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SOME TYPES OF CONVEX FUNCTIONS ON NETWORKS

DANIELA MARIAN*

Abstract. We present and study some kinds of convex functions defined on undirected networks. The relations between these concepts are also presented. We adopt the definition of network as metric space used by Dearing P. M. and Francis R. L. in 1974.

Keywords. E-d-convex functions, roughly d-convex functions, roughly E-d-convex functions.

1. INTRODUCTION

A class of sets and a class of functions called E-convex sets and E-convex functions are introduced in [35] by relaxing the definitions of convex sets and convex functions. This kind of generalized convexity is based on the effect of an operator E on the sets and domain of definition of the functions.

We recall the definitions of E-convex sets and E-convex functions.

We consider a map $E : \mathbb{R}^n \to \mathbb{R}^n$.

Definition 1. [35] *A set $A \subset \mathbb{R}^n$ is said to be E-convex if $\lambda E(x) + (1 - \lambda)E(y) \in A$, for each $x, y \in A$ and $0 \leq \alpha \leq 1$.*

Definition 2. [35] *A function $f : A \to \mathbb{R}$, $A \subset \mathbb{R}^n$ being E-convex, is said to be E-convex on A if for each $x, y \in A$ and $0 \leq \alpha \leq 1$ the following inequality is satisfied*

$$f(\lambda E(x) + (1 - \lambda)E(y)) \leq \lambda f(E(x)) + (1 - \lambda)f(E(y)).$$

We recall first the definitions of undirected networks as metric space introduced in [1] and also used in many other papers (see, e.g., [2], [9], [7], etc.).

We consider an undirected, connected graph $G = (W, A)$, without loops or multiple edges. To each vertex $w_i \in W = \{w_1, ..., w_n\}$ we associate a point v_i from an euclidean space X. This yields a finite subset $V = \{v_1, ..., v_n\}$ of X, called the vertex set of the network. We also associate to each edge $(w_i, w_j) \in A$ a rectifiable arc $[v_i, v_j] \subset X$ called edge of the network. We assume that any two edges have no interior common points. Consider that

*Department of Mathematics, Faculty of Automation and Computer Science, Technical University of Cluj-Napoca, Constantin Daicoviciu, no. 15, 400020 Cluj-Napoca, Romania, e-mail: daniela.marian@math.utcluj.ro.

$[v_i, v_j]$ has the positive length l_{ij} and denote by U the set of all edges. We define the network $N = (V, U)$ by

$$N = \{x \in X \mid \exists (w_i, w_j) \in A \text{ such that } x \in [v_i, v_j]\}.$$

It is obvious that N is a geometric image of G, which follows naturally from an embedding of G in X. Suppose that for each $[v_i, v_j] \in U$ there is a continuous one-to-one mapping $\theta_{ij} : [v_i, v_j] \to [0, 1]$ with $\theta_{ij}(v_i) = 0, \theta_{ij}(v_j) = 1$, and $\theta_{ij}([v_i, v_j]) = [0, 1]$. We denote by T_{ij} the inverse function of $\theta_{ij.}$

Any connected and closed subset of an edge bounded by two points x and y of $[v_i, v_j]$ is called a closed subedge and is denoted by $[x, y]$. If one or both of x, y are missing we say than the subedge is open in x, or in y or is open and we denote this by $(x, y]$, $[x, y)$ or (x, y), respectively. Using θ_{ij}, it is possible to compute the length of $[x, y]$ as

$$l([x, y]) = |\theta_{ij}(x) - \theta_{ij}(y)| \cdot l_{ij}.$$

Particularly we have

$$l([v_i, v_j]) = l_{ij}, l([v_i, x]) = \theta_{ij}(x)\, l_{ij}$$

and

$$l([x, v_j]) = (1 - \theta_{ij}(x)) \cdot l_{ij}.$$

A path $L(x,y)$ linking two points x and y in N is a sequence of edges and at most two subedges at extremities, starting at x and ending at y. If $x = y$ then the path is called cycle. The length of a path (cycle) is the sum of the lengths of all its component edges and subedges and will be denoted by $l(L(x, y))$. If a path (cycle) contains only distinct vertices then we call it elementary.

A network is connected if for any points $x, y \in N$ there exists a path $L(x, y) \subset N$.

A connected network without cycles is called tree.

Let $L^*(x, y)$ be a shortest path between the points $x, y \in N$. This path is also called geodesic.

Definition 3. [1] *For any $x, y \in N$, the distance from x to y, $d(x, y)$ in the network N is the length of a shortest path from x to y:*

$$d(x, y) = l(L^*(x, y)).$$

It is obvious that (N, d) is a metric space.

