REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 38 (2009) no. 1, pp. 45–53
ictp.acad.ro/jnaat

SOME FUNCTIONAL DIFFERENTIAL EQUATIONS WITH BOTH RETARDED AND ADVANCED ARGUMENTS

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Abstract. In this paper we shall study a functional differential equations of mixed type. This equation is a generalization of some equations from medicine. Related to this equation we study the existence of the solution by contraction's principle and Schauder's fixed point theorem.

MSC 2000. 47H10, 34H10.
Keywords. Functional-differential equations, boundary value problems, contraction's principle, Schauder's fixed point principle.

1. INTRODUCTION

Functional differential equations with advanced and retarded arguments (called here as functional differential equations of mixed type – MFDE) have had a slower contributions in mathematical researches, that is if we compare them with functional differential equations with delay. This is due the fact that in our type of equations we have in the same time both advanced and retarded argument, each of them having different behavior. These equations have the important feature that both the history and the future status of the system affect their change rate at the present time.

Here we have a general form of MFDE problem

$$x'(t) = f(t, x(t), x(t-h), x(t+h)) + \lambda, \quad t \in (a, b), \tag{1}$$

$$x(t) = \varphi(t), \quad t \in [a-h, a], \tag{2}$$

$$x(t) = \zeta(t), \quad t \in [b, b+h], \tag{3}$$

where $f \in C((a,b) \times \mathbb{R}^3; \mathbb{R})$, $\varphi \in C([a-h,a]; \mathbb{R})$, $\zeta \in C([b, b+h]; \mathbb{R})$ and $a,\ b \in \mathbb{R}$, $a < b$, $h > 0$.

The first part of the results from this paper appear in I.A. Rus-V. Dârzu-Ilea [6] and contains the existence and the uniqueness of the solution of the problem (1)–(2)–(3) studied in different spaces.

This type of equations came from different fields of applications. For example, A. Rustichini [7], [8] investigated a specific mixed functional differential

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equation arising in a special way from a competitive economy; Schulman [9] gave a physical justification to MFDE. Some other fields of interest for this type of equations would be: population genetic (D.G. Aronson and H.F. Weinberger [1]), population growth, mathematical biology. Some other names in the study of MFDE are: J. Mallet-Paret [2], [3], [4], J. Wu and X. Zou [10], R. Precup [5].

The biologic signification of a MFDE can be given as follows.

The equation (1) is a model for a certain disease. This depends on the physical state of the subject – the delay argument; the treatment that should be given to the patient – the advanced argument; the parameter is an outside factor that can influence the physical state of the subject; condition (2) – represents the statistics observations obtained before from other subjects; condition (3) – represents also, from a statistical point of view, the expectations of the evolution of the disease.

From the study of population growth we can explain the above model as follows:

φ-the state of some population in an chosen environment, ζ-the state that should have the population, λ-a control parameter, $x'(t)$ is the speed of grow of the population with the low $x' = f + \lambda$.

If some part of the population is sacrificed then $\lambda < 0$, and if the population is extending numerically then $\lambda > 0$.

2. HOW TO OBTAIN A FREDHOLM-VOLTERRA INTEGRAL EQUATION

Let (x, λ) a solution for (1)–(2)–(3). It follows that:

$$x(t) = \begin{cases} \varphi(t), & t \in [a-h, a], \\ \varphi(a) + \int_a^t f(s, x(s), x(s-h), x(s+h))\mathrm{d}s + \\ \quad +\lambda(t-a), & t \in [a, b], \\ \zeta(t) & t \in [b, b+h]. \end{cases} \tag{4}$$

From the continuity in $t = b$ we have

$$\lambda = \tfrac{\zeta(b)-\varphi(a)}{b-a} - \tfrac{1}{b-a}\int_a^b f(s, x(s), x(s-h), x(s+h))\mathrm{d}s. \tag{5}$$

