations (37), (41) and (47), using the hypothesis implying that the pair (C_1, C_2) verifies the system (28), we deduce that:

$$\frac{\|f(x_{n+1})\|_Y}{d^{\alpha^{n+1}}} \le \frac{L}{2} B^2 (1+C_2)^2 h^2(C_2)\, g^2(C_1,C_2) + \tag{48}$$
$$+ \left[C_2^{r+1} + LMBC_1(1+C_2)\, h(C_2)\right] g(C_1,C_2) \le C_1,$$

also:

$$\|f(x_{n+1})\|_Y \le C_1 d^{\alpha^{n+1}}.$$

For the second inequality from b) it is clear that:

$$\|\mathbf{I}_Y - f'(x_{n+1})\, A_{n+1}\| =$$
$$= \left\|\mathbf{I}_Y - f'(x_{n+1})\, \mathbf{S}^{(q+1)}_{f'(x_{n+1})}(A_n)\right\| \le \|\mathbf{I}_Y - f'(x_{n+1})\, A_n\|^{q+1} \le \tag{49}$$
$$\le \left[\|\mathbf{I}_Y - f'(x_n)\, A_n\| + \|f'(x_n) - f'(x_{n+1})\| \cdot \|A_n\|\right]^{q+1}.$$

In the last expression from (49) it is clear that:

$$\|\mathbf{I}_Y - f'(x_n)\, A_n\| \le C_1 d^{\alpha^n},\ \ \|A_n\| \le B(1+C_2),\ \ \|f'(x_n) - f'(x_{n+1})\| \le$$
$$\le L\|x_{n+1} - x_n\|_X \le L\left[\|y_n - x_n\|_X + \left\|\mathbf{S}^{(r+1)}_{f'(x_n)}(A_n)\right\| \cdot \|f(y_n)\|_Y\right] \le$$
$$\le L\left[M\|f(x_n)\|_Y + B(1+C_2)\, h(C_2)\, \|f(y_n)\|_Y\right] \le$$
$$\le L\left[MC_1 + B(1+C_2)\, h(C_2)\, g(C_1,C_2)\right] d^{\alpha^n}.$$

In this way, from (49) we obtain:

$$\frac{\|\mathbf{I}_Y - f'(x_{n+1})A_{n+1}\|}{d^{(q+1)\alpha^n}} \le \tag{50}$$
$$\le \left\{C_1 + BL(1+C_2)\left[MC_1 + B(1+C_2)\, h(C_2)\, g(C_1,C_2)\right]\right\}^{q+1}.$$

As $q+1 \ge \alpha$ and $d<1$ it is clear that $d^{(q+1)\alpha^n} < d^{\alpha^{n+1}}$, therefore, using again the fact that (C_1, C_2) is a solution of the system (28), from the inequality (50) we deduce that

$$\left\|\mathbf{I}_Y - f'(x_{n+1})\, A_{n+1}\right\| \le C_2 d^{\alpha^{n+1}}, \tag{51}$$

and so the second inequality from b) is also true.

c) As $x_{n+1} \in \mathcal{S}(x_0, R)$ it is clear that the mapping $f'(x_{n+1}) \in (X,Y)^*$ exists, this mapping is invertible, so the mapping $[f'(x_{n+1})]^{-1} \in (Y,X)^*$ exists and we have the inequality $\left\|[f'(x_{n+1})]^{-1}\right\| \le B$.

Thus we have the inequalities:

$$\|A_{n+1}\| = \left\|[f'(x_{n+1})]^{-1} + A_{n+1} - [f'(x_{n+1})]^{-1}\right\| \le$$
$$\le \left\|[f'(x_{n+1})]^{-1}\right\| + \left\|[f'(x_{n+1})]^{-1}(I_Y - f'(x_{n+1})\, A_{n+1})\right\| \le \tag{52}$$
$$\le \left\|[f'(x_{n+1})]^{-1}\right\|(1 + \|I_Y - f'(x_{n+1})\, A_{n+1}\|) \le B\left(1 + C_2 d^{\alpha^{n+1}}\right),$$

and as $\alpha > 1$ and $d<1$ it is clear that $d^{\alpha^{n+1}} < 1$, therefore
$\|A_{n+1}\| \le B(1+C_2)$.

d) It is clear as well that:

$$\|y_{n+1} - x_0\|_X \leq \|y_{n+1} - x_{n+1}\|_X + \|x_{n+1} - x_0\|_X .$$

From the theorem's hypothesis we have that:

$$\|y_{n+1} - x_{n+1}\|_X \leq M \|f(x_{n+1})\|_Y \leq MC_1 d^{\alpha^{n+1}}, \tag{53}$$

therefore:

$$\|y_{n+1} - x_0\|_X \leq [2MC_1 + B(1 + C_2) g(C_1, C_2) h(C_2)] \frac{d}{1-d^{\alpha-1}} \leq R,$$

and so $y_{n+1} \in \mathcal{S}(x_0, R)$.