For $x, y \in N$, we denote

$$\langle x, y\rangle = \{z \in N \mid d(x, z) + d(z, y) = d(x, y)\}, \tag{1}$$

and $\langle x, y\rangle$ is called the metric segment between x and y.

One of the natural analogs of convexity in a metric space (X, d) is d-convexity, which has been introduced independently by K. Menger [13], P. S. Soltan and K. F. Prisakaru [28], J. de Groot [3], and other authors. However, except for definitions, these papers contain almost no theorems on d-convex sets. The theory of d-convex sets was mainly developed by P. S. Soltan and his students, see, e.g., [25], [30]. In graphs, the theory of d-convex sets was developed by V. P. Soltan [33], [29] and in networks by M. E. Iacob[7], M. E. Iacob and V. P. Soltan [8].

We recall the definitions of d-convex sets.

We consider a metric space (X, d) and the points $x, y \in X$. The set

$$\{z \in X \mid d(x,z) + d(z,y) = d(x,y)\} \tag{2}$$

is called d-segment with endpoints x, y and is denoted with $\langle x, y\rangle$.

Definition 4. [13] *A set $D \subset X$ is called d-convex if $\langle x, y\rangle \subset D$ for every $x, y \in D$.*

The empty set is d-convex and also X is d-convex.

Consequently, we have from network the next definition.

Definition 5. [1] *A set $D \subset N$ is called d-convex if $\langle x, y\rangle \subset D$ for any $x, y \in D$.*

2. E-d-CONVEX SETS ON UNDIRECTED NETWORKS

We consider a network N and a map $E : N \to N$.

According to Definitions 1 we define:

Definition 6. *A set $A \subset N$ is said to be E-d-convex if $\langle E(x), E(y)\rangle \subset A$, for each $x, y \in A$.*

Example 7. We consider the network $N = (V, U)$ with $V = \{v_1, v_2, v_3, v_4\}$ and

$$U = \{[v_1, v_2], [v_1, v_3], [v_3, v_2], [v_4, v_2], [v_4, v_3]\}$$

such that

$$l([v_1, v_2]) = 1 = l([v_3, v_2]) = l([v_4, v_3]) \text{ and}$$
$$l([v_1, v_3]) = 2 = l([v_4, v_2]).$$

For every edge $[v_i, v_j] \in U$ we consider the corresponding function $\theta_{ij} : [v_i, v_j] \to [0, 1]$. For every $z \in [v_i, v_j]$ we denote by z' the point of the edge $[v_1, v_2]$ such that $d(v_1, z') = \theta_{ij}(z)$. We define now

$$E : N \to N,\ E(z) = \begin{cases} z', & \forall z \in N \backslash \{v_3\} \\ v_2, & z = v_3. \end{cases}$$

The set $A = [v_1, v_2] \cup [v_4, v_2]$ it is not d-convex since $v_3 \in \langle v_1, v_4\rangle$ but $v_3 \notin A$. But it is obviously that the set A is E-d-convex with E defined above. $\square$

Theorem 8. *If a set $A \subset N$ is E-d-convex then $E(A) \subseteq A$.*

Proof. If the set A is E-d-convex then for every $x, y \in A$ we have

$$\langle E(x), E(y)\rangle = \{z \in N \mid d(E(x), z) + d(z, E(y)) = d(E(x), E(y))\} \subseteq A.$$

But $E(x) \in \langle E(x), E(y)\rangle$ since $d(E(x), E(x)) + d(E(x), E(y)) = d(E(x), E(y))$. Consequently $E(A) \subseteq A$. $\square$

THEOREM 9. *If $E(A)$ is d-convex and $E(A) \subseteq A$ then A is E-d-convex.*

Proof. We suppose that $x, y \in A$. Then $E(x)$ and $E(y) \in A$. Since $E(A)$ is d-convex we have $\langle E(x), E(y)\rangle \subset A$. Consequently A is E-d-convex. $\square$

THEOREM 10. *If the sets $A_1 \subset N$ and $A_2 \subset N$ are E-d-convex then the set $A_1 \cap A_2$ is E-d-convex.*

Proof. Indeed, if $x, y \in A_1 \cap A_2$ then $x, y \in A_1$ and $x, y \in A_2$. Since the sets A_1 and A_2 are E-d-convex then $\langle E(x), E(y)\rangle \subset A_1$ and $\langle E(x), E(y)\rangle \subset A_2$. Consequently $\langle E(x), E(y)\rangle \subset A_1 \cap A_2$. $\square$

THEOREM 11. *If the set $A \subset N$ is E_1-d-convex and E_2-d-convex then it is $(E_1 \circ E_2)$-d-convex and $(E_2 \circ E_1)$-d-convex.*

Proof. We consider the points $x, y \in A$. We suppose there is

$$z \in \langle (E_1 \circ E_2)(x), (E_1 \circ E_2)(y)\rangle = \langle E_1(E_2(x)), (E_1(E_2(y))\rangle$$

such that $z \notin A$. From Theorem 8 we have $E_2(x) \in A$ and $E_2(y) \in A$. We denote $x' = E_2(x)$ and $y' = E_2(y)$. Consequently $z \in \langle E_1(x'), E_1(y')\rangle$ but $z \notin A$, wich contradict E_1-d-convexity of A. Hence A is $(E_1 \circ E_2)$-d-convex. $\square$

Similarly, A is an $(E_2 \circ E_1)$-d-convex set.