Thus the problem (1)–(2)–(3) is equivalent with

$$x = A(x) \ \text{ and } \lambda \ = \ \text{the right hand side of (5)},$$

where $A : C[a-h, b+h] \to C[a-h, b+h]$ and

(6)
$$A(x)(t) := \begin{cases} \varphi(t), & t \in [a-h, a] \\ \varphi(a) + \frac{t-a}{b-a}(\zeta(b) - \varphi(a)) - \\ \quad -\frac{t-a}{b-a}\int_a^b f(s, x(s), x(s-h), x(s+h))\mathrm{d}s + \\ \quad +\int_a^t f(s, x(s), x(s-h), x(s+h))\mathrm{d}s, & t \in [a, b] \\ \zeta(t), & t \in [b, b+h]. \end{cases}$$

By the contraction principle we obtain the following existence theorem.

THEOREM 1. *(I.A. Rus, V. Dârzu-Ilea* [6]*)*
If we have the following conditions

(i) *there exist* $L_f > 0$ *such as:*

$$\| f(t, u_1, v_1, w_1) - f(t, u_2, v_2, w_2) \| \leq L_f(\| u_1 - u_2 \| + \| v_1 - v_2 \| + \| w_1 - w_2 \|),$$

for all $t \in [a, b], u_i, v_i, w_i \in \mathbb{R}, i = 1, 2;$

(ii) $6L_f(b-a) < 1$.

Then the problem (1)–(2)–(3) *has a unique solution. More than that, if* (x^*, λ^*) *is the unique solution for* (1)–(2)–(3)*, then*

$$x^* = \lim_{n\to\infty} A^n(x), \quad for \quad all \quad x \in C[a-h, b+h],$$

and

(7)
$$\lambda^* = \frac{\zeta(b)-\varphi(a)}{b-a} - \frac{1}{b-a}\int_a^b f(s, x^*(s), x^*(s-h), x^*(s+h))\mathrm{d}s.$$

3. MAIN RESULT

We generalize the above problem considering the next Fredholm-Volterra integral equation

(8)
$$\begin{aligned} x(t) = &g(t) + \int_a^t K(t, s, x(s), x(s-h), x(s+h))\mathrm{d}s - \\ &- \frac{t-a}{b-a}\int_a^b K(t, s, x(s), x(s-h), x(s+h))\mathrm{d}s, \ t \in [a, b] \end{aligned}$$

with

(9)
$$\begin{cases} x(t) = \varphi(t), & t \in [a-h, a], \\ x(t) = \zeta(t), & t \in [b, b+h], \end{cases}$$

where $K \in C([a, b] \times [a, b] \times \mathbb{R}^3)$ and $g \in C([a, b]; \mathbb{R})$ given by the formula $(F1)$ $g(t) = \frac{\zeta(b)-\varphi(a)}{b-a}t + \frac{b\varphi(a)-a\zeta(b)}{b-a}$.

Our purpose here is to study the existence of the solution of the equation (8) with contraction's principle and Schauder's types theorems.

The problem (8)–(9) is equivalent with $x = T(x)$, where $T : C[a-h, b+h] \to C[a-h, b+h]$ and

(10)
$$T(x)(t) := \begin{cases} \varphi(t), & t \in [a-h, a] \\ \frac{\zeta(b)-\varphi(a)}{b-a}t + \frac{b\varphi(a)-a\zeta(b)}{b-a} - \\ \quad -\frac{t-a}{b-a}\int_a^b K(s,t,x(s),x(s-h),x(s+h))\mathrm{d}s + \\ \quad + \int_a^t K(s,t,x(s),x(s-h),x(s+h))\mathrm{d}s, & t \in [a,b] \\ \zeta(t), & t \in [b, b+h]. \end{cases}$$

By the contraction principle we obtain the following existence theorem.