Thus all the 4 assertions are true for n substituted by $n+1$. Therefore based on the principle of mathematical induction, these assertions will be true for every $n \geq 0$.

We will now prove that the sequences $(x_n)_{n\geq 0} \subseteq D \subseteq X$ and $(A_n)_{n\geq 0} \subseteq (Y, X)^*$ are Cauchy sequences in the respective spaces.

Indeed, for any $n \in \mathbb{N}$ we have:

$$\|x_{n+m} - x_n\| \leq \sum_{i=n}^{n+m-1} \|x_{i+1} - x_i\|_X \leq$$
$$\leq [MC_1 + B(1 + C_2) g(C_1, C_2) h(C_2)] \cdot \left(d^{\alpha^n} + d^{\alpha^{n+1}} + ... + d^{\alpha^{n+m-1}}\right).$$

But:

$$d^{\alpha^n} + d^{\alpha^{n+1}} + ... + d^{\alpha^{n+m-1}} = d^{\alpha^n}\left[1 + d^{\alpha^{n+1}-\alpha^n} + ... + d^{\alpha^{n+m-1}-\alpha^n}\right].$$

For this expression, for any $i = \overline{1, m-1}$, we have:

$$\alpha^{n+i} - \alpha^n = \alpha^n\left(\alpha^i - 1\right) = \alpha^n(\alpha - 1)\left(1 + \alpha + ... + \alpha^{i-1}\right).$$

As $\alpha \geq 1$ it is clear that $1 + \alpha + ... + \alpha^{i-1} \geq i$, therefore
$\alpha^{n+i} - \alpha^n \geq i\alpha^n(\alpha - 1)$ and as $d < 1$ we deduce in addition that:

$$d^{\alpha^{n+i}-\alpha^n} \leq \left(d^{\alpha^n(\alpha-1)}\right)^i,$$

therefore:

$$d^{\alpha^n} + d^{\alpha^{n+1}} + ... + d^{\alpha^{n+m-1}} < d^{\alpha^n} \sum_{i=0}^{\infty} \left(d^{\alpha^n(\alpha-1)}\right)^i = \frac{d^{\alpha^n}}{1-d^{\alpha^n(\alpha-1)}},$$

and so:

$$\|x_{n+m} - x_n\|_X \leq [MC_1 + B(1 + C_2) g(C_1, C_2) h(C_2)] \frac{d^{\alpha^n}}{1-d^{\alpha^n(\alpha-1)}}. \tag{54}$$

At the same time:

$$\|A_{n+1} - A_n\| \leq \|A_n\| \sum_{k=1}^{q} \left\|\mathbf{I}_Y - f'(x_{n+1}) A_n\right\|^k$$

and through the same proceeding as in the case of the relations (49) we will have:

$$\|A_{n+1} - A_n\| \leq B\left(1 + C_2\right) \sum_{k=1}^{q} \left(\beta d^{\alpha^n}\right)^k, \tag{55}$$

where β has the value from the enunciation.

It is clear that:

$$\sum_{k=1}^{q} \left(\beta d^{\alpha^n}\right)^k = \frac{\beta d^{\alpha^n} - \left(\beta d^{\alpha^n}\right)^{q+1}}{1-\beta d^{\alpha^n}},$$

therefore we will obtain the inequality (32).

From the inequality (55) we deduce immediately that:

$$\begin{aligned} \|A_{n+m} - A_n\| &\leq \sum_{i=n}^{n+m-1} \|A_{i+1} - A_i\| \leq \\ \leq B\left(1 + C_2\right) \sum_{i=n}^{n+m-1} \sum_{k=1}^{q} \left(\beta d^{\alpha^i}\right)^k &= B\left(1 + C_2\right) \sum_{k=1}^{q} \beta^k \sum_{i=n}^{n+m-1} d^{k\alpha^i}, \end{aligned} \tag{56}$$

therefore:

$$\|A_{n+m} - A_n\| \leq B\left(1 + C_2\right) \sum_{k=1}^{q} \frac{\left(\beta d^{\alpha^n}\right)^k}{1-d^{k\alpha^n(\alpha-1)}}. \tag{57}$$

As $d < 1$ and $\alpha \geq 1$ it is clear that:

$$\lim_{n\to\infty} \frac{d^{\alpha^n}}{1-d^{\alpha^n(\alpha-1)}} = \lim_{n\to\infty} \sum_{k=1}^{q} \frac{\left(\beta d^{\alpha^n}\right)^k}{1-d^{k\alpha^n(\alpha-1)}} = 0$$

and so the inequalities (54) and (57) indicate that the sequence $(x_n)_{n\geq 0}$ is a Cauchy sequence in the space X and the sequence $(A_n)_{n\geq 0}$ is a sequence of the same type in the space $(Y, X)^*$.