3. E-d-CONVEX FUNCTIONS ON UNDIRECTED NETWORKS

We consider again a network N and a map $E : N \to N$. We denote $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.

We recall first the definitions of d-convex functions introduced by Soltan in [27].

DEFINITION 12. [27] *The function $f : N \to \overline{\mathbb{R}}$ is called d-convex on N if for any pair of points $x, y \in N, x \neq y$ and for all $z \in \langle x, y\rangle$ is satisfied the inequality*

$$f(z) \leq \tfrac{d(z,y)}{d(x,y)} f(x) + \tfrac{d(x,z)}{d(x,y)} f(y). \tag{3}$$

This functions was studied in [33], [30], [31], [32], [7], [26], [6], etc.

According to Definitions 2 we define:

DEFINITION 13. *A function $f : A \to \overline{\mathbb{R}}$, $A \subset N$ being E-d-convex, is said to be E-d-convex on A if for each $x, y \in A$ is satisfied the inequality*

$$f(z) \leq \tfrac{d(z,E(y))}{d(E(x),E(y))} f(E(x)) + \tfrac{d(E(x),z)}{d(E(x),E(y))} f(E(y))$$

for any $z \in \langle E(x), E(y)\rangle$.

EXAMPLE 14. We consider the network $N=(V,U)$ with $V=\{v_1,v_2,v_3,v_4\}$ and

$$U=\{[v_1,v_2],[v_1,v_3],[v_1,v_4]\},$$

such that

$$l([v_1,v_2])=1=l([v_1,v_3])=l([v_1,v_4]).$$

For every edge $[v_i,v_j]\in U$ we consider the corresponding function $\theta_{ij}:[v_i,v_j]\to[0,1]$. For every $z\in[v_i,v_j]$ we denote by z' the point of the edge $[v_1,v_2]$ such that $d(v_1,z')=\theta_{ij}(z)$. We define now

$$E:N\to N, E(z)=z', \forall z\in N$$

and

$$f:N\to\overline{\mathbb{R}}, f(x)=\begin{cases}1, & \text{if } x\in N\setminus[v_1,v_2]\\ d(v_1,x), & \text{if } x\in[v_1,v_2]\end{cases}.$$

It is obviously that f is E-d-convex on N. But this function it is not d-convex. Indeed, if we consider a point $x\in[v_1,v_2]$ such that $d(v_1,x)=d(v_2,x)$, $y=v_4$ and a point z such that $d(v_1,z)=d(v_4,z)$ we see that $z\in\langle x,y\rangle$ but

$$f(z)=1>\tfrac{d(z,y)}{d(x,y)}f(x)+\tfrac{d(x,z)}{d(x,y)}f(y)=\tfrac{1}{3}\cdot\tfrac{1}{2}+\tfrac{2}{3}\cdot 1=\tfrac{5}{6}$$

$\square$

THEOREM 15. *If the function $f:N\to\overline{\mathbb{R}}$ is E-d-convex on N and $f(E(z))\leq f(z)$ for every $z\in N$ then for every $\alpha\in\mathbb{R}$ the sets $A=\{z\in N\mid f(E(z))\leq\alpha\}$ and $B=\{z\in N\mid f(E(z))<\alpha\}$ are E-d-convex.*

Proof. Let us verify, for example, that the set A is E-d-convex. Let $x,y\in A$ and $z\in\langle E(x),E(y)\rangle$. Then

$$\begin{aligned} f(z) &\leq \tfrac{d(z,E(y))}{d(E(x),E(y))}f(E(x))+\tfrac{d(E(x),z)}{d(E(x),E(y))}f(E(y))\\ &\leq \left[\tfrac{d(z,E(y))+d(E(x),z)}{d(E(x),E(y))}\right]\alpha=\alpha. \end{aligned}$$

But $f(E(z))\leq f(z)$ for every $z\in N$, hence $f(E(z))\leq\alpha$ and $\langle E(x),E(y)\rangle\subset A$. Consequently A is E-d-convex. $\square$

4. ROUGHLY d-CONVEX FUNCTIONS

Roughly d-convex functions are a generalization of roughly convex functions and respective of d-convex functions proposed by V. P. Soltan and P. S. Soltan in [27]. We recall that there are several kinds of roughly convex functions: ρ-convex functions, proposed by Klötzler and investigated by Hartwig and Söllner in [4], [24], δ-convex and midpoint δ-convex functions established by Hu, Klee, Larman in [5] and γ-convex, strictly γ-convex, lightly γ-convex, midpoint γ-convex, strictly r-convexlike functions, proposed and investigated by Phu in [14], [15], [16], [17], [18], [19] etc.