THEOREM 2. *If we have the following conditions:*

(i) *there exist $L_f > 0$ such as*

$$\| K(s,t,u_1,v_1,w_1) - K(s,t,u_2,v_2,w_2) \| \leq L_f(\| u_1 - u_2 \| + \| v_1 - v_2 \| + \| w_1 - w_2 \|),$$

for all $t \in [a,b], u_i, v_i, w_i \in \mathbb{R}, i = 1, 2;$

(ii) $6L_f(b-a) < 1$

Then the problem (8)–(9) *has a unique solution, moreover the solution x^* can be obtain by the method of successive approximation beginning from any element from the space $C[a-h, b+h]$.*

Proof. Let the operator

(11)
$$T(x)(t) := \begin{cases} \varphi(t), & t \in [a-h, a] \\ g(t) + \int_a^t K(t,s,x(s),x(s-h),x(s+h))\mathrm{d}s - \\ \quad -\frac{t-a}{b-a}\int_a^b K(t,s,x(s),x(s-h),x(s+h))\mathrm{d}s, & t \in [a,b] \\ \zeta(t), & t \in [b, b+h]. \end{cases}$$

$$|Tx(t) - Ty(t)| =$$
$$= \left| \int_a^t (K(t,s,x(s),x(s-h),x(s+h)) - K(t,s,y(s),y(s-h),y(s+h)))\mathrm{d}s - \right.$$
$$\left. - \frac{t-a}{b-a}\int_a^b (K(t,s,x(s),x(s-h),x(s+h)) - K(t,s,y(s),y(s-h),y(s+h)))\mathrm{d}s \right|$$
$$\leq 3L_f(b-a) \| x-y \| + 3L_f(b-a) \| x-y \|$$
$$= 6L_f(b-a) \| x-y \|$$

But $6L_f(b-a) < 1$, follows T is a contraction. We can apply now the principle of contraction and follows the conclusion from the theorem.

This is a classical problem of Krasnoselskii type, but by applying this type of theorem in space $C[a-h, b+h]$ we obtain to many conditions on the date K, thus we apply Schauder type theorem in order to obtain optimality of the conditions on equation's data.

Let the Banach space $C[a-h, b+h]$ with the Chebyshev norm, $\| \cdot \|$.

THEOREM 3. *If we have the following conditions*

(i) $K \in C([a,b]\times[a,b]\times J^3)$, *$J$ - is a compact interval,* $g \in C([a,b])$ *given by* $(F1)$*;*

(ii) *there exists* $M \in \mathbb{R}_+$ *such that*

$$\| K(t,s,u,v,w) \| \leq M, \quad t,s \in [a,b], \quad u,v,w \in J,$$

then the equation (8) *has at least one solution* $x^* \in C([a,b])$ *with the propriety that* $\| x^* \| \leq R$, *where R is a number greater than* $2M(b-a)$.

Proof. Let the operator $T : C[a-h,b+h] \to C[a-h,b+h]$ be defined by

$$(12) \quad x(t) = \begin{cases} \varphi(t), & t \in [a-h,a] \\ g(t) + \int_a^t K(t,s,x(s),x(s-h),x(s+h))\mathrm{d}s- & \\ \quad -\frac{t-a}{b-a}\int_a^b K(t,s,x(s),x(s-h),x(s+h))\mathrm{d}s, & t \in [a,b] \\ \zeta(t). & t \in [b,b+h] \end{cases}$$

From (i) the operator T is well defined and complete continuous.