As $(X, \|\cdot\|_X)$ is a Banach space, on account of a well-known theorem, the space $((Y, X)^*, \|\cdot\|)$ is a Banach space as well, so the sequence $(x_n)_{n\geq 0} \subseteq \subseteq X$ and $(A_n)_{n\geq 0} \subseteq (Y, X)^*$ are convergent in the respective spaces, therefore there exist $\overline{x} \in X$ and $\overline{A} \in (Y, X)^*$ so that $\overline{x} = \lim\limits_{n\to\infty} x_n$ and $\overline{A} =$ $= \lim\limits_{n\to\infty} A_n$.

If in the inequalities (54) and (57) we consider $m \to \infty$ we obtain the inequalities (33) and (34) from the theorem's conclusions.

From the inequality (32) with $n = 0$ we obtain:

$$\|\overline{x} - x_0\| \leq \left[MC_1 + B\left(1 + C_2\right) g\left(C_1, C_2\right) h\left(C_2\right)\right] \frac{d}{1-d^{\alpha-1}} \leq R$$

therefore $\overline{x} \in \mathcal{S}\left(x_0, R\right)$.

From the fact that $\mathcal{S}\left(x_0, R\right) \subseteq D$ we deduce $\overline{x} \in D$ therefore the mapping $f'\left(\overline{x}\right) \in (X, Y)^*$ exists and even $\left[f'\left(\overline{x}\right)\right]^{-1} \in (Y, X)^*$ together with the inequality $\left\|\left[f'\left(\overline{x}\right)\right]^{-1}\right\| \leq B$.

From the fact that for any $n \in \mathbb{N}$ we have the inequality $\|f(x_n)\|_Y \leq$ $\leq C_1 d^{\alpha^n}$ with $d < 1$ and $\alpha > 1$ we deduce that $\lim\limits_{n\to\infty} \|f(x_n)\|_Y = 0$, from where on account of the continuity of the norme and of the mapping $f : D \to Y$ in $\overline{x}$ (this last is even differentiable in $\overline{x}$) we deduce that $\left\|f\left(\lim\limits_{n\to\infty} x_n\right)\right\|_Y = 0$ that ip $\|f(\overline{x})\|_Y = 0$, from where $f(\overline{x}) = \theta_Y$, therefore $\overline{x}$ is a solution of the equation (1).

On the other hand:

$$\left[f'(x_n)\right]^{-1} - \left[f'(\overline{x})\right]^{-1} = \left[f'(x_n)\right]^{-1}\left(f'(\overline{x}) - f'(x_n)\right)\left[f'(\overline{x})\right]^{-1},$$

therefore:

$$\begin{aligned} &\left\|[f'(x_n)]^{-1} - [f'(\overline{x})]^{-1}\right\| \leq \\ \leq &\left\|[f'(x_n)]^{-1}\right\| \cdot \left\|[f'(\overline{x})]^{-1}\right\| \cdot \|f'(\overline{x}) - f'(x_n)\| \leq B^2 L \|\overline{x} - x_n\|_X . \end{aligned} \tag{58}$$

As $\lim\limits_{n\to\infty} \|\overline{x} - x_n\|_X = 0$ we deduce that $\lim\limits_{n\to\infty} \left\|[f'(x_n)]^{-1} - [f'(\overline{x})]^{-1}\right\| = 0$, therefore:

$$\lim_{n\to\infty} \left[f'(x_n)\right]^{-1} = \left[f'(\overline{x})\right]^{-1} \tag{59}$$

in the sense of the norm from $(Y, X)^*$.