In the following lines we consider a network $N=(V,U)$. We denote $\overline{\mathbb{R}}=\mathbb{R}\cup\{+\infty\}$.

Extending Phu's observation at this function, we remark in [12] that the inequality (3) can be satisfied just for the points $x, y \in N$ with $d(x, y) \geq r$, r being a fixed positive real number convenient selected.

We consider the positive real numbers $r_\rho, r_\delta, r_\gamma, r$ and a d-convex set $A \subset N$.

DEFINITION 16. [12] *The function $f : A \to \mathbb{R}$ is called:*

(1) ρ-d-convex on A with the roughness degree r_ρ if for any pair of points $x, y \in A$ with $d(x, y) \geq r_\rho$, is satisfied the inequality (3) for any $z \in \langle x, y\rangle$;

(2) δ-d-convex on A with the roughness degree r_δ if for any pair of points $x, y \in A$ with $d(x, y) \geq r_\delta$, is satisfied the inequality (3) for any $z \in \langle x, y\rangle$ with $d(x, z) \geq r_\delta/2$, and $d(z, y) \geq r_\delta/2$;

(3) midpoint δ-d-convex on A with the roughness degree r_δ if for any pair of points $x, y \in A$ with $d(x, y) \geq r_\delta$, is satisfied the inequality (3) for any $z \in \langle x, y\rangle$ with $d(x, z) = d(z, y) = d(x, y)/2$;

(4) γ-d-convex on A with the roughness degree r_γ if for any pair of points $x, y \in A$ with $d(x, y) \geq r_\gamma$, is satisfied the inequality

$$f\left(x'\right) + f\left(y'\right) \leq f(x) + f(y) \tag{4}$$

for any pair of points $x', y' \in \langle x, y\rangle$ with $d(x, x') = d(y, y') = r_\gamma$;

(5) lightly γ-d-convex on A with the roughness degree r_γ if for any pair of points $x, y \in A$ with $d(x, y) \geq r_\gamma$, is satisfied the inequality (3) for any $z \in \langle x, y\rangle$ with $d(x, z) = r_\gamma$ or for any $z \in \langle x, y\rangle$ with $d(z, y) = r_\gamma$;

(6) midpoint γ-d-convex on A with the roughness degree r_γ if for any pair of points $x, y \in A$ with $d(x, y) = 2r_\gamma$, is satisfied the inequality (3) for any $z \in \langle x, y\rangle$ with $d(x, z) = d(z, y) = r_\gamma$;

(7) strictly γ-d-convex on A with the roughness degree r_γ if for any pair of points $x, y \in A$ with $d(x, y) > r_\gamma$, is satisfied the inequality

$$f\left(x'\right) + f\left(y'\right) < f(x) + f(y), \tag{5}$$

for any pair of points $x', y' \in \langle x, y\rangle$ with $d(x, x') = d(y, y') = r_\gamma$;

(8) strictly r-d-convexlike (or strictly roughly d-convexlike) on A with the roughness degree r if for any pair of points $x, y \in A$ with $d(x, y) > r$ there is $z \in \langle x, y\rangle, z \neq x, z \neq y$ such that is satisfied the inequality:

$$f(z) < \tfrac{d(z,y)}{d(x,y)} f(x) + \tfrac{d(x,z)}{d(x,y)} f(y). \tag{6}$$

The functions who satisfy one of the conditions (1)-(8) are called roughly d-convex.

5. ROUGHLY E-d-CONVEX FUNCTIONS

In the following lines we will define and study roughly E-d-convex functions on networks, starting from roughly d-convex functions and E-d-convex functions respectively.

We consider the positive real numbers $r_\rho, r_\delta, r_\gamma, r$, a map $E : N \to N$ and a E-d-convex set $A \subset N$.