In what follows we use the conditions (ii) in order to prove the invariance on sphere

$\| T(x)(t) - g(t) \| \leq R$, with $\| x \| \leq \| g \| + R$ and $R > 0$ ($x \in \overline{B}(g;R) \Rightarrow x(t) \in J$, where $J = [-j,j]$, with $j = \| g \| + R$),

$$\begin{aligned} \| T(x)(t) - g(t) \| \leq & \int_a^t \| K(t,s,x(s),x(s-h),x(s+h)) \| \mathrm{d}s+ \\ & + \frac{t-a}{b-a}\int_a^b \| K(t,s,x(s),x(s-h),x(s+h)) \| \mathrm{d}s \leq \\ \leq & \int_a^t \| K(t,s,x(s),x(s-h),x(s+h)) \| \mathrm{d}s \\ & + \int_a^b \| K(t,s,x(s),x(s-h),x(s+h)) \| \mathrm{d}s \\ \leq & \, 2M(b-a) \leq R. \end{aligned}$$

Therefore

$$(13) \qquad \| T(x)(t) - g(t) \| \leq 2M(b-a).$$

Now we can say that for R greater than $2M(b-a)$ the operator T satisfy the invariance condition. Thus by applying Schauder's theorem it follows that there exist at least one solution x^* and for this solution we have established that

$$\| x^* \| \leq \varphi(b) + R.$$

□

Now we consider the following equation

$$(14)\qquad x(t) = g(t) + \tfrac{b-t}{b-a}\int_a^t K_1(t,s,x(s),x(s-h),x(s+h))\mathrm{d}s + \\ + \tfrac{(b-t)(t-a)}{b-a}\int_a^b K_2(t,s,x(s),x(s-h),x(s+h))\mathrm{d}s,\ t\in[a,b]$$

with the initial conditions

$$(15)\qquad \begin{cases} x(t)=\varphi(t), & t\in[a-h,a] \\ x(t)=\zeta(t), & t\in[b,b+h] \end{cases}$$

where $K_1, K_2 \in C([a,b]\times[a,b]\times\mathbb{R}^3;\mathbb{R})$ and $g \in C([a,b];\mathbb{R})$ given by the formula

$$(F1)\quad g(t) = \tfrac{\zeta(b)-\varphi(a)}{b-a}t + \tfrac{b\varphi(a)-a\zeta(b)}{b-a}.$$

The problem (14)–(15) is equivalent with $x = T(x)$, where $T : C[a-h,b+h] \to C[a-h,b+h]$ and

(16)

$$T(x)(t) := \begin{cases} \varphi(t), & t\in[a-h,a] \\ \frac{\zeta(b)-\varphi(a)}{b-a}t + \frac{b\varphi(a)-a\zeta(b)}{b-a} + \\ \quad + \frac{(b-t)(t-a)}{b-a}\int_a^b K_2(s,x(s),x(s-h),x(s+h))\mathrm{d}s + \\ \quad + \frac{b-t}{b-a}\int_a^t K_1(s,x(s),x(s-h),x(s+h))\mathrm{d}s, & t\in[a,b] \\ \zeta(t), & t\in[b,b+h]. \end{cases}$$

The existence of the solution of the equation (14) with contraction's principle is trivial.

Theorem 4. *If we have the following conditions:*

(i) *there exist $L_i > 0$ such as*

$$\| K_i(t,u_1,v_1,w_1) - K_i(t,u_2,v_2,w_2) \| \le L_i(\| u_1-u_2 \| + \| v_1-v_2 \| + \| w_1-w_2 \|),$$

for all $t\in[a,b]$, $u_j, v_j, w_j \in R$, $i,j=1,2$;

(ii) $3(L_1 + (b-a)L_2)(b-a) < 1$

Then the problem (14)–(15) *has a unique solution, more the solution x^* can be obtain by the method of successive approximation beginning from any element from the space $C[a-h,b+h]$.*

Now we consider the Banach space $C[a-h,b+h]$ with the Chebyshev norm, $\|\cdot\|$.