Afterwards we deduce that:

$$\begin{aligned} &\left\|A_n - [f'(\overline{x})]^{-1}\right\| \leq \\ \leq &\left\|A_n - [f'(x_n)]^{-1}\right\| + \left\|[f'(x_n)]^{-1} - [f'(\overline{x})]^{-1}\right\| \leq \\ \leq &\left\|[f'(x_n)]^{-1}\right\| \cdot \|\mathbf{I}_Y - f'(x_n) A_n\| + \left\|[f'(x_n)]^{-1} - [f'(\overline{x})]^{-1}\right\| \leq \\ \leq &BC_2 d^{\alpha^n} + \left\|[f'(x_n)]^{-1} - [f'(\overline{x})]^{-1}\right\|, \end{aligned} \tag{60}$$

from where, on account of the fact that $d < 1$ and $\alpha > 1$ together with the relation (59), we deduce that $\lim\limits_{n\to\infty} \left\|A_n - [f'(\overline{x})]^{-1}\right\| = 0$, therefore:

$$\lim_{n\to\infty} A_n = \left[f'(\overline{x})\right]^{-1}.$$

On account of the unicity of the limit we deduce that $\overline{A} = [f'(\overline{x})]^{-1}$.

On account of the inequality (53) we deduce that $\lim\limits_{n\to\infty} \|y_n - x_n\|_X = 0$, therefore:

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n = \overline{x}.$$

From the same inequality (53) we deduce that:

$$\|y_n - x_n\|_X \leq MC_1 \tfrac{d^{\alpha^n}}{1 - d^{\alpha^n(\alpha-1)}},$$

from where:

$$\|y_n - \overline{x}\|_X \leq \|y_n - x_n\|_X + \|\overline{x} - x_n\|_X ,$$

and in this way one obtain the inequality (35).

The theorem is proven. □

3. SPECIAL CASES

Let us consider the case in which we choose the sequence $(y_n)_{n\geq 0} \subseteq X$ defined through:

$$y_n = x_n - \mathbf{S}^{(p+1)}_{f'(x_n)}(A_n)\, f(x_n) \tag{61}$$

with a certain $p \in \mathbb{N}$. In this case the relations of recurrence (22) become:

$$\begin{cases} x_{n+1} = & x_n - \mathbf{S}^{(p+1)}_{f'(x_n)}(A_n)\, f(x_n) - \\ & -\mathbf{S}^{(r+1)}_{f'(x_n)}(A_n)\, f\left(x_n - \mathbf{S}^{(p+1)}_{f'(x_n)}(A_n)\, f(x_n)\right), \\ A_{n+1} = & \mathbf{S}^{(q+1)}_{f'(x_{n+1})}(A_n). \end{cases} \tag{62}$$

In this case the function $g : [0,+\infty[\times [0,+\infty[\to \mathbb{R}$ is defined through:

$$g(u,v) = \tfrac{LB^2}{2} u^2 (1+v)^2 \sum_{k=0}^{p} v^k + uv^{p+1} \tag{63}$$

where $L, B > 0$ are the constants that result from the verification of the hypothesis $\mathbf{I_1)}$ also needed in this case.

The convergence of the pair of sequences $(x_n)_{n\in\mathbb{N}\cup\{0\}} \subseteq D$ and $(A_n)_{n\geq 0} \subseteq (Y,X)^*$ is expressed through the following:

THEOREM 5. *If the following statements are true:*

i) *the linear normed space* $(X, \|\cdot\|_X)$ *is a Banach space and with regard to the function* $f : D \to Y$ *the hypothesis* $\mathbf{I_1)}$ *is true;*

ii) *the numbers* $C_1, C_2 > 0$, $r \in \mathbb{N}\cup\{0\}$ *and* $q \in \mathbb{N}$ *exist and the system* (28), *of the inequality* (29) *together with the relation of inclusion* $\mathcal{S}(x_0, R) \subseteq D$, *where* R *is expressed through the equality* (30) *and* $\alpha = \min\{3, q+1\}$ *are verified;*

then:

j) *if the sequence* $(y_n)_{n\geq 0}$ *is defined by relation* (61), *then for any* $n \geq 0$ *the following relations, of the same type as the relation* (25), *are true:*

$$\begin{cases} \|f(y_n)\|_Y \leq & \frac{LB^2}{2}(1+\delta_n)^2 \|f(x_n)\|_Y^2 \sum\limits_{k=0}^{p} \delta_n^k + \\ & +\delta_n^{p+1} \cdot \|f(x_n)\|_Y, \\ \|y_n - x_n\|_X \leq & B(1+C_2)\, h(C_2)\, \|f(x_n)\|_Y, \end{cases} \tag{64}$$

where $\delta_n = \|\mathbf{I}_Y - f'(x_n) A_n\|$;

jj) *the conclusions j)–jv) of the theorem 4 with* $g(u,v)$ *defined through* (62) *and* $M = B(1+C_2)\, h(C_2)$ *are true.*

Proof. We apply Theorem 4 in the case in which the sequence $(y_n)_{n\geq 0}$ is generated by the relation (61). For this it is enough to show that for any $n \geq 0$ the relations (64) are true.