Definition 17. *The function $f : A \to \mathbb{R}$ is called:*

(1) ρ-E-d-convex on A with the roughness degree r_ρ if for any pair of points $x, y \in A$ with $d\,(E(x), E(y)) \geq r_\rho$, is satisfied the inequality

$$f\,(z) \leq \tfrac{d(z,E(y))}{d(E(x),E(y))} f\,(E(x)) + \tfrac{d(E(x),z)}{d(E(x),E(y))} f\,(E(y)) \tag{7}$$

for any $z \in \langle E(x), E(y)\rangle$;

(2) δ-E-d-convex on A with the roughness degree r_δ if for any pair of points $x, y \in A$ with $d\,(E(x), E(y)) \geq r_\delta$, is satisfied the inequality (7) for any $z \in \langle E(x), E(y)\rangle$ with $d\,(E(x), z) \geq r_\delta/2$, and $d\,(z, E(y)) \geq r_\delta/2$;

(3) midpoint δ-E-d-convex on A with the roughness degree r_δ if for any pair of points $x, y \in A$ with $d\,(E(x), E(y)) \geq r_\delta$, is satisfied the inequality (7) for any $z \in \langle E(x), E(y)\rangle$ with $d\,(E(x), z) = d\,(z, E(y)) = d\,(E(x), E(y))\,/2$;

(4) γ-E-d-convex on A with the roughness degree r_γ if for any pair of points $x, y \in A$ with $d\,(E(x), E(y)) \geq r_\gamma$, is satisfied the inequality

$$f\left(x'\right) + f\left(y'\right) \leq f\,(E(x)) + f\,(E(y)) \tag{8}$$

for any pair of points $x', y' \in \langle E(x), E(y)\rangle$ with

$$d\left(E(x), x'\right) = d\left(E(y), y'\right) = r_\gamma;$$

(5) lightly γ-E-d-convex on A with the roughness degree r_γ if for any pair of points $x, y \in A$ with $d\,(E(x), E(y)) \geq r_\gamma$, is satisfied the inequality (7) for any $z \in \langle E(x), E(y)\rangle$ with $d\,(E(x), z) = r_\gamma$ or for any $z \in \langle E(x), E(y)\rangle$ with $d\,(z, E(y)) = r_\gamma$;

(6) midpoint γ-E-d-convex on A with the roughness degree r_γ if for any pair of points $x, y \in A$ with $d\,(E(x), E(y)) = 2r_\gamma$, is satisfied the inequality (7) for any $z \in \langle E(x), E(y)\rangle$ with $d\,(E(x), z) = d\,(z, E(y)) = r_\gamma$;

(7) strictly γ-E-d-convex on A with the roughness degree r_γ if for any pair of points $x, y \in A$ with $d\,(E(x), E(y)) > r_\gamma$, is satisfied the inequality

$$f\left(x'\right) + f\left(y'\right) < f\,(E(x) + f\,(E(y))\,, \tag{9}$$

for any pair of points $x', y' \in \langle E(x), E(y)\rangle$ with

$$d\left(E(x), x'\right) = d\left(E(y), y'\right) = r_\gamma\,;$$

(8) strictly r-E-d-convexlike (or strictly roughly E-d-convexlike) on A with the roughness degree r if for any pair of points $x, y \in A$ with $d(E(x), E(y)) > r$ there is $z \in \langle E(x), E(y)\rangle\,, z \neq E(x), z \neq E(y)$ such that is satisfied the inequality:

$$f\,(z) < \tfrac{d(z,E(y))}{d(E(x),E(y))} f\,(E(x)) + \tfrac{d(E(x),z)}{d(E(x),E(y))} f\,(E(y))\,. \tag{10}$$

The functions who satisfy one of the conditions (1)-(8) are called roughly E-d-convex.

Example 18. We consider the network $N = (V, U)$ from the Example (7). The function $f : N \to \mathbb{R}$,

$$f(z) = \begin{cases} \frac{2}{3} - d(v_1, z), & d(v_1, z) < \frac{1}{3} \\ 0, & \text{otherwise} \end{cases}$$

is ρ-E-d-convex on $A = [v_1, v_2] \cup [v_4, v_2]$ with the roughness degree $r_\rho \geq \frac{2}{3}$. □

We compared this kinds of roughly E-d-convex functions and we got the following scheme for the relation between them:

Theorem 19. *Between some different kinds of roughly E-d-convex functions there are the following relations:*

$$\begin{array}{ccccccc} f \text{ } E\text{-}d\text{-}convex & \overset{\forall r_\rho > 0}{\Longrightarrow} & f \text{ } \rho\text{-}E\text{-}d\text{-}convex & \overset{r_\rho \leq r_\delta}{\Longrightarrow} & f \text{ } \delta\text{-}E\text{-}d\text{-}convex & \Longrightarrow & \begin{array}{c} f \text{ } midpoint \\ \delta\text{-}E\text{-}d\text{-}convex \end{array} \\ & & \Downarrow r_\rho \leq r_\gamma & & & & \Downarrow r_\delta = 2r_\gamma \\ & & f \text{ } \gamma\text{-}E\text{-}d\text{-}convex & \Longrightarrow & \begin{array}{c} f \text{ } lightly \\ \gamma\text{-}E\text{-}d\text{-}convex \end{array} & \Longrightarrow & \begin{array}{c} f \text{ } midpoint \\ \gamma\text{-}E\text{-}d\text{-}convex \end{array} \end{array}$$