Theorem 5. *If we have the following conditions*

(i) $K_1, K_2 \in C([a,b]\times[a,b]\times J^3)$, *$J$ - is a compact interval, $g \in C([a,b];\mathbb{R})$ given by $(F1)$;*

(ii) *there exist real numbers $\alpha,\beta,\gamma,\delta$ such as* $\| K_1(t,s,u,v,w) \| \le \alpha \| u \| + \beta \| v \| + \gamma \| w \| + \delta$, $t,s\in[a,b]$, $u,v,w \in J$;

(iii) *there exists* $M \in \mathbb{R}$ *such that*

$$\| K_2(t,s,u,v,w) \| \leq M,\ t,s \in [a,b],\ u,v,w \in J;$$

then the equation (14) *has at least one solution* $x^* \in C([a,b];\mathbb{R})$ *with the propriety that* $\| x^* \| \leq R$, *where* R *is a number greater than*

$$\frac{(b-a)[\varphi(b)(\alpha+\beta+\gamma)+M(b-a)+\delta]}{1-(\alpha+\beta+\gamma)(b-a)}$$

with $(\alpha+\beta+\gamma)(b-a) < 1$.

Proof. Let the operator $T : C[a-h,b+h] \to C[a-h,b+h]$ be defined by

(17)
$$x(t) = \begin{cases} \varphi(t), & t \in [a-h,a] \\ g(t) + \frac{b-t}{b-a}\int_a^t K_1(t,s,x(s),x(s-h),x(s+h))\mathrm{d}s + & \\ \quad + \frac{(b-t)(t-a)}{b-a}\int_a^b K_2(t,s,x(s),x(s-h),x(s+h))\mathrm{d}s, & t \in [a,b] \\ \zeta(t). & t \in [b,b+h] \end{cases}$$

From (i) the operator T is well defined and complete continuous.

In what follows we use the conditions (ii), (iii) in order to prove the invariance on sphere

$\| T(x)(t) - g(t) \| \leq R$, with $\| x \| \leq \| g \| + R$ and $R > 0$ ($x \in \overline{B}(g;R) \Rightarrow x(t) \in J$, where $J = [-j,j]$, with $j = \| g \| + R$),

$$\begin{aligned}
&\| T(x)(t) - g(t) \| \leq \\
&\leq \frac{b-t}{b-a}\int_a^t \| K_1(t,s,x(s),x(s-h),x(s+h)) \| \mathrm{d}s + \\
&\quad + \frac{(b-t)(t-a)}{b-a}\int_a^b \| K_2(t,s,x(s),x(s-h),x(s+h)) \| \mathrm{d}s \\
&\leq \int_a^t [\alpha \| x(s) \| + \beta \| x(s-h) \| + \gamma \| x(s+h) \| + \delta]\mathrm{d}s + (b-a)\int_a^b M\mathrm{d}s \leq \\
&\leq (\alpha+\beta+\gamma) \| x \| (b-a) + \delta(b-a) + M(b-a)^2 \leq \\
&\leq (\alpha+\beta+\gamma)(\| g \| + R)(b-a) + (M(b-a)+\delta)(b-a) \\
&\leq (\alpha+\beta+\gamma)(\varphi(b)+R)(b-a) + (M(b-a)+\delta)(b-a).
\end{aligned}$$

Therefore

$$\| T(x)(t) - g(t) \| \leq (\alpha+\beta+\gamma)(\varphi(b)+R)(b-a) + (M(b-a)+\delta)(b-a). \tag{18}$$

Now we can say that if $(\alpha+\beta+\gamma)(b-a) < 1$, for R greater than

$$\frac{(b-a)[\varphi(b)(\alpha+\beta+\gamma)+M(b-a)+\delta]}{1-(\alpha+\beta+\gamma)(b-a)}$$

the operator T satisfy the invariance condition. Thus by applying the Schauder theorem it follows that there exist at least one solution x^* and for this solution we have established that $\| x^* \| \leq \varphi(b) + R$. □

EXAMPLE 6. Let the equation

$$(19)\quad x(t) = t + \tfrac{b-t}{b-a} \int_a^t [x(s) + x(s-1)]\mathrm{d}s + \tfrac{(b-t)(t-a)}{b-a} \int_a^b x(s+1)\mathrm{d}s,\ t \in [a,b]$$

with the initial conditions

$$(20)\qquad \begin{cases} x(t) = t, & t \in [a-1, a], \\ x(t) = t, & t \in [b, b+1]. \end{cases}$$