Through mathematical induction we will show that for any $n \geq 0$ the propositions a)–c) from the proof of Theorem 4 followed by the inequalities (64) are true.

The relations a)–c) for a certain number $n \in \mathbb{N} \cup \{0\}$ are proved in the same manner as in the proof of Theorem 4.

Afterwards for the same number $n \geq 0$ we have:

$$\|y_n - x_n\|_X \leq \left\|\mathbf{S}^{(p+1)}_{f'(x_n)}(A_n)\right\| \cdot \|f(x_n)\|_Y .$$

But:

$$\left\|\mathbf{S}^{(p+1)}_{f'(x_n)}(A_n)\right\| \leq \|A_n\| \sum_{k=0}^{p} \left\|I_Y - f'(x_n) A_n\right\|^k$$

and:

$$\|A_n\| = \left\|\left[f'(x_n)\right]^{-1} + A_n - \left[f'(x_n)\right]^{-1}\right\| \leq$$

$$\leq \left\|\left[f'(x_n)\right]^{-1}\right\| \left(1 + \left\|I_Y - f'(x_n) A_n\right\|\right) \leq B\left(1 + \left\|I_Y - f'(x_n) A_n\right\|\right).$$

From here:

$$\|f(y_n)\|_Y \leq$$

$$\leq \|f(y_n) - f(x_n) - f'(x_n)(y_n - x_n)\|_Y + \|f(x_n) + f'(x_n)(y_n - x_n)\|_Y ,$$

for which:

$$\|f(y_n) - f(x_n) - f'(x_n)(y_n - x_n)\|_Y \leq \tfrac{L}{2}\|y_n - x_n\|_X^2 \leq$$

$$\leq \tfrac{LB^2}{2}\left(1 + \|I_Y - f'(x_n) A_n\|\right)^2 \|f(x_n)\|_Y^2 \cdot \sum_{k=1}^{p} \|I_Y - f'(x_n) A_n\|^k ,$$

and:

$$\|f(x_n) + f'(x_n)(y_n - x_n)\|_Y \leq \left\|I_Y - S^{(p+1)}_{f'(x_n)}(A_n)\right\| \cdot \|f(x_n)\|_Y +$$

$$+ \|I_Y - f'(x_n) A_n\|^{p+1} \cdot \|f(x_n)\|_Y .$$

In this way:

$$\|f(y_n)\|_Y \leq$$

$$\leq \tfrac{LB^2}{2}\left(1 + \|I_Y - f'(x_n) A_n\|\right)^2 \|f(x_n)\|_Y^2 \cdot \sum_{k=1}^{p} \|I_Y - f'(x_n) A_n\|^k +$$

$$+ \|I_Y - f'(x_n) A_n\|^{p+1} \cdot \|f(x_n)\|_Y .$$

For the finishing of the second inequality from (64) we have:

$$\|y_n - x_n\|_X \leq \left[B(1 + C_2)\sum_{k=0}^{p} C_2^k\right] \|f(x_n)\|_Y = B(1 + C_2)\, h(C_2)\, \|f(x_n)\|_Y .$$

The theorem is thus proven. □

Remark 6. From the conclusions of Theorem 4 it results that the convergence speed of the sequence $(x_n)_{n\geq 0}$ does not depend on r and p. So the order of this speed is $\alpha = 3$ for any $q \geq 2$. In this way, for the simplicity of the calculation, the most efficient methods are the ones for which $r = p = 0$ and $q = 2$. The relations (62) will become:

$$\text{(65)} \qquad \begin{cases} x_{n+1} = x_n - A_n f(x_n) - A_n f(x_n - A_n f(x_n)), \\ A_{n+1} = A_n \left[3I_Y - 3f'(x_{n+1}) A_n + (f'(x_{n+1}) A_n)^2 \right]. \end{cases}$$

□

If we introduce this values in the expression of $g(u,v)$ and $h(v)$ we will obtain:

$$g(u,v) = \tfrac{LB^2}{2} u^2 (1+v)^2 + uv, \quad h(v) = 1.$$

The pair $(C_1, C_2) \in [0, +\infty[\times [0, +\infty[$ must be a solution of the system in u and v :

$$\begin{cases} z\left[z(z-v) + w\right] \leq 1 \\ \left[w + z(2z-v)\right]^3 \leq v \end{cases}$$

where $z = \frac{LB^2}{2} u (1+v)^2 + v$ and $w = v + LMBu(1+v)$.

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