Proof. The implications f E-d-convex on $A \overset{\forall r_\rho > 0}{\Longrightarrow}$ f ρ-E-d-convex on $A \overset{r_\rho \leq r_\delta}{\Longrightarrow}$ f δ-E-d-convex on $A \Longrightarrow$ f midpoint δ-E-d-convex on $A \overset{r_\delta = 2r_\gamma}{\Longrightarrow}$ f midpoint γ-E-d-convex on A and f lightly γ-E-d-convex on $A \Longrightarrow$ f midpoint γ-E-d-convex on A follow directly from Definition 17.

We verify now f ρ-E-d-convex on $A \overset{r_\rho \leq r_\gamma}{\Longrightarrow}$ f γ-E-d-convex on A. So let f be ρ-E-d-convex on A and $r_\rho \leq r_\gamma$. Then for any pair of points $x, y \in A$ with $d(E(x), E(y)) \geq r_\gamma$ and for all pair of points $x', y' \in \langle E(x), E(y) \rangle$ with $d(E(x), x') = d(y', E(y)) = r_\gamma$ we have

$$\begin{array}{l} f(x') \leq \frac{d(x', E(y))}{d(E(x), E(y))} f(E(x)) + \frac{d(E(x), x')}{d(E(x), E(y))} f(E(y)), \\ f(y') \leq \frac{d(y', E(y))}{d(E(x), E(y))} f(E(x)) + \frac{d(E(x), y')}{d(E(x), E(y))} f(E(y)). \end{array}$$

By addition the two inequalities yield

$$\begin{array}{l} f(x') + f(y') \leq \\ \leq \frac{d(x', E(y)) + d(y', E(y))}{d(E(x), E(y))} f(E(x)) + \frac{d(E(x), x') + d(E(x), y')}{d(E(x), E(y))} f(E(y)) \\ = \frac{d(x', E(y)) + d(x', E(x))}{d(E(x), E(y))} f(E(x)) + \frac{d(y', E(y)) + d(E(x), y')}{d(E(x), E(y))} f(E(y)) \\ = f(E(x)) + f(E(y)). \end{array}$$

Hence, f is γ-E-d-convex on A when $r_\rho \leq r_\gamma$.

Remain to verify f γ-E-d-convex on $A \Longrightarrow f$ lightly γ-E-d-convex on A. For that we suppose f is γ-E-d-convex on A but f is not lightly γ-E-d-convex on A, that means there exist two points $x, y \in A$ with $d(E(x), E(y)) \geq r_\gamma$ and a pair of points $x', y' \in \langle E(x), E(y) \rangle$ with $d\left(E(x), x'\right) = d\left(y', E(x)\right) = r_\gamma$ such that

$$f\left(x'\right) > \tfrac{d(x', E(y))}{d(E(x), E(y))} f\left(E(x)\right) + \tfrac{d(E(x), x')}{d(E(x), E(y))} f\left(E(y)\right)$$

and

$$f\left(y'\right) > \tfrac{d(y', E(y))}{d(E(x), E(y))} f\left(E(x)\right) + \tfrac{d(E(x), y')}{d(E(x), E(y))} f\left(E(y)\right).$$

By addition this two inequalities yield

$$\begin{aligned} f\left(x'\right) + f\left(y'\right) &> \tfrac{d(x', E(y)) + d(y', E(y))}{d(x,y)} f\left(E(x)\right) + \\ &+ \tfrac{d(E(x), x') + d(E(x), y')}{d(x,y)} f\left(E(y)\right) = \\ &= f\left(E(x)\right) + f\left(E(y)\right). \end{aligned}$$

hence f is not γ-E-d-convex on A, contradiction. $\square$

Theorem 20. *If the function $f : A \to \mathbb{R}$ is strictly γ-E-d-convex on A, with the roughness degree r_γ, then f is strictly r-E-d-convexlike, with any roughness degree $r \geq r_\gamma$.*