The problem (19)–(20) is equivalent with $x = T(x)$, where $T : C[a-h, b+h] \to C[a-h, b+h]$ and

$$(21)\qquad T(x)(t) := \begin{cases} t, & t \in [a-1, a] \\ \frac{(b-t)(t-a)}{b-a} \int_a^b [x(s) + x(s-1)]\mathrm{d}s + & \\ \quad + \frac{b-t}{b-a} \int_a^t x(s+1)\mathrm{d}s, & t \in [a, b] \\ \zeta(t), & t \in [b, b+1]. \end{cases}$$

The existence of the solution of the equation (19) with contraction's principle is trivial. □

THEOREM 7. *If we have the condition:*

(ii) $(b-a)(2+b-a) < 1$ *(i.e.* $a = 0, b = 1/3$*)*

Then the problem (19)–(20) *has a unique solution, more the solution* x^* *can be obtain by the method of successive approximation beginning from any element from the space* $C[a-1, b+1]$.

Proof. We have to estimate if the operator T is Lipschitz

$$\begin{aligned} |Tx - Ty| &\le |\tfrac{b-t}{b-a} \int_a^t [x(s) - y(s) + x(s-1) - y(s-1)]\mathrm{d}s| + \\ &\quad + |\tfrac{(b-t)(t-a)}{b-a} \int_a^b [x(s+1) - y(s+1)]\mathrm{d}s| \\ &\le (b-a)(2+b-a). \end{aligned}$$

□

Let now the Banach space $C[a-1, b+1]$ with the Chebyshev norm.

THEOREM 8. *The equation* (19) *with the condition* (20) *has at least one solution* $x^* \in C([a,b];\mathbb{R})$ *with the propriety that* $\| x^* \| \le R$*, where* R *is a number greater than* $\frac{b(b-a)(1+b-a)}{1-(b-a)(1+b-a)}$ *with* $(b-a)(1+b-a) < 1$ *(i.e.* $a = 0, b = \frac{1}{3}$*).*

Proof. Let the operator $T : C[a-1, b+1] \to C[a-1, b+1]$ defined by

$$(22) \qquad x(t) = \begin{cases} t, & t \in [a-1, a] \\ t + \frac{b-t}{b-a} \int_a^t [x(s) + x(s-h)] \mathrm{d}s + \\ \quad + \frac{(b-t)(t-a)}{b-a} \int_a^b x(s+h) \mathrm{d}s, & t \in [a, b] \\ t. & t \in [b, b+1] \end{cases}$$

T is well defined and complete continuous.

In what follows we prove the invariance on sphere

$\| T(x)(t) - t \| \leq R$, with $\| x \| \leq \| g \| + R$ and $R > 0$ $(x \in \overline{B}(g; R) \Rightarrow x(t) \in J$, where $J = [-j, j]$, with $j = \| g \| + R)$,

$$\begin{aligned} &\| T(x)(t) - t \| \leq \\ &\leq \tfrac{b-t}{b-a} \int_a^t [|x(s)| + |x(s-h)|] \mathrm{d}s + \tfrac{(b-t)(t-a)}{b-a} \int_a^b |x(s+h)| \mathrm{d}s \\ &\leq (b-a)(1+b-a) \| x \| \leq (b-a)(1+b-a)(b+R) < R. \end{aligned}$$

Now we can say that if $(b-a)(1+b-a) < 1$, for R greater than $\frac{b(b-a)(1+b-a)}{1-(b-a)(1+b-a)}$ the operator T satisfy the invariance condition. Thus by applying the Schauder theorem it follows that there exist at least one solution x^* and for this solution we have established that $\| x^* \| \leq \varphi(b) + R$. □

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Received by the editors: March 26, 2009.