Proof. We consider a function $f : A \to \mathbb{R}$, strictly γ-E-d-convex on A. We suppose that f it is not strictly r-E-d-convexlike. We consider the points $x, y \in A$ such that $d\left(E(x), E(y)\right) > r \geq r_\gamma$ and the pair of points $x', y' \in \langle E(x), E(y) \rangle$ with $d\left(E(x), x'\right) = d\left(E(y), y'\right) = r_\gamma$. Since f it is not strictly r-E-d-convexlike on A, we have:

$$\begin{aligned} f\left(x'\right) &\geq \tfrac{d(x', E(y))}{d(E(x), E(y))} f\left(E(x)\right) + \tfrac{d(E(x), x')}{d(E(x), E(y))} f\left(E(y)\right) = \\ &= \tfrac{d(E(x), E(y)) - r_\gamma}{d(E(x), E(y))} f\left(E(x)\right) + \tfrac{r_\gamma}{d(E(x), E(y))} f\left(E(y)\right) \end{aligned}$$

$$\begin{aligned} f\left(y'\right) &\geq \tfrac{d(y', E(y))}{d(E(x), E(y))} f\left(E(x)\right) + \tfrac{d(E(x), y')}{d(E(x), E(y))} f\left(E(y)\right) = \\ &= \tfrac{r_\gamma}{d(E(x), E(y))} f\left(E(x)\right) + \tfrac{d(E(x), E(y)) - r_\gamma}{d(E(x), E(y))} f\left(E(y)\right) \end{aligned}$$

By addition this two inequalities yield

$$f\left(x'\right) + f\left(y'\right) \geq f\left(E(x)\right) + f\left(E(y)\right),$$

and this is in contradiction with the assumption that f strictly γ-E-d-convex on A. Hence f is strictly r-E-d-convexlike, with any roughness degree $r \geq r_\gamma$. $\square$

We consider the function $f : A \to \mathbb{R}$. We denote

$$E - \arg\min f = \left\{x^* \in A \mid \ f\left(E(x)\right) \geq \ f\left(E(x)\right)^*, \forall x \in A\right\}.$$

Theorem 21. *If the function $f : A \to \mathbb{R}$ is strictly r-E-d-convexlike on A with the roughness degree r and $E(A)$ is d-convex then*

$$d\left(E(x), E(y)\right) \leq r$$

for every $x, y \in E - \arg\min f$.

Proof. We suppose the contrary, that there is the points $x, y \in E-\arg\min f$ such that $d\left(E(x), E(y)\right) > r$. Since the set $E(A)$ is d-convex, we have $\langle E(x), E(y)\rangle \subset E(A)$. Hence there is $t \in A$ such that $E(t) = z$, for every $z \in \langle E(x), E(y)\rangle$.

Consequently

$$\begin{aligned} f(z) &= f(E(t)) \geq f(E(x) = f(E(y) = \\ &= \tfrac{d(z,E(y))}{d(E(x),E(y))} f\left(E(x)\right) + \tfrac{d(E(x),z)}{d(E(x),E(y))} f\left(E(y)\right) \end{aligned}$$

for every $z \in \langle E(x), E(y)\rangle$ and this contradict 10. Hence $d\left(E(x), E(y)\right) \leq r$ for every $x, y \in E - \arg\min f$. □

We consider now a set $A \subset N$, *E-d-convex*, a real number $r > 0$ and a function $f : A \to R$.

Definition 22. *We say that the function $f : A \to \mathbb{R}$ attains a r-E-d-local minimum at a point $x^* \in A$ if*

$$f\left(E(x)\right) \geq f\left(E(x)^*\right)$$

for any $x \in A$ satisfying $d\left(E(x), E(x^*)\right) < r$.

Definition 23. *We say that the function $f : A \to \mathbb{R}$ attains a E-d-global minimum at a point $x^* \in A$ if*

$$f\left(E(x)\right) \geq f\left(E(x^*)\right)$$

for any $x \in A$.

We denote

$$E - B(x^*, r_\gamma) = \{x \in A \ \mid d\left(E(x), E(x^*)\right) < r\}$$

and

$$\overline{E - B(x^*, r_\gamma)} = \{x \in A \ \mid d\left(E(x), E(x^*)\right) \leq r\}.$$

We consider a a tree network N, a set $A \subset N$, *E-d-convex* and a function $f : A \to R$. We suppose that $E(A)$ is d-convex.

Theorem 24. *If the function $f : A \to \mathbb{R}$ is a midpoint δ-E-d-convex on A with the roughness degree $r_\delta > 0$, $x^* \in A$ and*

$$f\left(E(x^*)\right) \leq f\left(E(x)\right) \tag{11}$$

for any $x \in E - B(x^*, r_\gamma)$ then $f\left(E(x^*)\right) \leq f\left(E(x)\right)$ for any $x \in A$ (f attains its global minimum in A at x^*).

Proof. We consider a midpoint δ-E-d-convex function on A with the roughness degree $r_\delta > 0$ such that $f\left(E(x^*)\right) \leq f\left(E(x)\right)$ for any $x \in E-B(x^*, r_\gamma)$.We suppose that f does not attain its global minimum at x^*. Then there is $x_0 \in A \backslash E - B(x^*, r_\gamma)$ such that $f\left(E(x^*)\right) > f\left(E(x_0)\right)$. We consider now a point $z_1 \in \langle E(x_0), E(x^*)\rangle$ such that $d(E(x_0), z_1) = d(z_1, E(x^*))$. Since f is midpoint *δ-E-d-convex* function on A, we have

$$\begin{aligned} f(z_1) &\leq \tfrac{d(E(x_0),z_1)}{d(E(x_0),E(x^*))} f\left(E(x^*)\right) + \tfrac{d(z_1,E(x^*))}{d(E(x_0),E(x^*))} f\left(E(x_0)\right) < \\ &< \tfrac{d(E(x_0),z_1)}{d(E(x_0),E(x^*))} f\left(E(x^*)\right) + \tfrac{d(z_1,E(x^*))}{d(E(x_0),E(x^*))} f\left(E(x^*)\right) = f(E(x^*)). \end{aligned}$$

Since the set $E(A)$ is $d-$convex, there is the point $x_1 \in A$ such that $E(x_1) = z_1$. Consequently $f(E(x_1)) < f(E(x^*))$. We repeat this construction, and we get $z_i \in E(A)$ and $x_i \in A, i \in I \subset N$, such that $E(x_i) = z_i$ and $f(E(x_i)) < f(E(x^*))$ for any $i \in I$. Since $d\left(E(x_i), E(x^*)\right) = d\left(E(x_{i-1}), E(x^*)\right)/2$, there is $i^* \in I$ such that $d\left(E(x_{i^*}), E(x^*)\right) < r_\delta$. Hence $x_{i^*} \in E - B(x^*, r_\gamma)$, consequently $f\left(E(x_{i^*})\right) \geq f\left(E(x^*)\right)$, which contradicts the relation $f\left(E(x^*)\right) > f\left(E(x_i)\right)$ for any $i \in I$. This contradiction completes our proof. □

REMARK 25. Since ρ-E-d-convexity and δ-E-d-convexity imply midpoint δ-E-d-convexity, ρ-E-d-convex functions and δ-E-d-convex functions have this property, too. □

We consider a tree network N, a set $A \subset N$, *E-d-convex* and a function $f : A \to \mathbb{R}$. We suppose that $E(A)$ is d-convex.

THEOREM 26. *If the function $f : A \to \mathbb{R}$ is a lightly γ-E-d-convex on A with the roughness degree $r_\gamma > 0$, $x^* \in A$ and*

$$f\left(E(x^*)\right) \leq f\left(E(x)\right)$$

for any $x \in \overline{E - B(x^*, r_\gamma)}$ then $f\left(E(x^*)\right) \leq f\left(E(x)\right)$ for any $x \in A$.

Proof. Assume the contrary that f does not attains its global minimum at x^*, then there is $x_0 \in A \backslash \overline{E - B(x^*, r_\gamma)}$ such that $f\left(E(x^*)\right) > f\left(E(x_0)\right)$. We consider now the points $s, z_1 \in \langle E(x_0), E(x^*)\rangle$ such that

$$d(E(x^*), s) = r_\gamma \ \text{ and } d(z_1, E(x_0)) = r_\gamma.$$

Since the set $E(A)$ is $d-$convex, there are the points $t, x_1 \in A$ such that $E(t) = s$ and $E(x_1) = z_1$. Since $f(E(x_0)) < f(E(x^*)) \leq f(E(t))$, the definition of lightly γ-E-d-convexity implies

$$f(z_1) \leq \tfrac{d(E(x_0),z_1)}{d(E(x_0),E(x^*))} f\left(E(x^*)\right) + \tfrac{d(z_1,E(x^*))}{d(E(x_0),E(x^*))} f\left(E(x_0)\right) < f(E(x^*)).$$

We repeat this construction, and we get $z_i \in \langle E(x_0), E(x^*)\rangle, x_i \in A, i \in I \subset N$, such that $E(x_i) = z_i$ and $f(E(x_i)) < f(E(x^*))$ for any $i \in I$. Since $d\left(E(x_i), E(x^*)\right) = d\left(E(x_{i-1}), E(x^*)\right) - r_\gamma$, there is $i^* \in I$ such that $d\left(E(x_{i^*}), E(x^*)\right) < r_\delta$ and hence for x_{i^*} we have $f\left(E(x_{i^*})\right) \geq f\left(E(x^*)\right)$, which contradicts the relation $f\left(E(x^*)\right) > f\left(E(x_i)\right)$ for any $i \in I$. This contradiction completes our proof. □

REMARK 27. Since every γ-E-d-convex function is lightly γ-E-d-convex, this conclusion holds for γ-E-d-convex functions, too. □

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